PASSIVE STATE/SIGNAL SYSTEMS AND CONSERVATIVE BOUNDARY RELATIONS

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Contents

1. Introduction

This chapter is a continuation and deepening of the introductory Chapter ??. Here we extend the state/signal theory (s/s theory) beyond boundary control, and the main aim of this chapter is to clarify the basic connections between the s/s theory and that of boundary relations, in a way similar to Theorem ?? in Chapter ??.

The theory of boundary relations has been developed by a number of authors in the framework of the theory of self-adjoint extensions of symmetrical operators and relations in Hilbert spaces; see e.g. the recent articles [Derkach et al., 2006], [Derkach et al., 2009], and [Behrndt et al., 2009].

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One way of introducing the notion of a state/signal (s/s) system is to start from an input/state/output (i/s/o) system. By a standard i/s/o system we mean a system of equations of the type

$$\Sigma_{i/s/o}: \quad \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t), \end{cases} \quad t \in \mathbb{R}^+, \ x(0) = x_0 \text{ given}, \tag{1.1}$$

where $\dot{x}$ stands for the time derivative of x. Here x, u and y take values in the Hilbert spaces $\mathcal{X}$, $\mathcal{U}$ and $\mathcal{Y}$, that are called the "state space", the "input space" and the "output space", respectively. For now the linear operators A, B, C, and D are assumed to be bounded, but we soon drop this restrictive assumption.

The system (1.1) can be viewed as an i/s/o representation of a s/s system by setting $\mathcal{W} := \left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right]$ and using the graph V of the operator $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$:

$$\begin{aligned} &\Sigma: \quad \begin{bmatrix} \dot{x}(t) \\ x(t) \\ \left[\begin{smallmatrix} u(t) \\ y(t) \end{smallmatrix}\right] \end{bmatrix} \in V, \quad t \in \mathbb{R}^+, \ x(0) = x_0, \quad \text{where} \\ &V := \left\{ \begin{bmatrix} z \\ x \\ \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right] \end{bmatrix} \in \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix} \;\middle|\; \begin{matrix} z = Ax + Bu \\ y = Cx + Du \end{matrix} \right\}. \end{aligned} \tag{1.2}$$

This reformulation might seem trivial, but many concepts, such as that of a passive or a conservative system, becomes much simpler in the s/s framework than in the i/s/o formulation, see Remark 18 below. Moreover, the input/output-free approach of the s/s theory permits the study of a physical system *as such* by looking at the geometric properties of V instead of merely studying *a particular representation* $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ of the system, cf. Remark ??.

We give the general definition of a s/s system, which does not a priori assume a representation (1.2), and we discuss a particular class of i/s/o representations in Section 2. In Section 3 we study passive and conservative systems in more detail. The topic of Section 4 is frequency domain theory, and here we introduce the characteristic node bundle of a s/s system, which extends the notions of the γ-field and the Weyl family of a boundary relation. We make the precise connection in Section 5, where we also describe exactly how to transform a conservative s/s system into a boundary relation and vice versa.

2. Continuous-time state/signal systems

In this section we extend the ideas in Section ?? to more general generating subspaces V than those arising from either an i/s/o representation of the type (1.2) or from a boundary control s/s system.

2.1. **General definitions.** We first introduce the s/s node and the s/s system that it induces.

Definition 1. Let $\mathcal{X}$ be a Hilbert space, let $\mathcal{W}$ be a Kreĭn space, and let V be a *closed* subspace of the node space $\mathfrak{K} = \mathcal{X} \times \mathcal{X} \times \mathcal{W}$ equipped with the indefinite inner product induced by the quadratic form

$$\left[\left[\begin{smallmatrix} z_1 \\ x_1 \\ w_1 \end{smallmatrix}\right], \left[\begin{smallmatrix} z_2 \\ x_2 \\ w_2 \end{smallmatrix}\right]\right]_{\mathfrak{K}} = -(z_1, x_2)_{\mathcal{X}} - (x_1, z_2)_{\mathcal{X}} + [w_1, w_2]_{\mathcal{W}}.$$

The pair $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is a *classical trajectory* generated by V on $\mathbb{R}^+$ if $x \in C^1(\mathbb{R}^+; \mathcal{X})$, $w \in C(\mathbb{R}^+; \mathcal{W})$, and

$$\left[\begin{smallmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{smallmatrix}\right] \in V, \quad t \in \mathbb{R}^+, \tag{2.1}$$

where $\dot{x}$ stands for the time derivative of x (at $t = 0$ this is the right-sided derivative of x at zero). The closure of the set of classical trajectories on $\mathbb{R}^+$ in $\left[\begin{smallmatrix} C(\mathbb{R}^+;\mathcal{X}) \\ L^2_{\text{loc}}(\mathbb{R}^+;\mathcal{W}) \end{smallmatrix}\right]$ is the set of *generalized trajectories* on $\mathbb{R}^+$ generated by V.

Moreover, $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is a *state/signal node (s/s node)* if V has the following properties in addition to being closed:

(1) The generating subspace V satisfies the condition

$$\left[\begin{smallmatrix} z \\ 0 \\ 0 \end{smallmatrix}\right] \in V \quad \Longrightarrow \quad z = 0. \tag{2.2}$$

(2) For every $\left[\begin{smallmatrix} z_0 \\ x_0 \\ w_0 \end{smallmatrix}\right] \in V$ there exists a classical trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ of Σ on $\mathbb{R}^+ := [0, \infty)$ that satisfies $\left[\begin{smallmatrix} \dot{x}(0) \\ x(0) \\ w(0) \end{smallmatrix}\right] = \left[\begin{smallmatrix} z_0 \\ w_0 \\ w_0 \end{smallmatrix}\right]$.

By the *state/signal system (s/s system)* induced by a s/s node $(V; \mathcal{X}, \mathcal{W})$ we mean the s/s node itself together with its sets of classical and generalized trajectories on $\mathbb{R}^+$ generated by V.

It follows immediately from part (2) of Definition 1 that a space of classical trajectories determines its generating subspace uniquely through

$$V = \left\{ \left[\begin{smallmatrix} \dot{x}(0) \\ x(0) \\ w(0) \end{smallmatrix}\right] \;\middle|\; \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \text{ is a classical trajectory} \right\}.$$

It is less obvious, but still true, that a space of generalized trajectories determines its generating s/s node uniquely. This is because the space of generalized trajectories of a s/s node determines the space of classical trajectories uniquely. Indeed, a generalized trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is in fact a classical trajectory if and only if $x \in C^1(\mathbb{R}^+; \mathcal{X})$ and $w \in C(\mathbb{R}^+; \mathcal{W})$. For proof, see [Kurula and Staffans, 2011, Cor. 3.2].

Example 2. Let $(L, \Gamma; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a boundary control s/s node as given in Definition ??. This does in general not imply that $\Sigma :=$

$(V;\mathcal{X},\mathcal{W})$ is a s/s system, where V is given by

$$V=\left\{\begin{bmatrix} Lx\\ x\\ \Gamma x\end{bmatrix}\,\middle|\, x\in\operatorname{dom}(L)\right\},$$

because V might not have property (2) of Definition 1. We prove in Example 8 below that Σ is indeed a s/s system when L and Γ arise from a boundary control i/s/o system of the type described in Definition ??.

The fact that the generating subspace V is independent of the time variable t means that the state/signal system is *time invariant*. Moreover, condition (2.2) means that V is the graph of some linear operator $G\colon \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{W}\end{smallmatrix}\right]\to\mathcal{X}$ with domain $\operatorname{dom}(G)\subset\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{W}\end{smallmatrix}\right]$, i.e., that

$$V=\left\{\begin{bmatrix} z\\ x\\ w\end{bmatrix}\,\middle|\, z=G\left[\begin{smallmatrix}x\\ w\end{smallmatrix}\right],\ \left[\begin{smallmatrix}x\\ w\end{smallmatrix}\right]\in\operatorname{dom}(G)\right\}.$$

The assumption that V is closed means that G is a closed operator. Now (2.1) can alternatively be written in the form

$$\begin{bmatrix} x(t)\\ w(t)\end{bmatrix}\in\operatorname{dom}(G)\quad\text{and}\quad \dot x(t)=G\begin{bmatrix} x(t)\\ w(t)\end{bmatrix},\qquad t\in\mathbb{R}^+, \tag{2.3}$$

and all classical trajectories generated by V satisfy this condition.

Example 3. If V is given by (1.2) then the operator G defined above is given by

$$\begin{aligned} G&=\begin{bmatrix} A & \begin{bmatrix} B & 0\end{bmatrix}\end{bmatrix}\Big|_{\operatorname{dom}(G)}\quad\text{with}\\ \operatorname{dom}(G)&=\left\{\begin{bmatrix} x\\ \left[\begin{smallmatrix}u\\ Cx+Du\end{smallmatrix}\right]\end{bmatrix}\,\middle|\,\left[\begin{smallmatrix}x\\ u\end{smallmatrix}\right]\in\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right]\right\}.\end{aligned}\tag{2.4}$$

Note, however, that the operators A, B, C, and D in (1.2), and therefore also in (2.4), by construction depend on a particular choice of i/o (input/output) decomposition $\mathcal{W}=\left[\begin{smallmatrix}\mathcal{U}\\ \mathcal{Y}\end{smallmatrix}\right]$, whereas (2.3) does not. In this sense (2.3) is a truly coordinate-free differential-equation representation of a s/s system.

Let us now go back to the general case. Condition (2) in Definition 1 means that there for all $\left[\begin{smallmatrix}x_0\\ w_0\end{smallmatrix}\right]\in\operatorname{dom}(G)$ exists a classical trajectory $\left[\begin{smallmatrix}x\\ w\end{smallmatrix}\right]\in\begin{bmatrix} C^1(\mathbb{R}^+;\mathcal{X})\\ C(\mathbb{R}^+;\mathcal{W})\end{bmatrix}$ such that $\begin{bmatrix} x(0)\\ w(0)\end{bmatrix}=\left[\begin{smallmatrix}x_0\\ w_0\end{smallmatrix}\right]$. From the condition $\begin{bmatrix}\dot x(0)\\ x(0)\\ w(0)\end{bmatrix}\in V$ we immediately obtain that this trajectory also satisfies $\dot x(0)=G\left[\begin{smallmatrix}x_0\\ w_0\end{smallmatrix}\right]$.

It is an interesting observation that we can represent an arbitrary s/s system by a closed operator G, and it helps to build intuition, but we shall not make any significant use of this operator in this exposition. For our present purposes it is more convenient to use (2.1).

As is well-known, an arbitrary Kreĭn space $\mathcal{W}$ can be interpreted as a Hilbert space consisting of the same vectors as $\mathcal{W}$. This is done by equipping $\mathcal{W}$ with an *admissible Hilbert-space inner product*; see Remark 11 below. An important consequence is that, from a topological point of view, every closed subspace of a Kreĭn space can be regarded as a Hilbert space, and we make frequent use of this.

Definition 4. A direct-sum decomposition $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$ of a Kreĭn space is *i/s/o well-posed* for the s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ if the following two conditions hold:

(1) For every $x_0 \in \mathcal{X}$ and $u \in L^2_{\rm loc}(\mathbb{R}^+; \mathcal{U})$ there exists a generalized trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \left[\begin{smallmatrix} C(\mathbb{R}^+;\mathcal{X}) \\ L^2_{\rm loc}(\mathbb{R}^+;\mathcal{W}) \end{smallmatrix}\right]$ of Σ on $\mathbb{R}^+$ with $x(0) = x_0$ and $P^{\mathcal{Y}}_{\mathcal{U}} w = u$.

(2) There exists a positive nondecreasing function K on $\mathbb{R}^+$ such that every generalized trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \left[\begin{smallmatrix} C(\mathbb{R}^+;\mathcal{X}) \\ L^2_{\rm loc}(\mathbb{R}^+;\mathcal{W}) \end{smallmatrix}\right]$ of Σ on $\mathbb{R}^+$ satisfies

$$\begin{aligned} &\|x(t)\|^2_{\mathcal{X}} + \int_0^t \|P^{\mathcal{U}}_{\mathcal{Y}} w(s)\|^2_{\mathcal{W}}\,\mathrm{d}s \\ &\qquad \le K(t)\left(\|x(0)\|^2_{\mathcal{X}} + \int_0^t \|P^{\mathcal{Y}}_{\mathcal{U}} w(s)\|^2_{\mathcal{W}}\,\mathrm{d}s\right), \quad t \in \mathbb{R}^+. \end{aligned} \tag{2.5}$$

Here $\|\cdot\|_{\mathcal{W}}$ stands for an arbitrary admissible norm in $\mathcal{W}$.

The s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is *well-posed* if there exists at least one i/s/o well-posed decomposition $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$ of the signal space $\mathcal{W}$.

For more details on well-posed s/s systems, see [Kurula and Staffans, 2009]. In the next section we elaborate on the topic of representing state/signal systems by i/s/o systems.

2.2. **Input/state/output representations.** The simplest example of a s/s system may be constructed by starting from a bounded classical linear i/s/o continuous-time system $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ as we did in the introduction.

However, applications often require that the operator $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is unbounded. In the unbounded case the operator $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ in (1.1) can be replaced by an *i/s/o system node* operator $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$. Here the top and bottom rows are denoted by $A\&B$ and $C\&D$ in order to indicate the connection to (1.1), but this notation is purely symbolic. In general it is possible to extend $A\&B$ into an operator $\left[\begin{matrix} A_{-1} & B \end{matrix}\right]$ which maps $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]$ continuously into a larger extrapolation space $\mathcal{X}_{-1}$. The operator A_{-1} is the continuous extension to $\mathcal{X}$ of the generator A of a C_0-semigroup on $\mathcal{X}$. Unfortunately, $C\&D$ does not split correspondingly. One can define an operator C, whose domain is a subspace of $\mathcal{X}$ containing the domain of A, but there is no uniquely defined operator corresponding to D in the general unbounded case. See [Staffans, 2005, Chapter 5] for details.

We now give a definition of an abstract system node, which is based on [Staffans, 2005, Lem. 4.7.7].

Definition 5. By an *i/s/o-system node* $(S;\mathcal{X},\mathcal{U},\mathcal{Y})$ we mean a triple of Hilbert spaces $\mathcal{X}$ (the state space), $\mathcal{U}$ (the input space), and $\mathcal{Y}$ (the output space), together with a linear operator

$$S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right] : \mathrm{dom}\,(S) \to \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right], \quad \mathrm{dom}\,(S) \subset \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right],$$

with the following properties:

(1) The operator $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right] : \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$ is closed as an operator mapping $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]$ into $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$ with domain $\mathrm{dom}\,(S)$.

(2) The operator $A\&B : \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \to \mathcal{X}$ is closed with domain $\mathrm{dom}\,(S)$.

(3) The *main operator* A of $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$, defined by

$$Ax = A\&B \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] \quad \text{on} \quad \mathrm{dom}\,(A) = \left\{ x \in \mathcal{X} \;\middle|\; \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] \in \mathrm{dom}\,(S) \right\},$$

generates a strongly continuous semigroup $t \mapsto \mathfrak{A}^t$ on $\mathcal{X}$.

(4) For all $u \in \mathcal{U}$ there exists an $x \in \mathcal{X}$ such that $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \mathrm{dom}\,(S)$.

The triple (u,x,y) is said to be a *classical i/s/o trajectory* of the i/s/o system node $(S;\mathcal{X},\mathcal{U},\mathcal{Y})$ if $u \in C(\mathbb{R}^+;\mathcal{U})$, $x \in C^1(\mathbb{R}^+;\mathcal{X})$, $y \in C(\mathbb{R}^+;\mathcal{Y})$, and

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad t \in \mathbb{R}^+. \tag{2.6}$$

As we can see from the above definition, in the unbounded case (1.1) is replaced by (2.6). The i/s/o system (2.6) can again be interpreted as an i/s/o representation of a s/s system $\Sigma = (V;\mathcal{X},\mathcal{W})$ by taking $\mathcal{W} := \left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right]$ and defining

$$V := \left\{ \left[\begin{smallmatrix} z \\ x \\ \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right] \end{smallmatrix}\right] \subset \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right] \;\middle|\; \left[\begin{smallmatrix} z \\ y \end{smallmatrix}\right] = S \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right],\ \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \mathrm{dom}\,(S) \right\}. \tag{2.7}$$

Here $\left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right]$ stands for the product of $\mathcal{U}$ and $\mathcal{Y}$ which can be turned into a Kreĭn space by equipping it with any of several indefinite inner products. A few important choices of inner product will be described later.

Recall that the component spaces $\mathcal{U}$ and $\mathcal{Y}$ of every direct-sum decomposition $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$ of a Kreĭn space can be interpreted as Hilbert spaces with inner products inherited from some admissible inner product in $\mathcal{W}$.

Remark 6. In the sequel we call $\Sigma_{i/s/o} = (S;\mathcal{X},\mathcal{U},\mathcal{Y})$, where $\mathcal{U}$ and $\mathcal{Y}$ are arbitrary closed subspaces of some Kreĭn spaces, an i/s/o system node if $\Sigma_{i/s/o}$ is an i/s/o system node in the sense of Definition 5 with $\mathcal{U}$ and $\mathcal{Y}$ equipped with admissible inner products.

We now define an i/s/o representation of a general s/s system $(V;\mathcal{X},\mathcal{W})$.

Definition 7. Let $\Sigma = (V;\mathcal{X},\mathcal{W})$ be a s/s system and let $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$ be an arbitrary direct-sum decomposition of the signal space.

Assume that V can be written on the form (2.7), where $\Sigma_{i/s/o} := (S;\mathcal{X},\mathcal{U},\mathcal{Y})$ is an i/s/o system node. Then we call $\Sigma_{i/s/o}$ the *i/s/o representation* of Σ corresponding to the i/o (input/output) decomposition $\mathcal{W} = \left[\begin{smallmatrix}\mathcal{U}\\ \{0\}\end{smallmatrix}\right] \dotplus \left[\begin{smallmatrix}\{0\}\\ \mathcal{Y}\end{smallmatrix}\right]$, and we call the i/o decomposition *system-node admissible*, or shortly just *admissible*.

The i/s/o representation $\Sigma_{i/s/o}$ is uniquely determined by the s/s system $\Sigma = (V;\mathcal{X},\mathcal{W})$ and the decomposition $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$ (except for the fact that the norms and inner products in $\mathcal{U}$ and $\mathcal{Y}$ are determined only up to equivalence), since V is the graph of S in the sense of (2.7). In general, a s/s system Σ has several i/s/o representations, one induced by every admissible i/o decomposition of $\mathcal{W}$.

Example 8. Let $\Sigma_{i/s/o} = (L,\Gamma_0,\Gamma_1;\mathcal{X},\mathcal{U},\mathcal{Y})$ be an i/s/o boundary control system of the type in Definition ??. We let $\mathcal{W} := \left[\begin{smallmatrix}\mathcal{U}\\ \mathcal{Y}\end{smallmatrix}\right]$, equipped with an arbitrary Kreĭn-space inner product, e.g. the standard Hilbert-space inner product, and we define

$$V := \left\{ \left[\begin{smallmatrix} Lx \\ x \\ \left[\begin{smallmatrix}\Gamma_0 x\\ \Gamma_1 x\end{smallmatrix}\right]\end{smallmatrix}\right] \,\middle|\, x \in \operatorname{dom}(L) \right\}. \tag{2.8}$$

We now prove that $\Sigma = (V;\mathcal{X},\mathcal{W})$ is a s/s system with admissible i/o decomposition $\mathcal{W} = \left[\begin{smallmatrix}\mathcal{U}\\ \{0\}\end{smallmatrix}\right] \dotplus \left[\begin{smallmatrix}\{0\}\\ \mathcal{Y}\end{smallmatrix}\right]$. We find the corresponding i/s/o representation $(S;\mathcal{X},\mathcal{U},\mathcal{Y})$ by identifying $\mathcal{U} = \left[\begin{smallmatrix}\mathcal{U}\\ \{0\}\end{smallmatrix}\right]$ and $\mathcal{Y} = \left[\begin{smallmatrix}\{0\}\\ \mathcal{Y}\end{smallmatrix}\right]$, and by noting that the map from $\left[\begin{smallmatrix}x\\ u\end{smallmatrix}\right]$ to $\left[\begin{smallmatrix}z\\ y\end{smallmatrix}\right]$, where $\left[\begin{smallmatrix} z \\ x \\ \left[\begin{smallmatrix}u\\ y\end{smallmatrix}\right]\end{smallmatrix}\right] \in V$, is given by

$$S = \left[\begin{smallmatrix}L\\ \Gamma_1\end{smallmatrix}\right]\left[\begin{smallmatrix}1\\ \Gamma_0\end{smallmatrix}\right]^{-1}, \quad \operatorname{dom}(S) = \left\{ \left[\begin{smallmatrix}x\\ \Gamma_0 x\end{smallmatrix}\right] \,\middle|\, x \in \operatorname{dom}(L) \right\}.$$

A detailed investigation of the connections between $\left[\begin{smallmatrix}L\\ \Gamma_0\\ \Gamma_1\end{smallmatrix}\right]$ and S can be found in Section 2 of [Malinen and Staffans, 2006]. In particular, S is a system node by [Malinen and Staffans, 2006, Thm 2.3], and from [Kurula, 2010, Prop. 2.7] it then follows that Σ is a s/s node, as we claimed above. This shows how nicely boundary control can be incorporated into the general s/s framework.

If $\Sigma_{i/s/o} = (S;\mathcal{X},\mathcal{U},\mathcal{Y})$ is an i/s/o representation of $\Sigma = (V;\mathcal{X},\mathcal{W})$, so that V is given by (2.7), then the well-posedness condition (2.5) is equivalent to the condition that every classical trajectory (u,x,y) of $\Sigma_{i/s/o}$ satisfies the following inequality for all $t \in \mathbb{R}^+$:

$$\|x(t)\|_{\mathcal{X}}^2 + \int_0^t \|y(s)\|_{\mathcal{Y}}^2 \,\mathrm{d}s \le K(t)\left(\|x(0)\|_{\mathcal{X}}^2 + \int_0^t \|u(s)\|_{\mathcal{U}}^2 \,\mathrm{d}s \right). \tag{2.9}$$

Definition 9. Input/state/output systems whose classical trajectories (u,x,y) satisfy (2.9) for a positive nondecreasing function K, which does not depend on the trajectory, are called *well-posed*.

It follows directly from Definitions 4 and 7 that if an i/o decomposition is both admissible and well-posed for a s/s system then the corresponding i/s/o representation is i/s/o-well-posed. In fact, every well-posed i/o decomposition is admissible by [Kurula and Staffans, 2009, Thms 4.9 and 6.4]. For more detailed information on i/s/o system nodes and well-posed i/s/o systems we refer the reader to [Staffans, 2005].

3. Passive and conservative state/signal systems

In this section we describe the concepts of passivity and conservativity within the state/signal system framework.

We need the notion of an *anti-Hilbert space*. A subspace $\mathcal{Y}$ of a Kreĭn space $\mathcal{W}$, equipped with the inherited indefinite inner product $[y_1, y_2]_\mathcal{Y} = [y_1, y_2]_\mathcal{W}$ for $y_j \in \mathcal{Y}$, is an anti-Hilbert space if $-\mathcal{Y}$, i.e., the space of all vectors in $\mathcal{Y}$ equipped with the inner product $-[\cdot,\cdot]_\mathcal{Y}$, is a Hilbert space.

Definition 10. A direct-sum decomposition $\mathcal{W} = \mathcal{W}_1 \dotplus \mathcal{W}_2$ of a Kreĭn space is called:

(1) *orthogonal* if every vector $w_1 \in \mathcal{W}_1$ is orthogonal to every vector in $w_2 \in \mathcal{W}_2$: $[w_1, w_2]_\mathcal{W} = 0$, and we write this as $\mathcal{W} = \mathcal{W}_1 \boxplus \mathcal{W}_2$.
(2) *fundamental* if $\mathcal{W} = \mathcal{W}_1 \boxplus \mathcal{W}_2$, where $\mathcal{W}_1$ is a Hilbert space and $\mathcal{W}_2$ is an anti-Hilbert space in the inner product inherited from $\mathcal{W}$. In this case we denote $\mathcal{W}_+ := \mathcal{W}_1$ and $\mathcal{W}_- := -\mathcal{W}_2$, so that $\mathcal{W}_+$ always is the Hilbert space component and $-\mathcal{W}_-$ is the anti-Hilbert space component, and we write $\mathcal{W} = \mathcal{W}_+ \boxplus -\mathcal{W}_-$.
(3) *Lagrangian* if $\mathcal{W}_1$ and $\mathcal{W}_2$ are both Lagrangian: $\mathcal{W}_j = \mathcal{W}_j^{[\perp]}$. We introduce and explain the special notation $\mathcal{W} = \mathcal{W}_1 \overset{\Psi}{\dotplus} \mathcal{W}_2$ for Lagrangian decompositions in Definition 21 below.

When $\mathcal{W}$ is the signal space of a s/s system we typically use $\mathcal{W}_1$ as input space and $\mathcal{W}_2$ as output space in i/s/o representations. In this connection we do not always use the inner products inherited from $\mathcal{W}$ in $\mathcal{W}_1$ and $\mathcal{W}_2$. In a Lagrangian decomposition the subspaces do not even inherit a unique inner product from $\mathcal{W}$. In the case of orthogonal (and fundamental) decompositions we throughout take the input space to be $\mathcal{U} := \mathcal{W}_1$ with the inner product inherited from $\mathcal{W}$ and the output space to be $\mathcal{Y} := -\mathcal{W}_2$. Thus in the case of a fundamental decomposition both $\mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces.

If $\mathcal{W} = \mathcal{U} \boxplus \mathcal{Y}$, then in fact $\mathcal{Y} = \mathcal{U}^{[\perp]}$ and both $\mathcal{U}$ and $\mathcal{Y}$ are themselves Kreĭn spaces. Every Kreĭn space, which is neither a Hilbert space nor an anti-Hilbert space, has an uncountable number of fundamental decompositions $\mathcal{W} = \mathcal{W}_+ \boxplus -\mathcal{W}_-$. For every fundamental decomposition it holds that

$$\begin{aligned}
[w_+, w_+]_\mathcal{W} &= (w_+, w_+)_{\mathcal{W}_+} > 0, \qquad && w_+ \in \mathcal{W}_+,\ w_+ \neq 0, \\
[w_-, w_-]_\mathcal{W} &= -(w_-, w_-)_{\mathcal{W}_-} < 0, \qquad && w_- \in -\mathcal{W}_-,\ w_- \neq 0.
\end{aligned}$$

Remark 11. Let $\mathcal{W} = \mathcal{W}_+ \boxplus -\mathcal{W}_-$ be a fundamental decomposition of a Kreĭn space. Then $\mathcal{W}$ can be viewed as a Hilbert space with the inner product

$$(w_{1,+} + w_{1,-}, w_{2,+} + w_{2,-})_{\mathcal{W}} = (w_{1,+}, w_{2,+})_{\mathcal{W}_+} + (w_{1,-}, w_{2,-})_{\mathcal{W}_-}, \qquad w_{1,+}, w_{2,+} \in \mathcal{W}_+, \quad w_{1,-}, w_{2,-} \in -\mathcal{W}_-.$$

This inner product is called an *admissible inner product* and the norm induced by this inner product is called an *admissible norm.*

If $\mathcal{W}$ is either a Hilbert space or an anti-Hilbert space, then $\mathcal{W}$ has one unique fundamental decomposition, but in all other case $\mathcal{W}$ has infinitely many fundamental decompositions, and consequently also infinitely many admissible norms. However, all of these norms are equivalent.

Thus, once a fundamental decomposition $\mathcal{W} = \mathcal{W}_+ \boxplus -\mathcal{W}_-$ has been fixed, each $w \in \mathcal{W}$ has a unique decomposition $w = w_+ + w_-$ with $w_\pm \in \mathcal{W}_\pm$, and

$$[w, w]_{\mathcal{W}} = (w_+, w_+)_{\mathcal{W}_+} - (w_-, w_-)_{\mathcal{W}_-} = \|w_+\|^2_{\mathcal{W}_+} - \|w_-\|^2_{\mathcal{W}_-}. \tag{3.1}$$

The dimensions of $\mathcal{W}_\pm$ do not depend on the choice of fundamental decomposition $\mathcal{W} = \mathcal{W}_+ \boxplus -\mathcal{W}_-$. They are called the positive and negative indices of $\mathcal{W}$ and are denoted by $\mathrm{ind}_\pm \mathcal{W}$. A Lagrangian decomposition of $\mathcal{W}$ exists if and only if $\mathrm{ind}_+ \mathcal{W} = \mathrm{ind}_- \mathcal{W}$.

3.1. **Passive s/s systems and scattering representations.**

We first recall that a subspace V of a Kreĭn space $\mathfrak{K}$ is called *non-negative, non-positive*, or *neutral* if every vector $v \in V$ satisfies

$$[v, v]_{\mathfrak{K}} \geq 0, \quad [v, v]_{\mathfrak{K}} \leq 0, \quad \text{or} \quad [v, v]_{\mathfrak{K}} = 0,$$

respectively. A non-negative (or non-positive) subspace is called maximal non-negative (or maximal non-positive) if it is not strictly contained in any other non-negative (or non-positive) subspace. Such a subspace is automatically closed. A subspace V is *Lagrangian* if $V = V^{[\perp]}$, where $V^{[\perp]}$ is given by

$$V^{[\perp]} := \bigl\{ k \in \mathfrak{K} \bigm| [k, k']_{\mathfrak{K}} = 0 \text{ for all } k' \in V \bigr\}.$$

Since many physical systems lack internal energy sources, it is natural to require the generating subspace V to be non-negative in the node space $\mathfrak{K} := \mathcal{X} \times \mathcal{X} \times \mathcal{W}$ which is equipped with the inner product

$$\left[\begin{bmatrix} z_1 \\ x_1 \\ w_1 \end{bmatrix}, \begin{bmatrix} z_2 \\ x_2 \\ w_2 \end{bmatrix} \right]_{\mathfrak{K}} = -(z_1, x_2)_{\mathcal{X}} - (x_1, z_2)_{\mathcal{X}} + [w_1, w_2]_{\mathcal{W}}; \tag{3.2}$$

cf. Definition 1.

The node space $\mathfrak{K}$ is a Kreĭn space with the fundamental decomposition $\mathfrak{K} = \mathfrak{K}_+ \boxplus -\mathfrak{K}_-$, where

$$\mathfrak{K}_\pm = \left\{ \begin{bmatrix} \mp x \\ x \\ w_\pm \end{bmatrix} \middle| \, x \in \mathcal{X}, \ w_\pm \in \mathcal{W}_\pm \right\}$$

and $\mathcal{W} = \mathcal{W}_+ \boxplus -\mathcal{W}_-$ is an arbitrary fundamental decomposition of $\mathcal{W}$. As an immediate consequence, we have that $\operatorname{ind}_\pm \mathfrak{K} = \dim \mathcal{X} + \operatorname{ind}_\pm \mathcal{W}$.

Just as in the case of boundary control, it is immediate that all classical trajectories on $\mathbb{R}^+$ generated by a non-negative V satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|_{\mathcal{X}}^2 \le [w(t), w(t)]_{\mathcal{W}}, \quad t \in \mathbb{R}^+, \quad \text{and} \tag{3.3}$$

$$\|x(t)\|_{\mathcal{X}}^2 - \|x(s)\|_{\mathcal{X}}^2 \le \int_s^t [w(v), w(v)]_{\mathcal{W}} \,\mathrm{d}v, \quad s,t \in \mathbb{R}^+,\ s \le t, \tag{3.4}$$

where the second inequality holds also for the generalized trajectories.

However, non-negativity of V does not yet imply that $(V; \mathcal{X}, \mathcal{W})$ is a s/s node. The situation is analogous to the situation in semigroup theory: the generator of a contraction semigroup is not just dissipative, but even *maximal* dissipative; see the Lumer-Phillips Theorem [Staffans, 2005, Thm 3.4.8].

Definition 12. A s/s system $\Sigma = (V; \mathcal{X}; \mathcal{W})$ is said to be *passive* if V is a maximal non-negative subspace of the node space $\mathfrak{K}$, i.e., with respect to the inner product (3.2). The system Σ is *conservative* if $V = V^{[\perp]}$.

I/s/o representations corresponding to fundamental decompositions of the signal space of a passive s/s system are exceptionally well-behaved, and we now investigate these in more detail.

Definition 13. Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a s/s system and let $\Sigma_{i/s/o} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be an i/s/o representation of Σ in the sense of Definition 7. Then $\Sigma_{i/s/o}$ is called a *scattering representation* of Σ if $\mathcal{U} = \mathcal{W}_+$ and $\mathcal{Y} = \mathcal{W}_-$, where $\mathcal{W} = \mathcal{W}_+ \boxplus -\mathcal{W}_-$ is a fundamental decomposition.

Let $\mathcal{W} = \mathcal{W}_+ \boxplus -\mathcal{W}_-$ be a fundamental decomposition, and set $\mathcal{U} := \mathcal{W}_+$ and $\mathcal{Y} := \mathcal{W}_-$. Combining (3.4) and (3.1) we obtain that every classical trajectory of a passive s/s system satisfies (with $u(v) \in \mathcal{U}$ and $y(v) \in \mathcal{Y}$):

$$\|x(t)\|_{\mathcal{X}}^2 - \|x(s)\|_{\mathcal{X}}^2 \le \int_s^t \|u(v)\|_{\mathcal{U}}^2 - \|y(v)\|_{\mathcal{Y}}^2 \,\mathrm{d}v \tag{3.5}$$

for every $s, t \in \mathbb{R}^+$ such that $s \le t$. This is the well-known *scattering-passivity inequality.* Note that (3.5) implies (2.5) with $K(t) = 1$, $t \in \mathbb{R}^+$.

The first part of the following further development of the above ideas was proved as Theorem 4.5 and Proposition 5.8 in [Kurula, 2010]. The second part follows from the first part and Definition 4.

Theorem 14. *Assume that V is a maximal non-negative subspace of $\mathfrak{K}$ satisfying (2.2): $\left[\begin{smallmatrix} z \\ 0 \\ 0 \end{smallmatrix}\right] \in V$ only if $z = 0$. Then $(V; \mathcal{X}, \mathcal{W})$ is a passive well-posed s/s node for which every fundamental decomposition*

$\mathcal{W} = \mathcal{W}_+ \boxplus -\mathcal{W}_-$ is (admissible and) well-posed and the corresponding scattering representation with input space $\mathcal{U} = \mathcal{W}_+$ and output space $\mathcal{Y} = \mathcal{W}_-$ is well-posed.

In particular, for every $x_0 \in \mathcal{X}$ and $u \in L^2_{\mathrm{loc}}(\mathbb{R}^+;\mathcal{U})$ there exists a unique generalized trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \left[\begin{smallmatrix} C(\mathbb{R}^+;\mathcal{X}) \\ L^2_{\mathrm{loc}}(\mathbb{R}^+;\mathcal{W}) \end{smallmatrix}\right]$ of Σ on $\mathbb{R}^+$ with $x(0) = x_0$ and $P^{\mathcal{Y}}_{\mathcal{U}} w = u$.

Thus a triple $(V;\mathcal{X},\mathcal{W})$ is a passive s/s system if and only if V is a maximal non-negative subspace of $\mathfrak{K}$ with the property (2.2).

Let $\Sigma = (V;\mathcal{X},\mathcal{W})$ be a passive s/s system. Each different fundamental decomposition $\mathcal{W} = \mathcal{W}_+ \boxplus -\mathcal{W}_-$ gives rise to a different scattering representation, so there always exist uncountably many scattering representations of a given *passive* s/s system (except for the degenerate cases where the energy exchange through the external signal is unidirectional).

Now suppose that $\Sigma = (V;\mathcal{X},\mathcal{W})$ has the property that V is *maximal non-positive*. Then (3.3) is replaced by

$$\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|^2_{\mathcal{X}} \geq [w(t), w(t)]_{\mathcal{W}}, \quad t \in \mathbb{R}^+,$$

and an analogue of Theorem 14 can be formulated for Σ, which says that Σ is *well-posed in the backward time direction*, and that every fundamental decomposition $\mathcal{W} = \mathcal{W}_+ \boxplus -\mathcal{W}_-$ yields a well-posed i/s/o representation if we take the *output* space to be $\mathcal{Y} = \mathcal{W}_+$ and the *input* space to be $\mathcal{U} = \mathcal{W}_-$.

Definition 15. We call a triple $\Sigma = (V;\mathcal{X},\mathcal{W})$ with a maximal non-positive generating subspace V satisfying (2.2) an *anti-passive s/s node* (in the backward time direction), i.e., it has properties (1) and (2) in Definition 1 with $\mathbb{R}^+$ replaced by $\mathbb{R}^-$.

It is well-known that $V = V^{[\perp]}$ if and only if V is both maximal non-negative and maximal non-positive. A conservative s/s system is thus one that is at the same time both passive and anti-passive. We conclude that conservative s/s systems are *i/s/o well-posed both in the forward and in the backward time directions*. This does *not* imply that the signal space $\mathcal{W}$ has a direct sum decomposition $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$ which is i/s/o well-posed both in the forward and backward time direction. We provided a conservative system for which no decomposition of the signal space is admissible both in the forward and backward time directions in Example ??. Indeed, when that system is solved in forward time, $x(0) \in \mathbb{C}$ is the output and there is no input, and when the system is solved in backward time, $x(0)$ is the input and there is no output. See [Kurula, 2010, Thm 4.11] for more details on conservative s/s systems.

Remark 16. The *maximal* non-negativity of V in a passive s/s system $\Sigma = (V;\mathcal{X},\mathcal{W})$ intuitively means that it has "enough" trajectories to make sense as a system.

More precisely, the maximal non-negativity of V implies that the *state/signal dual* $(V^{[\perp]};\mathcal{X},\mathcal{W})$ of Σ is anti-passive and thus very well-structured. If V is replaced by a smaller space then the dual becomes larger, and in particular, if $V=\{0\}$, then $V^{[\perp]}=\mathfrak{K}$ which has no meaning as a s/s system at all.

We also note that a s/s system is conservative if and only if it coincides with its own s/s dual.

See [Kurula, 2010, Sec. 3] for more details on the dual s/s system and its i/s/o representations. We now return to i/s/o representations of passive s/s systems. It is well known that the adjoint S^* of an i/s/o system node operator S is also an i/s/o system node operator which represents the adjoint system; see [Staffans, 2005, Lemma 6.2.14]. If $\mathcal{U}$ is the input space and $\mathcal{Y}$ is the output space of S then $\mathcal{Y}$ is the input space and $\mathcal{U}$ is the output space of S^*.

Definition 17. An i/s/o system node $\Sigma_{i/s/o}=(S;\mathcal{X},\mathcal{U},\mathcal{Y})$ is *scattering passive* if all its classical trajectories (u,x,y) satisfy (3.5) for every $s,t\in\mathbb{R}^+$ such that $s\le t$.

The i/s/o system node $\Sigma_{i/s/o}$ is *scattering conservative* if all classical trajectories (u,x,y) and (y^d,x^d,u^d) on $\mathbb{R}^+$ of S and S^*, respectively, satisfy

$$\begin{aligned}\|x(t)\|_{\mathcal{X}}^2-\|x(s)\|_{\mathcal{X}}^2&=\int_s^t\|u(v)\|_{\mathcal{U}}^2-\|y(v)\|_{\mathcal{Y}}^2\,\mathrm{d}v\quad\text{and}\\ \|x^d(t)\|_{\mathcal{X}}^2-\|x^d(s)\|_{\mathcal{X}}^2&=\int_s^t\|y^d(v)\|_{\mathcal{Y}}^2-\|u^d(v)\|_{\mathcal{U}}^2\,\mathrm{d}v\end{aligned}$$

for every $s,t\in\mathbb{R}^+$ such that $s\le t$. (Compare this to (3.5).)

Remark 18. Recall that a triple $(V;\mathcal{X},\mathcal{W})$ is a passive s/s system if and only if V is a maximal non-negative subspace of $\mathfrak{K}$ with the property (2.2). Comparing this to Definitions 5 and 17, which are necessary for defining only a special class of passive i/s/o systems, we see that the s/s definition is both more general and considerably simpler. Moreover, the definitions of conservative i/s/o systems are even more complicated, since we need to formulate conditions on the *dual system* but a conservative s/s system is very elegantly characterized by the properties $V=V^{[\perp]}$ and (2.2).

The following proposition was proved in [Kurula, 2010, Prop. 5.6].

Proposition 19. *All scattering representations of a* passive (conservative) *s/s system are scattering passive (conservative) i/s/o systems.*

Conversely, let $\Sigma_{i/s/o}=(S;\mathcal{X},\mathcal{U},\mathcal{Y})$ be a scattering passive (conservative) i/s/o system node, so that $\mathcal{U}$ and $\mathcal{Y}$ are both Hilbert spaces. Define $\mathcal{W}:=\left[\begin{smallmatrix}\mathcal{U}\\ -\mathcal{Y}\end{smallmatrix}\right]$ with inner product $[[\left[\begin{smallmatrix}u_1\\ y_1\end{smallmatrix}\right],\left[\begin{smallmatrix}u_2\\ y_2\end{smallmatrix}\right]]]:=(u_1,u_2)_{\mathcal{U}}-(y_1,y_2)_{\mathcal{Y}}$. Then $\mathcal{W}$ is a Kreĭn space with fundamental decomposition $\Sigma=\left[\begin{smallmatrix}\mathcal{U}\\ \{0\}\end{smallmatrix}\right]\boxplus$

$\left[\begin{smallmatrix}\{0\}\\ -\mathcal{Y}\end{smallmatrix}\right]$. *Moreover,* $(V;\mathcal{X},\mathcal{W})$ *with* V *given by* (2.7)*, is the unique passive (conservative) s/s system whose scattering representation induced by the above fundamental decomposition is* $\Sigma_{i/s/o}$.

Scattering passive i/s/o systems are discussed in, e.g., [Arov and Nudelman, 1996] and [Staffans, 2005, Chapter 11]. The connection between different well-posed i/s/o representations of a s/s system, and thus in particular, between different scattering representations of a passive s/s system, is described in [Kurula and Staffans, 2009, Section 4].

3.2. **Impedance and transmission representations.** In the context of boundary relations, another type of i/s/o representation is in fact more important than the scattering representation, namely the *impedance* representation.

Definition 20. An *impedance representation* of a s/s system $\Sigma = (V;\mathcal{X},\mathcal{W})$ is an i/s/o representation corresponding to a system-node admissible *Lagrangian* decomposition $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$ of the signal space $\mathcal{W}$.

Since not all Kreĭn spaces have Lagrangian decompositions, there exist passive s/s systems which have no impedance representations. However, assume that $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$ indeed is a Lagrangian decomposition of $\mathcal{W}$, i.e., that $\mathcal{U}$ and $\mathcal{Y}$ are both Lagrangian subspaces of $\mathcal{W}$. By [Arov and Staffans, 2007a, Lemma 2.3] there exist admissible Hilbert-space inner products on $\mathcal{U}$ and $\mathcal{Y}$ and a unitary operator $\Psi : \mathcal{Y} \to \mathcal{U}$, such that the Kreĭn-space inner product on $\mathcal{W}$ is given by the following (where $u_1, u_2 \in \mathcal{U},\ y_1, y_2 \in \mathcal{Y}$):

$$[y_1 + u_1, y_2 + u_2]_{\mathcal{W}} = (\Psi y_1, u_2)_{\mathcal{U}} + (u_1, \Psi y_2)_{\mathcal{U}}. \tag{3.6}$$

Definition 21. By writing $\mathcal{W} = \mathcal{U} \overset{\Psi}{\dotplus} \mathcal{Y}$ we mean that the Kreĭn space $\mathcal{W}$ is decomposed into the direct sum of $\mathcal{U}$ and $\mathcal{Y}$, and that the inner product $[\cdot,\cdot]_{\mathcal{W}}$ in $\mathcal{W}$ may be written in the form (3.6), where Ψ is a unitary operator from $\mathcal{Y}$ to $\mathcal{U}$.

It follows from (3.6) that both $\mathcal{U}$ and $\mathcal{Y}$ are Lagrangian subspaces of $\mathcal{W}$, i.e., that the decomposition in Definition 21 is always Lagrangian. See Section 2 of [Arov and Staffans, 2007a] for more details on Lagrangian decompositions of $\mathcal{W}$. If $\mathcal{W} = \mathcal{U} \overset{\Psi}{\dotplus} \mathcal{Y}$, then the inequality (3.3) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|^2_{\mathcal{X}} \le 2\mathrm{Re}\,(u(t), \Psi y(t))_{\mathcal{U}}, \quad t \in \mathbb{R}^+.$$

Moreover, the inequality (3.4) takes the form

$$\|x(t)\|^2_{\mathcal{X}} - \|x(s)\|^2_{\mathcal{X}} \le 2\mathrm{Re}\int_s^t (u(v), \Psi y(v))_{\mathcal{U}}\,\mathrm{d}v \quad s, t \in \mathbb{R}^+,\ t \ge s, \tag{3.7}$$

and this is the *impedance-passivity inequality.*

Definition 22. An i/s/o system node $\Sigma_{i/s/o} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ is *impedance passive* if all its classical trajectories (u, x, y) satisfy (3.7) for some unitary operator $\Psi : \mathcal{Y} \to \mathcal{U}$. (One commonly has $\mathcal{Y} = \mathcal{U}$ and $\Psi = 1_{\mathcal{U}}$.)

The i/s/o system node $\Sigma_{i/s/o}$ is *impedance conservative* if all classical trajectories (u, x, y) and (y, x, u) on $\mathbb{R}^+$ of S and S^*, respectively, satisfy (3.7) with equality instead of inequality.

An analogue of Proposition 19 relating impedance representations and impedance passive i/s/o systems can be formulated simply by replacing "scattering" by "impedance" and the fundamental decomposition $\mathcal{W} = \left[\begin{smallmatrix} \mathcal{U} \\ \{0\} \end{smallmatrix}\right] \boxplus \left[\begin{smallmatrix} \{0\} \\ -\mathcal{Y} \end{smallmatrix}\right]$ by a Lagrangian decomposition $\mathcal{W} = \mathcal{U} \overset{\Psi}{\dotplus} \mathcal{Y}$, where $\Psi : \mathcal{Y} \to \mathcal{U}$ is an arbitrary unitary operator.

Theorem 23. *The following claims are true for an impedance conservative i/s/o system node $\Sigma_{i/s/o} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$:*

(1) *The main operator A of S, see item (3) of Definition 5, is skew-adjoint and A generates a unitary group $t \mapsto \mathfrak{A}^t$, $t \in \mathbb{R}$, on $\mathcal{X}$.*

(2) *For every $u \in W^{2,1}_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{U})$ and initial state $x_0 \in \mathcal{X}$, such that $\left[\begin{smallmatrix} x_0 \\ u(0) \end{smallmatrix}\right] \in \mathrm{dom}\,(S)$, the system*

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad t \in \mathbb{R}^+, \tag{3.8}$$

has a unique classical *trajectory (u, x, y) with $x(0) = x_0$, where $W^{2,1}_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{U})$ denotes the space of functions that together with their first and second distribution derivatives lie in $L^1_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{U})$.*

(3) *For every $x_0 \in \mathcal{X}$ there exists a* generalized *trajectory (u, x, y) of $\Sigma_{i/s/o}$, such that $x(0) = x_0$. This trajectory is uniquely determined by the initial state x_0 and the input u.*

(4) *The system (3.8) can also be solved in backwards time, i.e., for $t \in \mathbb{R}^- = (-\infty, 0]$, with the initial state $x_0 \in \mathcal{X}$ given at $t = 0$. In particular, every trajectory (u, x, y) of $\Sigma_{i/s/o}$ with $x(0) = x_0$ and $u = 0$ satisfies $x(t) = \mathfrak{A}^t x_0$ for all $t \in \mathbb{R}$, and if $x_0 \in \mathrm{dom}\,(A)$, then this trajectory is classical. This trajectory is the unique trajectory of $\Sigma_{i/s/o}$ with the given state x_0 at time 0 and input $u(t) = 0$, $t \in \mathbb{R}$.*

Proof. One can verify that the i/s/o system node $\Sigma_{i/s/o}$ is impedance conservative with some given Ψ if and only if V defined in (2.7) is a Lagrangian subspace of $\mathfrak{K}$: $V = V^{[\perp]}$, where $\mathcal{W} := \mathcal{U} \overset{\Psi}{\dotplus} \mathcal{Y}$.

(1) According to [Staffans, 2002a, Thm 4.7(4)] we have $A = -A^*$, and thus A generates a unitary group by Stone's theorem [Pazy, 1983, Thm 10.8].

(2) This follows from [Staffans, 2005, Lem. 4.7.8].

(3) The s/s system induced by an impedance conservative i/s/o system node is conservative, and therefore in particular passive. By Theorem 14, $(V; \mathcal{X}, \mathcal{W})$ is well-posed, and according

to condition (1) of Definition 4, every $x_0 \in \mathcal{X}$ can be taken as the initial state of some generalized trajectory.

Moreover, if $x_0 = 0$ and $u(t) = 0$ for all $t \in \mathbb{R}^+$, then $x(t) = 0$ and $y(t) = 0$ for all ≥ 0 by claim (2).

(4) This is a consequence of Remark 16, Theorem 3.8.2 in [Staffans, 2005], and the previous claims in this theorem.

□

There are several ways to add dynamics to a boundary relation. Using Theorem 23 is one way, as we will show later in Section 5.3.

Remark 24. Note that the input u in Theorem 23 corresponds to a system-node admissible Lagrangian decomposition of the signal space of a conservative s/s system, and that this decomposition need not be well-posed in general. Indeed, the corresponding impedance representation $(S;\mathcal{X},\mathcal{U},\mathcal{Y})$ need not be well-posed, i.e., the i/s/o system node S in (2.7) need not satisfy (2.9).

If the decomposition $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$ happens to be well-posed then we have from Definition 4 that the set

$$\big\{u \mid (u,x,y) \text{ is a generalized trajectory of } \Sigma_{i/s/o} \text{ with } x(0) = x_0\big\}$$

equals all of $L^2_{\mathrm{loc}}(\mathbb{R}^+;\mathcal{U})$ for all $x_0 \in \mathcal{X}$, but in the ill-posed case we can make no such conclusion.

On the contrary, every scattering representation of a passive s/s system is well-posed, cf. Theorem 14. This explains why the scattering formalism is sometimes useful for solving technical difficulties in the boundary relations theory, cf. [Behrndt et al., 2009], where this technique is used extensively.

There exist conservative s/s systems for which no Lagrangian decompositions are system-node admissible, see [Arov and Staffans, 2007a, Ex. 5.13], which can also be formulated for continuous time with trivial modifications. It follows from Theorem 39 and Proposition 40 below that the following two conditions together are sufficient and necessary for a Lagrangian decomposition $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$ to be admissible for a conservative s/s system $(V;\mathcal{X},\mathcal{W})$:

(1) $\begin{bmatrix} z \\ 0 \\ \left[\begin{smallmatrix} 0 \\ y \end{smallmatrix}\right] \end{bmatrix} \in V$ only if $y = 0$ and
(2) for each $u \in \mathcal{U}$ there exist $z, x \in \mathcal{X}$ and $y \in \mathcal{Y}$ such that $\begin{bmatrix} z \\ x \\ \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right] \end{bmatrix} \in V$.

Well-posed impedance passive i/s/o systems have been studied in [Staffans, 2002a] and the ill-posed impedance case is considered in [Staffans, 2002b].

Remark 25. The energy inequalities (3.5) and (3.7) correspond to fundamental and Lagrangian decompositions of $\mathcal{W}$, respectively, but the *property of passivity* is characterized by the maximal non-negativity of V. Thus passivity is a *state/signal characteristic*, i.e., passivity *does not depend on any particular decomposition of the signal space into an input space and an output space.*

A third, fairly common, type of representation is the *transmission representation*.

Definition 26. An i/s/o representation of a passive s/s system corresponding to an admissible *orthogonal* decomposition $\mathcal{W} = \mathcal{W}_1 \boxplus \mathcal{W}_2$ of the signal space, with input space $\mathcal{U} = \mathcal{W}_1$ and output space $\mathcal{Y} = -\mathcal{W}_2$ is called a *transmission representation*.

Every scattering representation can also be interpreted as a transmission representation.

Example 27. We continue the transmission line example in Section ??. As we saw there, this is a conservative boundary control system. The choice of input and output maps Γ_0, Γ_1 in (??) corresponds to the Lagrangian decomposition $\mathcal{W} = \begin{bmatrix} \mathbb{C} \\ \{0\} \\ \mathbb{C} \\ \{0\} \end{bmatrix} \stackrel{\Psi}{+} \begin{bmatrix} \{0\} \\ \mathbb{C} \\ \{0\} \\ \mathbb{C} \end{bmatrix}$, where $\Psi = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, and according to Example 8, this decomposition is admissible. The choice $\widetilde{\Gamma}_0, \widetilde{\Gamma}_1$ made in (??) corresponds to the fundamental decomposition $\mathcal{W} = \left\{ \left[\begin{smallmatrix} a \\ a \\ b \\ b \end{smallmatrix}\right] \middle| \, a, b \in \mathbb{C} \right\} \boxplus \left\{ \left[\begin{smallmatrix} -a \\ a \\ -b \\ b \end{smallmatrix}\right] \middle| \, a, b \in \mathbb{C} \right\}$, and according to Theorem 14, also this decomposition is admissible. The choice $\widehat{\Gamma}_0, \widehat{\Gamma}_1$ in (??) corresponds to the orthogonal (but non-fundamental) decomposition $\mathcal{W} = \begin{bmatrix} \mathbb{C} \\ \mathbb{C} \\ \{0\} \\ \{0\} \end{bmatrix} \boxplus \begin{bmatrix} \{0\} \\ \{0\} \\ \mathbb{C} \\ \mathbb{C} \end{bmatrix}$, which is *not* admissible.

The non-admissible orthogonal and Lagrangian decompositions which do not yield i/s/o representations can be treated using continuous-time analogues of the *affine representations* developed in [Arov and Staffans, 2007b].

4. The frequency domain characteristics of a state/signal system

4.1. **The input/state/output resolvent matrix.** Suppose that x, $\dot{x}$, y, and u are all Laplace transformable, with the Laplace transforms converging in the whole right half-plane $\mathbb{C}^+ = \{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda > 0\}$. Take Laplace transforms in the i/s/o equation

$$\Sigma_{i/s/o}: \quad \begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad t \in \mathbb{R}^+, \quad x(0) = x_0,$$

in order to get

$$\begin{bmatrix} \lambda \hat{x}(\lambda) - x_0 \\ \hat{y}(\lambda) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \hat{x}(\lambda) \\ \hat{u}(\lambda) \end{bmatrix}, \quad \lambda \in \mathbb{C}^+. \tag{4.1}$$

At least in the case where $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is a bounded operator in a scattering representation of a passive s/s system it is possible to solve $\left[\begin{smallmatrix} \hat{x}(\lambda) \\ \hat{y}(\lambda) \end{smallmatrix}\right]$ in terms of $\left[\begin{smallmatrix} x_0 \\ \hat{u}(\lambda) \end{smallmatrix}\right]$ from the identity (4.1) for all $\lambda \in \mathbb{C}^+$. The map $\left[\begin{smallmatrix} x_0 \\ \hat{u}(\lambda) \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} \hat{x}(\lambda) \\ \hat{y}(\lambda) \end{smallmatrix}\right]$ turns out to be a bounded linear operator that we denote by $\widehat{\mathfrak{S}}(\lambda) = \left[\begin{smallmatrix} \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \\ \widehat{\mathfrak{C}}(\lambda) & \widehat{\mathfrak{D}}(\lambda) \end{smallmatrix}\right]$. More explicitly,

$$
\begin{aligned}
\begin{bmatrix} \hat{x}(\lambda) \\ \hat{y}(\lambda) \end{bmatrix} &= \begin{bmatrix} \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \\ \widehat{\mathfrak{C}}(\lambda) & \widehat{\mathfrak{D}}(\lambda) \end{bmatrix} \begin{bmatrix} x_0 \\ \hat{u}(\lambda) \end{bmatrix}, \quad \lambda \in \mathbb{C}^+, \quad \text{where} \\
\begin{bmatrix} \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \\ \widehat{\mathfrak{C}}(\lambda) & \widehat{\mathfrak{D}}(\lambda) \end{bmatrix} &= \begin{bmatrix} (\lambda - A)^{-1} & (\lambda - A)^{-1}B \\ C(\lambda - A)^{-1} & C(\lambda - A)^{-1}B + D \end{bmatrix}.
\end{aligned} \tag{4.2}
$$

Definition 28. The operator $\widehat{\mathfrak{S}} := \left[\begin{smallmatrix} \widehat{\mathfrak{A}} & \widehat{\mathfrak{B}} \\ \widehat{\mathfrak{C}} & \widehat{\mathfrak{D}} \end{smallmatrix}\right]$ is called the *i/s/o resolvent matrix of* $\Sigma_{i/s/o}$. The different components of this resolvent matrix are named as follows:

(1) $\widehat{\mathfrak{A}}$ is the *state/state* resolvent function,
(2) $\widehat{\mathfrak{B}}$ is the *input/state* resolvent function,
(3) $\widehat{\mathfrak{C}}$ is the *state/output* resolvent function, and
(4) $\widehat{\mathfrak{D}}$ is the *input/output* resolvent function.

Of course, the state/state resolvent function is the familiar *resolvent* of the main operator A. The other components of $\widehat{\mathfrak{S}}$ has different names in different parts of the literature, and we make the connections to the corresponding notions in the theory of boundary relations in Theorem 38 below. In the i/s/o tradition the input/output resolvent function is usually called the *transfer function* of $\Sigma_{i/s/o}$.

Remark 29. A significant part of formula (4.2) remains valid with the appropriate interpretation of the operators A, B, and C if we replace $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ by a *system node* operator S of the type described in Definition 5; see [Staffans, 2002b, Sec. 2].

In the case of a scattering passive i/s/o system $\Sigma_{i/s/o}$, the function $\widehat{\mathfrak{D}}$ is often called the *scattering matrix* of $\Sigma_{i/s/o}$. If in addition, $\Sigma_{i/s/o}$ is *conservative*, then $\widehat{\mathfrak{D}}$ is also called the *characteristic function* of the corresponding i/s/o system node, or of its main operator A; see Definition 5. In this case A is a maximal dissipative operator in $\mathcal{X}$.

In the case where $\Sigma_{i/s/o}$ is transmission passive, $\widehat{\mathfrak{D}}$ is called the *transmission matrix* of $\Sigma_{i/s/o}$. Also here $\widehat{\mathfrak{D}}$ is called the *characteristic function* if $\Sigma_{i/s/o}$ is conservative; see e.g. [Tsekanovskiĭ and Šmuljan, 1977]. In a transmission passive i/s/o system, the main operator A is often not dissipative, and this lack of dissipativity causes many of the technical problems associated with transmission passive systems.

Finally, in the case where $\Sigma_{i/s/o}$ is impedance passive, $\widehat{\mathfrak{D}}$ is called the *impedance matrix* of $\Sigma_{i/s/o}$. If $\Sigma_{i/s/o}$ is conservative then the main operator A is skew-adjoint, cf. Theorem 23.

See [Šmuljan, 1986], [Salamon, 1987], [Curtain and Weiss, 1989], [Arov and Nudelman, 1996] or [Staffans, 2005] for more information on transfer functions (input/output resolvent functions).

4.2. **The characteristic node bundle.** In order to derive the analogue of an i/s/o resolvent matrix for a s/s system, we rewrite the identity (4.1) so that it uses the generating subspace V instead of the system node operator S.

Suppose therefore that $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is a classical trajectory of a s/s node, and that x, $\dot{x}$, and w are all Laplace transformable with the Laplace transforms converging in the whole right half-plane $\mathbb{C}^+$. Taking Laplace transforms in $\left[\begin{smallmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{smallmatrix}\right] \in V$, $t \in \mathbb{R}^+$, we get

$$\begin{bmatrix} \lambda\hat{x}(\lambda) - x_0 \\ \hat{x}(\lambda) \\ \widehat{w}(\lambda) \end{bmatrix} \in V, \quad \lambda \in \mathbb{C}^+.$$

Definition 30. Let $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$ be a direct sum decomposition of $\mathcal{W}$. The domain of the *generalized i/s/o resolvent matrix* with respect to this decomposition and the generalized i/s/o resolvent matrix itself are defined by

$$\operatorname{dom}(\widehat{\mathfrak{S}}) = \left\{ \lambda \in \mathbb{C} \;\middle|\; \begin{array}{l} \text{for all } \left[\begin{smallmatrix} x_0 \\ u \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \text{ there exists} \\ \text{a unique pair } \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right] \\ \text{such that } \left[\begin{smallmatrix} \lambda x - x_0 \\ x \\ \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right] \end{smallmatrix}\right] \in V \end{array} \right\}, \tag{4.3}$$

$$\begin{aligned} &\widehat{\mathfrak{S}}(\lambda) \left[\begin{smallmatrix} x_0 \\ u \end{smallmatrix}\right] := \begin{bmatrix} \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \\ \widehat{\mathfrak{C}}(\lambda) & \widehat{\mathfrak{D}}(\lambda) \end{bmatrix} \left[\begin{smallmatrix} x_0 \\ u \end{smallmatrix}\right] := \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right], \ \lambda \in \operatorname{dom}(\widehat{\mathfrak{S}}), \\ &\text{where } \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \text{ is the unique pair for which } \left[\begin{smallmatrix} \lambda x - x_0 \\ x \\ \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right] \end{smallmatrix}\right] \in V. \end{aligned} \tag{4.4}$$

Of course, in this definition only those decompositions $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$ of the signal space for which the domain of the generalized resolvent matrix $\widehat{\mathfrak{S}}$ is nonempty are interesting.

Example 31. We continue Example 8 by computing the i/s/o resolvent matrix $\widehat{\mathfrak{S}} = \left[\begin{smallmatrix} \widehat{\mathfrak{A}} & \widehat{\mathfrak{B}} \\ \widehat{\mathfrak{C}} & \widehat{\mathfrak{D}} \end{smallmatrix}\right]$ of the boundary control i/s/o system $\Sigma_{i/s/o} = (L, \Gamma_0, \Gamma_1; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ in Definition ??. Therefore we again let V be given by (2.8) and we carry out the following computations:

$$\begin{gathered} \left[\begin{smallmatrix} \lambda x - x_0 \\ x \\ \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right] \end{smallmatrix}\right] \in V = \left\{ \left[\begin{smallmatrix} Lx \\ x \\ \left[\begin{smallmatrix} \Gamma_0 x \\ \Gamma_1 x \end{smallmatrix}\right] \end{smallmatrix}\right] \;\middle|\; x \in \operatorname{dom}(L) \right\} \iff \\ x_0 = (\lambda - L)x, \quad y = \Gamma_1 x, \quad \text{and} \quad u = \Gamma_0 x, \\ \text{so that} \quad \widehat{\mathfrak{S}}(\lambda) : \begin{bmatrix} (\lambda - L)x \\ \Gamma_0 x \end{bmatrix} \mapsto \begin{bmatrix} x \\ \Gamma_1 x \end{bmatrix}, \quad x \in \operatorname{dom}(L). \end{gathered}$$

One can show that $\mathbb{C}^+ \subset \operatorname{dom}(\widehat{\mathfrak{S}})$ if the system $\Sigma_{i/s/o}$ is *passive*, i.e., if V is maximal non-negative, and if $\mathcal{U} = \mathcal{W}_+$ for some fundamental decomposition $\mathcal{W} = \mathcal{W}_+ \boxplus -\mathcal{W}_-$.

It is possible to further extend the notion of a generalized i/s/o resolvent matrix by allowing $\widehat{\mathfrak{S}}(\lambda)$ to be a *relation* instead of a function. This extension is implemented by the following notion:

Definition 32. The *characteristic node bundle* of the (not necessarily passive) s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is the family $\{\widehat{\mathfrak{E}}(\lambda)\}_{\lambda\in\mathbb{C}}$ of subspaces of the node space $\mathfrak{K}$, where each $\widehat{\mathfrak{E}}(\lambda)$ is given by

$$\widehat{\mathfrak{E}}(\lambda) = \left\{ \left[\begin{smallmatrix} x \\ x_0 \\ w \end{smallmatrix}\right] \middle| \left[\begin{smallmatrix} \lambda x - x_0 \\ x \\ w \end{smallmatrix}\right] \in V \right\}. \tag{4.5}$$

The subspace $\widehat{\mathfrak{E}}(\lambda)$ is called the *fiber of* $\widehat{\mathfrak{E}}$ *at* $\lambda \in \mathbb{C}$.

By using the above state/signal characteristic node bundle we can reformulate the definition of the generalized i/s/o resolvent matrix $\left[\begin{smallmatrix} \widehat{\mathfrak{A}} & \widehat{\mathfrak{B}} \\ \widehat{\mathfrak{C}} & \widehat{\mathfrak{D}} \end{smallmatrix}\right]$ as follows.

Remark 33. Let $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$ be a direct sum decomposition of $\mathcal{W}$. The domain of the *generalized i/s/o resolvent matrix* $\widehat{\mathfrak{S}}$ of the passive s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ with respect to this decomposition consists of those points $\lambda \in \mathbb{C}$ for which *the fiber* $\widehat{\mathfrak{E}}(\lambda)$ *of the characteristic node bundle is the graph of a bounded linear operator* $\left[\begin{smallmatrix} 0 \\ \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathcal{X} \\ 0 \\ \mathcal{Y} \end{smallmatrix}\right]$, and $\widehat{\mathfrak{S}}(\lambda) = \left[\begin{smallmatrix} \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \\ \widehat{\mathfrak{C}}(\lambda) & \widehat{\mathfrak{D}}(\lambda) \end{smallmatrix}\right]$ is defined to be this operator. Note that we require that $\operatorname{dom}(\widehat{\mathfrak{S}}(\lambda)) = \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]$ for all $\lambda \in \operatorname{dom}(\widehat{\mathfrak{S}})$.

However, even if $\widehat{\mathfrak{E}}(\lambda)$ is not the graph of an operator, it can always be interpreted *as the graph of a closed relation* $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$. With this interpretation it makes sense to call this relation the *i/s/o resolvent relation at the point* $\lambda \in \mathbb{C}$. This resolvent relation is *defined for all* $\lambda \in \mathbb{C}$ but now $\operatorname{dom}(\widehat{\mathfrak{E}}(\lambda))$ may depend on λ.

Observe that unlike the above mentioned resolvent matrices and resolvent relations, *the fiber* $\widehat{\mathfrak{E}}(\lambda)$ *is a state/signal characteristic*, i.e., it does not depend on any particular decomposition $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$ of the signal space. Thus, although the s/s system Σ has many different resolvent relations, each corresponding to a different decomposition $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$, *all resolvent relations have the same graph.* The different resolvent relations are simply different *representations* of the characteristic node bundle corresponding to different input/output decompositions.

We refer the reader to [Arov and Staffans, 2011] for more information on characteristic node bundles.

5. Conservative boundary relations

As we showed in Section ??, boundary triplets can be obtained as the i/s/o representations of conservative boundary control systems in case the boundary mapping Γ is surjective and the external signal space $\mathcal{W}$ has equal positive and negative indices. Here we show that a Lagrangian decomposition of the signal space of a conservative s/s system gives rise to a *boundary relation*, even if the decomposition of the signal space does not induce an i/s/o representation. We also prove the converse: every conservative boundary relation can be interpreted as a conservative state/signal system.

5.1. **Definitions.** The following definition of a boundary relation has been adapted from [Derkach et al., 2009, Def. 3.1], with some minor change of notation.

Definition 34. Let R be a closed symmetric linear relation in a Hilbert space $\mathcal{X}$ (with arbitrary defect numbers), and let $\mathcal{U}$ be an auxiliary Hilbert space. A linear relation $\Gamma\colon \mathcal{X}^2 \to \mathcal{U}^2$ is called a *conservative boundary relation* for R^* if

(1) $\mathrm{dom}\,(\Gamma)$ is dense in R^*,
(2) the identity

$$(z_1, x_2)_{\mathcal{X}} - (x_1, z_2)_{\mathcal{X}} = (y_1, u_2)_{\mathcal{U}} - (u_1, y_2)_{\mathcal{U}} \tag{5.1}$$

holds for every $\{\left[\begin{smallmatrix} x_1 \\ z_1 \end{smallmatrix}\right], \left[\begin{smallmatrix} u_1 \\ y_1 \end{smallmatrix}\right]\}$, $\{\left[\begin{smallmatrix} x_2 \\ z_2 \end{smallmatrix}\right], \left[\begin{smallmatrix} u_2 \\ y_2 \end{smallmatrix}\right]\} \in \Gamma$, and
(3) Γ is maximal in the sense that if $\{\left[\begin{smallmatrix} x_1 \\ z_1 \end{smallmatrix}\right], \left[\begin{smallmatrix} u_1 \\ y_1 \end{smallmatrix}\right]\} \in \mathcal{X}^2 \times \mathcal{U}^2$ satisfies (5.1) for every $\{\left[\begin{smallmatrix} x_2 \\ z_2 \end{smallmatrix}\right], \left[\begin{smallmatrix} u_2 \\ y_2 \end{smallmatrix}\right]\} \in \Gamma$, then $\{\left[\begin{smallmatrix} x_1 \\ z_1 \end{smallmatrix}\right], \left[\begin{smallmatrix} u_1 \\ y_1 \end{smallmatrix}\right]\} \in \Gamma$.

We remark that what we here call "conservative boundary relation" is simply called "boundary relation" in [Derkach et al., 2009]. We have added the word "conservative" because of the close resemblance to conservative s/s systems. As shown in [Derkach et al., 2009, Proposition 3.1], $\ker\,(\Gamma) = R$ for the relation Γ and the operator R in Definition 34.

The following definition is an adaptation of [Derkach et al., 2009, Defs 3.4 and 3.5].

Definition 35. Let R be a closed symmetric linear relation in the Hilbert space $\mathcal{X}$ and let $\Gamma : \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{X} \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{U} \end{smallmatrix}\right]$ be a conservative boundary relation for R^*.

The *Weyl family* (of $R = \ker\,(\Gamma)$) corresponding to Γ is the family

$$M(\lambda) := \left\{\{u, y\} \;\middle|\; \{\left[\begin{smallmatrix} x \\ \lambda x \end{smallmatrix}\right], \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]\} \in \Gamma\right\}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

The *γ-field* (of $R = \ker\,(\Gamma)$) corresponding to Γ is the relation

$$\gamma(\lambda) := \left\{\{u, x\} \;\middle|\; \{\left[\begin{smallmatrix} x \\ \lambda x \end{smallmatrix}\right], \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]\} \in \Gamma\right\}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

By [Derkach et al., 2006, Sec. 4.2], the γ-field of a boundary relation is in fact single-valued for $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Note that

$$\mathrm{dom}\,(M(\lambda)) = \mathrm{dom}\,(\gamma(\lambda)) = \left\{u \;\middle|\; \{\left[\begin{smallmatrix} x \\ \lambda x \end{smallmatrix}\right], \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]\} \in \Gamma\right\}$$

in general depends on λ. This is analogous to the dependence of the domain of the i/s/o resolvent relation of a s/s system on λ in the general case. In [Derkach et al., 2006, Sect. 4.3] it is studied in which cases $\operatorname{dom}(M(\lambda))$ is independent of λ.

5.2. **Connections to conservative state/signal systems.** We now proceed essentially in the same way as we did in Section ?? in order to explain the connection between a conservative boundary relation and a conservative s/s system.

Let R be a closed symmetric linear relation in $\mathcal{X}$ and let $\Gamma : \mathcal{X}^2 \to \mathcal{U}^2$ be a conservative boundary relation for R^*. We construct a s/s system by taking the signal space $\mathcal{W}$ to be $\mathcal{W} := \left[\begin{smallmatrix}\mathcal{U}\\ \mathcal{U}\end{smallmatrix}\right]$ with the indefinite inner product

$$[\left[\begin{smallmatrix}u_1\\ y_1\end{smallmatrix}\right], \left[\begin{smallmatrix}u_2\\ y_2\end{smallmatrix}\right]]_{\mathcal{W}} := (u_1, y_2)_{\mathcal{U}} + (y_1, u_2)_{\mathcal{U}}, \tag{5.2}$$

corresponding to the Lagrangian decomposition $\mathcal{W} = \left[\begin{smallmatrix}\mathcal{U}\\ \{0\}\end{smallmatrix}\right] \overset{\Psi}{\dotplus} \left[\begin{smallmatrix}\{0\}\\ \mathcal{U}\end{smallmatrix}\right]$ with $\Psi = 1_{\mathcal{U}}$, and defining

$$V := \left\{ \begin{bmatrix} iz \\ x \\ \left[\begin{smallmatrix}u\\ iy\end{smallmatrix}\right] \end{bmatrix} \in \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix} \,\middle|\, \{\left[\begin{smallmatrix}x\\ z\end{smallmatrix}\right], \left[\begin{smallmatrix}u\\ y\end{smallmatrix}\right]\} \in \Gamma \right\}. \tag{5.3}$$

We will prove in Lemma 37 below that V is a Lagrangian subspace of the Kreĭn space $\mathfrak{K} := \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix}$ equipped with the inner product (3.2).

Thus, if we knew that also (2.2) holds, then $\Sigma = (V; \mathcal{X}, \mathcal{W})$ would be a conservative s/s system. However, conditions (2) and (3) of Definition 34 alone do not yet imply that V satisfies (2.2). Indeed, let $\mathcal{X}$ be an arbitrary nontrivial Hilbert space, and set $\mathcal{W} = \{0\}$ and $\Gamma := \left\{ \{\left[\begin{smallmatrix}0\\ z\end{smallmatrix}\right], \left[\begin{smallmatrix}0\\ 0\end{smallmatrix}\right]\} \mid z \in \mathcal{X} \right\}$. Then $V = \left\{ \begin{bmatrix} z \\ 0 \\ \left[\begin{smallmatrix}0\\ 0\end{smallmatrix}\right] \end{bmatrix} \,\middle|\, z \in \mathcal{X} \right\} = V^{[\perp]}$ in $\mathfrak{K}$.

Fortunately, it is possible to meet condition (2.2) by replacing the state space $\mathcal{X}$ by a smaller space, and this can be done without essential loss of generality. The following proposition follows from [Kurula, 2010, Prop. 4.7].

Proposition 36. *Let V be a maximal non-negative subspace of $\mathfrak{K}$. Denote*

$$\widetilde{\mathcal{X}} := \mathcal{X} \ominus \left\{ z \,\middle|\, \begin{bmatrix} z \\ 0 \\ 0 \end{bmatrix} \in V \right\} \quad \text{and} \quad \widetilde{V} = V \cap \begin{bmatrix} \widetilde{\mathcal{X}} \\ \widetilde{\mathcal{X}} \\ \mathcal{W} \end{bmatrix}. \tag{5.4}$$

Then $\widetilde{\Sigma} := (\widetilde{V}; \widetilde{\mathcal{X}}, \mathcal{W})$ is a passive s/s system and the sets of classical and generalized trajectories generated by V and $\widetilde{V}$ are the same.

The s/s system $\widetilde{\Sigma}$ is conservative if and only if $V = V^{[\perp]}$.

In this way every conservative boundary relation induces a unique conservative s/s system. See Theorem 38 below for an exact statement.

Conversely, let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a conservative s/s node, such that the signal space has a Lagrangian decomposition $\mathcal{W} = \mathcal{U} \overset{\Psi}{\dotplus} \mathcal{Y}$ with the inner product (3.6). Define a linear relation on $\mathcal{X}^2 \times \mathcal{U}^2$ by

$$\Gamma := \left\{ \left\{ \begin{bmatrix} x \\ -iz \end{bmatrix}, \begin{bmatrix} P_{\mathcal{U}}^{\mathcal{Y}} w \\ -i\Psi P_{\mathcal{Y}}^{\mathcal{U}} w \end{bmatrix} \right\} \mid \left[\begin{smallmatrix}z\\ x\\ w\end{smallmatrix}\right] \in V \right\}. \tag{5.5}$$

In order to prove that Γ is a conservative boundary relation, we need to recall the *main transform* $\mathcal{J}(\Gamma)$ of Γ defined in [Derkach et al., 2006, Sect. 2.4] by

$$\mathcal{J}(\Gamma) := \left\{\{\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right], \left[\begin{smallmatrix} z \\ -y \end{smallmatrix}\right]\} \mid \{\left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right], \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]\} \in \Gamma\right\}, \tag{5.6}$$

and to state the following lemma:

Lemma 37. *The space $\mathcal{U}^2$ with the indefinite inner product* (5.2) *is a Kreĭn space. Moreover, the following claims are equivalent for an arbitrary Lagrangian decomposition $\mathcal{W} = \mathcal{U} \overset{\Psi}{\dotplus} \mathcal{Y}$:*

(1) *The subspace $V \subset \mathfrak{K}$ satisfies $V = V^{[\perp]}$.*
(2) *The relation $\Gamma : \mathcal{X}^2 \to \mathcal{U}^2$ given by* (5.5) *satisfies conditions (2) and (3) of Definition 34.*
(3) *The relation $\mathcal{J}(\Gamma)$ in $\mathcal{X} \times \mathcal{U}$ is self-adjoint.*

Proof. The reader may verify that that $\mathcal{U}^2 = \left[\begin{smallmatrix} 1_\mathcal{U} \\ 1_\mathcal{U} \end{smallmatrix}\right]\mathcal{U} \boxplus -\left[\begin{smallmatrix} -1_\mathcal{U} \\ 1_\mathcal{U} \end{smallmatrix}\right]\mathcal{U}$ is a fundamental decomposition, and therefore $\mathcal{U}^2$ is a Kreĭn space with the given indefinite inner product.

In order to prove the equivalence of the three listed claims, first note that

$$\begin{aligned}\left[\begin{smallmatrix} z \\ x \\ \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right] \end{smallmatrix}\right] \in V \quad &\iff \quad \{\left[\begin{smallmatrix} x \\ -iz \end{smallmatrix}\right], \left[\begin{smallmatrix} u \\ -i\Psi y \end{smallmatrix}\right]\} \in \Gamma \\ &\iff \quad \left\{\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right], \left[\begin{smallmatrix} -iz \\ i\Psi y \end{smallmatrix}\right]\right\} \in \mathcal{J}(\Gamma).\end{aligned}$$

Moreover, $\left[\begin{smallmatrix} z \\ x \\ \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right] \end{smallmatrix}\right] \in V^{[\perp]}$ if and only if

$$\begin{aligned}(u, \Psi\widetilde{y}) + (\Psi y, \widetilde{u}) - (z, \widetilde{x}) - (x, \widetilde{z}) = 0, \quad \left[\begin{smallmatrix} \widetilde{z} \\ \widetilde{x} \\ \left[\begin{smallmatrix} \widetilde{u} \\ \widetilde{y} \end{smallmatrix}\right] \end{smallmatrix}\right] \in V \quad &\iff \\ (-iz, \widetilde{x}) - (x, -i\widetilde{z}) = (-i\Psi y, \widetilde{u}) - (u, -i\Psi\widetilde{y}), \quad & \\ \left\{\left[\begin{smallmatrix} \widetilde{x} \\ -i\widetilde{z} \end{smallmatrix}\right], \left[\begin{smallmatrix} \widetilde{u} \\ -i\Psi\widetilde{y} \end{smallmatrix}\right]\right\} \in \Gamma \quad &\iff \\ \left(\left[\begin{smallmatrix} -iz \\ i\Psi y \end{smallmatrix}\right], \left[\begin{smallmatrix} \widetilde{x} \\ \widetilde{u} \end{smallmatrix}\right]\right) = \left(\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right], \left[\begin{smallmatrix} -i\widetilde{z} \\ i\Psi\widetilde{y} \end{smallmatrix}\right]\right), \quad \left\{\left[\begin{smallmatrix} \widetilde{x} \\ \widetilde{u} \end{smallmatrix}\right], \left[\begin{smallmatrix} -i\widetilde{z} \\ i\Psi\widetilde{y} \end{smallmatrix}\right]\right\} \in \mathcal{J}(\Gamma), &\end{aligned}$$

where the last line is equivalent to $\left\{\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right], \left[\begin{smallmatrix} -iz \\ i\Psi y \end{smallmatrix}\right]\right\} \in \mathcal{J}(\Gamma)^*$.

Thus $V \subset V^{[\perp]}$ if and only if condition (2) of Definition 34 holds, which in turn is true if and only if $\mathcal{J}(\Gamma) \subset \mathcal{J}(\Gamma)^*$. Analogously, $V^{[\perp]} \subset V$ if and only if condition (3) of Definition 34 holds, which in turn is true if and only if $\mathcal{J}(\Gamma)^* \subset \mathcal{J}(\Gamma)$. $\square$

If the signal space $\mathcal{W}$ has no Lagrangian decomposition, which is the case, e.g., when the dimension of $\mathcal{W}$ is finite and odd, then Σ is not induced by any conservative boundary relation, cf. Example ??. We collect our observations in the following theorem:

Theorem 38. *The following claims are true:*

(1) *Let $(V; \mathcal{X}, \mathcal{W})$ be a* conservative *s/s node and assume that there exists a Lagrangian decomposition $\mathcal{W} = \mathcal{U} \overset{\Psi}{\dotplus} \mathcal{Y}$. Define Γ by* (5.5) *and set $R := \ker(\Gamma)$.*

Then R is a closed symmetric operator *in $\mathcal{X}$, R^* is the closure of* $\operatorname{dom}(\Gamma)$ *in $\mathcal{X}^2$, Γ is a conservative boundary relation for R^*, and V can be recovered using the following expression, which reduces to* (5.3) *when $\mathcal{Y}=\mathcal{U}$ and $\Psi=1_{\mathcal{U}}$:*

$$V=\left\{\begin{bmatrix} iz\\ x\\ \begin{bmatrix} u\\ i\Psi^* y\end{bmatrix}\end{bmatrix}\in\begin{bmatrix}\mathcal{X}\\ \mathcal{X}\\ \mathcal{W}\end{bmatrix}\,\middle|\,\{[\begin{smallmatrix} x\\ z\end{smallmatrix}],[\begin{smallmatrix} u\\ y\end{smallmatrix}]\}\in\Gamma\right\}. \tag{5.7}$$

(2) *Conversely, let R be a closed symmetric linear relation in the Hilbert space $\mathcal{X}$ and let $\Gamma:\mathcal{X}^2\to\mathcal{U}^2$ be a conservative boundary relation for R^*. Let $\mathcal{W}:=\mathcal{U}^2$ be the Kreĭn space with the indefinite inner product* (5.2) *(corresponding to $\Psi=1_{\mathcal{U}}$). Define V by* (5.3)*, and $\widetilde{\mathcal{X}}$ and $\widetilde{V}$ by* (5.4).

Then $\widetilde{\Sigma}=(\widetilde{V};\widetilde{\mathcal{X}},\mathcal{W})$ is a conservative s/s node with state space $\widetilde{\mathcal{X}}=\mathcal{X}\ominus\operatorname{mul}(R)$, where $\operatorname{mul}(R)=\{z\mid\{0,z\}\in R\}$ *is the* multi-valued part *of R. Moreover, if we define $\widetilde{\Gamma}$ by the right-hand side of* (5.5) *with V replaced by $\widetilde{V}$ and $\Psi=1_{\mathcal{U}}$, then*

$$\widetilde{\Gamma}=\Gamma\big|_{\operatorname{dom}(\Gamma)\cap\widetilde{\mathcal{X}}^2}=\Gamma\big|_{\operatorname{dom}(\Gamma)\cap\left[\begin{smallmatrix}\widetilde{\mathcal{X}}\\ \mathcal{X}\end{smallmatrix}\right]}. \tag{5.8}$$

(3) *Let the conservative boundary relation Γ and the conservative s/s node $\Sigma=(\widetilde{V};\widetilde{\mathcal{X}},\mathcal{W})$ be related as in* (5.7) *and* (5.4). *Denote the Weyl family and γ-field of Γ by M and γ, respectively, and let $\widehat{\mathfrak{E}}$ be the characteristic node bundle of Σ. Then*

$$\begin{aligned} M(\lambda)&=\left\{\{u,-i\Psi y\}\,\middle|\,\begin{bmatrix} x\\ 0\\ \begin{bmatrix} u\\ y\end{bmatrix}\end{bmatrix}\in\widehat{\mathfrak{E}}(i\lambda)\right\}\quad\textit{and}\\ \gamma(\lambda)&=\left\{\{u,x\}\,\middle|\,\begin{bmatrix} x\\ 0\\ \begin{bmatrix} u\\ y\end{bmatrix}\end{bmatrix}\in\widehat{\mathfrak{E}}(i\lambda)\right\},\quad\lambda\in\mathbb{C}\setminus\mathbb{R}.\end{aligned} \tag{5.9}$$

Proof. First note that if $R=\ker(\Gamma)$ then (5.5) implies that

$$z\in\operatorname{mul}(R)\quad\Longleftrightarrow\quad\{[\begin{smallmatrix}0\\ z\end{smallmatrix}],[\begin{smallmatrix}0\\ 0\end{smallmatrix}]\}\in\Gamma\quad\Longleftrightarrow\quad\begin{bmatrix} z\\ 0\\ 0\end{bmatrix}\in V. \tag{5.10}$$

(1) Since every Lagrangian V is closed and neutral, Γ and its kernel R are also closed, and from (5.1) it follows that R is symmetric. By (5.10), R is single-valued. Lemma 37 yields that $\mathcal{J}(\Gamma)$ is self-adjoint, and applying [Derkach et al., 2006, Prop. 3.5], we obtain that Γ is a conservative boundary relation for R^*. Condition (1) of Definition 34 says that R^* is the closure of $\operatorname{dom}(\Gamma)$ in $\mathcal{X}^2$. It is easy to verify that (5.5) and (5.7) are equivalent.
(2) Setting $\mathcal{Y}=\mathcal{U}$ and $\Psi=1_{\mathcal{U}}$ in Lemma 37, we obtain that $\mathcal{W}$ is a Kreĭn space and that $V=V^{[\perp]}$, and according to Proposition 36, $\widetilde{\Sigma}$ is then a conservative s/s system. By [Derkach et al., 2006, Prop. 3.2], $R=\ker(\Gamma)$, and therefore (5.10) and (5.4) imply that $\widetilde{\mathcal{X}}=\mathcal{X}\ominus\operatorname{mul}(R)$.

The first equality in (5.8) follows by noting that

$$\{[\begin{smallmatrix} x \\ z \end{smallmatrix}],[\begin{smallmatrix} u \\ y \end{smallmatrix}]\} \in \widetilde{\Gamma} \quad \Longleftrightarrow \quad \begin{bmatrix} iz \\ x \\ [\begin{smallmatrix} u \\ iy \end{smallmatrix}] \end{bmatrix} \in \widetilde{V} \quad \Longleftrightarrow$$

$$\begin{bmatrix} iz \\ x \\ [\begin{smallmatrix} u \\ iy \end{smallmatrix}] \end{bmatrix} \in V,\ z, x \in \widetilde{\mathcal{X}} \quad \Longleftrightarrow \quad \{[\begin{smallmatrix} x \\ z \end{smallmatrix}],[\begin{smallmatrix} u \\ y \end{smallmatrix}]\} \in \Gamma,\ z, x \in \widetilde{\mathcal{X}}.$$

The second equality holds, since we by (3.2) always have

$$\begin{bmatrix} iz \\ x \\ [\begin{smallmatrix} u \\ iy \end{smallmatrix}] \end{bmatrix} \in V = V^{[\perp]} \quad \Longrightarrow \quad (x, \widetilde{z})_{\mathcal{X}} = 0,\ \begin{bmatrix} \widetilde{z} \\ 0 \\ 0 \end{bmatrix} \in V, \tag{5.11}$$

i.e., $x \in \widetilde{\mathcal{X}}$ automatically when $\begin{bmatrix} iz \\ x \\ [\begin{smallmatrix} u \\ iy \end{smallmatrix}] \end{bmatrix} \in V$ for a Lagrangian V.

(3) The equalities (5.9) now follow from Definition 35 once we observe that

$$\begin{bmatrix} x \\ 0 \\ [\begin{smallmatrix} u \\ y \end{smallmatrix}] \end{bmatrix} \in \widehat{\mathfrak{E}}(i\lambda) \quad \Longleftrightarrow \quad \begin{bmatrix} i\lambda x \\ x \\ [\begin{smallmatrix} u \\ y \end{smallmatrix}] \end{bmatrix} \in V,\ x \in \widetilde{\mathcal{X}} \quad \Longleftrightarrow \quad \begin{bmatrix} i\lambda x \\ x \\ [\begin{smallmatrix} u \\ y \end{smallmatrix}] \end{bmatrix} \in V$$
$$\Longleftrightarrow \quad \{[\begin{smallmatrix} x \\ \lambda x \end{smallmatrix}],[\begin{smallmatrix} u \\ -i\Psi y \end{smallmatrix}]\} \in \Gamma$$

where we used Definition 32, (5.11), and (5.7), respectively.

□

Claim (3) of Theorem 38 shows that $\widehat{\mathfrak{E}} \cap \begin{bmatrix} \mathcal{X} \\ 0 \\ \mathcal{W} \end{bmatrix}$ can be identified with the product of the γ-field and the Weyl family of the conservative boundary relation Γ in (5.5). Note, however, that there is an extra rotation of the complex plane in (5.9), due to the fact that in the boundary relation theory one works with self-adjoint operators that have $\mathbb{C} \setminus \mathbb{R}$ in their resolvent set, whereas in the s/s theory the convention is to use skew-adjoint operators whose resolvent sets contain $\mathbb{C} \setminus i\mathbb{R}$. Also note that the ordering of the two internal variables z and x is different on the left-hand and the right-hand sides of (5.7), which is due to different conventions in different fields of mathematics.

5.3. **A systems theory interpretation.** We now introduce dynamics to a conservative boundary relation by giving a systems and control theory interpretation. At the same time, the following results show that boundary relations, in spite of their name, are much more closely related to the general i/s/o systems in Section 2.2 than to the boundary control systems in Chapter ??.

Theorem 39. *Assume that $\Gamma \subset \mathcal{X}^2 \times \mathcal{U}^2$ is a conservative boundary relation with the following properties:*

(1) *If $\left\{[\begin{smallmatrix} 0 \\ z \end{smallmatrix}],[\begin{smallmatrix} 0 \\ y \end{smallmatrix}]\right\} \in \Gamma$ then $z = 0$ and $y = 0$.*

(2) *The set $V_u := \left\{u \;\middle|\; \{[\begin{smallmatrix} x \\ z \end{smallmatrix}],[\begin{smallmatrix} u \\ y \end{smallmatrix}]\} \in \Gamma\right\}$ equals $\mathcal{U}$.*

Then Γ has the representation

$$\Gamma = \left\{\{[\begin{smallmatrix} x \\ -iz \end{smallmatrix}],[\begin{smallmatrix} u \\ -iy \end{smallmatrix}]\} \;\middle|\; [\begin{smallmatrix} x \\ u \end{smallmatrix}] \in \mathrm{dom}\,(S),\ [\begin{smallmatrix} z \\ y \end{smallmatrix}] = S\,[\begin{smallmatrix} x \\ u \end{smallmatrix}]\right\}, \tag{5.12}$$

where $(S;\mathcal{X},\mathcal{U},\mathcal{U})$ *is an impedance conservative i/s/o system node.*

Moreover, $\Sigma := (V;\mathcal{X},\mathcal{W})$ *is a conservative s/s node, where* V *is defined by* (5.3) *and* $\mathcal{W} = \mathcal{U}^2$ *with the inner product* (5.2)*. The Lagrangian decomposition* $\mathcal{W} = \left[\begin{smallmatrix}\mathcal{U}\\ \{0\}\end{smallmatrix}\right] \overset{\Psi}{+} \left[\begin{smallmatrix}\{0\}\\ \mathcal{U}\end{smallmatrix}\right]$, $\Psi = 1_{\mathcal{U}}$*, is admissible, and* $(S;\mathcal{X},\mathcal{U},\mathcal{U})$ *is the corresponding impedance representation.*

Proof. From claim (2) of Theorem 38 it follows that V defined in (5.3) generates a conservative s/s system. The representation (5.12) for some (single-valued) operator S follows from assumption (1). Then $V = V^{[\perp]}$, assumption (2), and [Ball and Staffans, 2006, Prop. 4.11] imply that S is an impedance conservative i/s/o system node operator, which is an impedance representation of V:

$$\left[\begin{smallmatrix} z\\ x\\ \left[\begin{smallmatrix}u\\ y\end{smallmatrix}\right]\end{smallmatrix}\right] \in V \quad\Longleftrightarrow\quad \left\{\left[\begin{smallmatrix}x\\ -iz\end{smallmatrix}\right], \left[\begin{smallmatrix}u\\ -iy\end{smallmatrix}\right]\right\} \in \Gamma \quad\Longleftrightarrow\quad \left[\begin{smallmatrix}z\\ y\end{smallmatrix}\right] = S\left[\begin{smallmatrix}x\\ u\end{smallmatrix}\right],$$

where we have used Definition 7, (5.3) and (5.12). $\square$

It follows from Theorems 23 and 39 that for every $u \in W^{2,1}_{\mathrm{loc}}(\mathbb{R}^+;\mathcal{U})$ and every initial state $x_0 \in \mathcal{X}$ with $\left\{\left[\begin{smallmatrix}x_0\\ z\end{smallmatrix}\right], \left[\begin{smallmatrix}u(0)\\ y\end{smallmatrix}\right]\right\} \in \Gamma$, the system

$$\left\{\begin{bmatrix} x(t)\\ -i\dot{x}(t)\end{bmatrix}, \begin{bmatrix} u(t)\\ -iy(t)\end{bmatrix}\right\} \in \Gamma, \quad t \in \mathbb{R}^+, \tag{5.13}$$

has a unique *classical* solution (u,x,y) with $x(0) = x_0$.

We have the following converse to Theorem 39:

Proposition 40. *If* $(S;\mathcal{X},\mathcal{U},\mathcal{U})$ *is an impedance-conservative i/s/o system node then* Γ *in* (5.12) *is a conservative boundary relation for* R^**, where* $R := \ker(\Gamma)$*. Moreover,* Γ *has properties (1) and (2) in Theorem 39.*

Proof. If $S = \left[\begin{smallmatrix}A\&B\\ C\&D\end{smallmatrix}\right]$ is an impedance conservative i/s/o system node operator then $S = \left[\begin{smallmatrix}A\&B\\ -C\&D\end{smallmatrix}\right]$ is skew-adjoint by [Staffans, 2002b, Thm 4.3], and this is equivalent to $S = \left[\begin{smallmatrix}-iA\&B\\ iC\&D\end{smallmatrix}\right]$ being self-adjoint. By (5.6) and (5.12),

$$\mathcal{J}(\Gamma) = \left\{\left\{\left[\begin{smallmatrix}x\\ u\end{smallmatrix}\right], \left[\begin{smallmatrix}-iA\&B\\ iC\&D\end{smallmatrix}\right]\left[\begin{smallmatrix}x\\ u\end{smallmatrix}\right]\right\} \;\middle|\; \left[\begin{smallmatrix}x\\ u\end{smallmatrix}\right] \in \mathrm{dom}\left(\left[\begin{smallmatrix}-iA\&B\\ iC\&D\end{smallmatrix}\right]\right)\right\},$$

and we obtain from [Derkach et al., 2006, Prop. 3.5] that Γ is a conservative boundary relation for R^*.

Moreover, by condition (4) of Definition 5, for all $u \in \mathcal{U}$ there exists some $x \in \mathcal{X}$ such that $\left[\begin{smallmatrix}x\\ u\end{smallmatrix}\right] \in \mathrm{dom}\left(\left[\begin{smallmatrix}A\&B\\ C\&D\end{smallmatrix}\right]\right)$. From (5.12) it now follows that condition (2) in Theorem 39 is met, and also that $\left\{\left[\begin{smallmatrix}0\\ z\end{smallmatrix}\right], \left[\begin{smallmatrix}0\\ y\end{smallmatrix}\right]\right\} \in \Gamma$ implies $\left[\begin{smallmatrix}z\\ y\end{smallmatrix}\right] = 0$. $\square$

Using Proposition 36, one can reformulate Theorem 39 slightly in such a way that condition (1) is replaced by the weaker condition that

$$\left\{\left[\begin{smallmatrix}0\\ z\end{smallmatrix}\right], \left[\begin{smallmatrix}0\\ y\end{smallmatrix}\right]\right\} \in \Gamma \quad\Longrightarrow\quad y = 0. \tag{5.14}$$

Moreover, condition (1) implies that $\mathrm{dom}\,(S)$ is dense in $\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right]$ when Γ is a boundary relation, and therefore, condition (2) can be weakened to the condition that V_u is closed. We formulate the result but we leave the proof to the reader.

Corollary 41. *Assume that* Γ *is a conservative boundary relation such that* (5.14) *holds and the set* V_u *in Theorem 39 is closed. Let* $\mathcal{W} := \left[\begin{smallmatrix}\mathcal{U}\\ \mathcal{U}\end{smallmatrix}\right]$ *with the indefinite inner product* (5.2)*, let* V *be given by* (5.3)*, and let* $\widetilde{\mathcal{X}}, \widetilde{V}$ *be given by* (5.4)*.*

Then Γ *has the representation*

$$\Gamma = \left\{\left\{\left[\begin{smallmatrix}x\\ -iz\end{smallmatrix}\right], \left[\begin{smallmatrix}u\\ -iy\end{smallmatrix}\right]\right\} \,\middle|\, \left[\begin{smallmatrix}x\\ u\end{smallmatrix}\right] \in \operatorname{dom}(S),\ z \in \mathcal{X},\ \left[\begin{smallmatrix}Pz\\ y\end{smallmatrix}\right] = S\left[\begin{smallmatrix}x\\ u\end{smallmatrix}\right]\right\},$$

where P *is the orthogonal projection of* $\mathcal{X}$ *onto* $\widetilde{\mathcal{X}}$*, and* $(S; \mathcal{X}, \mathcal{U}, \mathcal{U})$ *is the impedance representation corresponding to the admissible Lagrangian decomposition* $\mathcal{W} = \left[\begin{smallmatrix}\mathcal{U}\\ \{0\}\end{smallmatrix}\right] \overset{\Psi}{\dotplus} \left[\begin{smallmatrix}\{0\}\\ \mathcal{U}\end{smallmatrix}\right]$, $\Psi = 1_{\mathcal{U}}$*, of the conservative s/s system* $\Sigma = (\widetilde{V}; \widetilde{\mathcal{X}}, \mathcal{W})$*.*

We could also have introduced dynamics to the boundary relation simply by considering the classical and generalized solutions of (5.13). However, without using claim (2) of Theorem 23, or changing to a scattering representation and using Theorem 14, we would not know that the sets of classical and generalized trajectories in fact are large.

6. Conclusions

We have presented the fundamentals of the state/signal approach to systems theory and we have made the basic connections between this theory and that of conservative boundary relations. We can conclude that the main objects of the two fields, namely the s/s system and the (conservative) boundary relation, are very closely related.

Sometimes technical complications arise from the way a s/s system is represented by an i/s/o system and not from the s/s system itself. For instance, the characteristic node bundle of a s/s system is much cleaner and more general than an i/s/o resolvent matrix. Moreover, in many cases it is useful to change from a impedance representation to a scattering representations in order to obtain a well-posed system which describes the dynamics of the system in a clear way; see Remark 24. The s/s formalism provides a firm basis for doing this. In particular, the families of all classical and generalized trajectories of a passive s/s system are in general more easily characterized by means of a scattering representation than by means of an impedance representation.

Passivity is a good example of a property which refers to a physical system, and not to any one of its input/output representations. Indeed, the property of passivity of a s/s system $(V; \mathcal{X}, \mathcal{W})$ simply means that the generating subspace V is maximal non-negative, whereas different i/s/o *representations* of the s/s system *are passive in different senses*, cf. Definitions 17 and 22.

For instance the flexibility in choosing i/s/o representations, the introduction of dynamics, the connection to control theory made in Theorem 39, and the work done on passive nonconservative s/s systems could potentially turn out to be useful for future research in the theory of boundary relations.

Conversely, it is interesting to look for new directions for the future development of the state/signal theory by studying the theory of boundary relations. In particular, the realization results [Derkach et al., 2006, Thm 3.9] and [Behrndt et al., 2009, Thm 6.1] can be utilized directly for conservative s/s systems in the case where a Lagrangian decomposition of the external signal space exists, i.e., when the signal space has equal positive and negative indices. An intriguing question is exactly how these realizations are related to those developed in [Arov et al., 2011] and their frequency-domain counterparts. A related question is to what extent the available results on Weyl families and their connections to the associated boundary relation can be employed in order to explore the properties of the characteristic node bundles of s/s systems.

It is our sincere hope that this exposition will increase the interaction between researchers of boundary relations and state/signal systems, thus preventing overlapping research, and that it gives rise to future co-operation on common research interests.

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