BOUNDARY CONTROL STATE/SIGNAL SYSTEMS AND BOUNDARY TRIPLETS

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1. Introduction

We give an introduction to the basic theory of state/signal systems via boundary control. More precisely, we discuss the connection between some basic notions of boundary control state/signal systems on one hand, and classical boundary triplets on the other hand. Boundary triplets and their generalizations have been extensively utilized in the theory of self-adjoint extensions of symmetrical operators in Hilbert spaces, see e.g. [Gorbachuk and Gorbachuk, 1991], [Derkach and Malamud, 1995], [Behrndt and Langer, 2007], and the references therein.

The notions related to standard input/state/output boundary control systems are discussed in Section 2, where we also introduce the boundary control state/signal system. In Section 3 we briefly discuss the concept of conservativity in the state/signal framework and in Section 4 we illustrate the abstract concepts using the example of a finite-length conservative $\mathcal{LC}$-transmission line with distributed inductance and capacitance.

We conclude this chapter in Section 5, where we recall the definition of a boundary triplet for a symmetric operator and compare this object to a boundary control state/signal system. In particular, we show that every boundary triplet can be transformed into a conservative boundary control state/signal system in impedance form, but that the converse is not true. We make a few final remarks about common generalizations of boundary triplets, which leads over to Chapter ??,

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where we treat more general passive state/signal systems, not only conservative systems or systems of boundary-control type. There we show how conservative state/signal systems are related to *boundary relations*.

2. Boundary control systems

In this section we introduce boundary control state/signal systems by first describing their predecessors, input/state/output systems of boundary-control type.

In boundary control one often investigates systems that can be abstractly written in the form

$$\Sigma_{i/s/o}: \quad \begin{cases} \dot{x}(t) = Lx(t), \\ u(t) = \Gamma_0 x(t), \quad t \in \mathbb{R}^+, \ x(0) = x_0 \text{ given}, \\ y(t) = \Gamma_1 x(t), \end{cases} \tag{2.1}$$

where $\mathbb{R}^+ = [0, \infty)$ and $\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t}$. Here the *initial state* x_0 and the *current state* $x(t)$ belong to the Hilbert state space $\mathcal{X}$, the *input* $u(t)$ belongs to the Hilbert input space $\mathcal{U}$, and the *output* $y(t)$ belongs to the Hilbert output space $\mathcal{Y}$. The *main operator* L is an unbounded operator in $\mathcal{X}$ with domain $\mathrm{dom}\,(L)$, and the *boundary control operator* Γ_0 is an unbounded operator $\mathcal{X} \to \mathcal{U}$ with the same domain as L. The *observation operator* $\Gamma_1 \colon \mathcal{X} \to \mathcal{Y}$ may be bounded or unbounded, and it is defined at least on $\mathrm{dom}\,(L)$. All of these operators are linear. We denote the system (2.1) with these properties by $\Sigma_{i/s/o} = (L, \Gamma_0, \Gamma_1; \mathcal{X}, \mathcal{U}, \mathcal{Y})$.

In order for (2.1) to generate a dynamical system with good properties at least the properties listed in the following definition need to be assumed; see e.g. [Salamon, 1987], [Staffans, 2005], or [Malinen and Staffans, 2006] for details.

Definition 1. Assume that $\Sigma_{i/s/o} = (L, \Gamma_0, \Gamma_1; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ is as described above. Then $\Sigma_{i/s/o}$ is a *boundary control input/state/output (i/s/o) node* if $\Sigma_{i/s/o}$ satisfies the following conditions:

(1) The input operator Γ_0 is surjective and *strictly unbounded* in the sense that $\ker(\Gamma_0)$ *is dense in* $\mathcal{X}$.
(2) The restriction $A := L|_{\ker(\Gamma_0)}$ *of* L *to* $\ker(\Gamma_0)$ *generates a* C_0-*semigroup* $t \mapsto \mathfrak{A}^t$, $t \in \mathbb{R}^+$.

A *boundary control state/signal system* is analogous to a boundary control i/s/o system, but we no longer specify which part of the "boundary signal" $w(t) := \begin{bmatrix} u(t) \\ y(t) \end{bmatrix}$ is the input, and which part is the output. Instead we combine the input and output spaces into one signal space $\mathcal{W} := \begin{bmatrix} \mathcal{U} \\ \mathcal{Y} \end{bmatrix} = \mathcal{U} \times \mathcal{Y}$, and denote $\Gamma := \begin{bmatrix} \Gamma_0 \\ \Gamma_1 \end{bmatrix}$. Then $\Gamma \colon \mathrm{dom}\,(L) \to \mathcal{W}$, and (2.1) can be rewritten in the form

$$\Sigma: \begin{cases} \dot{x}(t) = Lx(t), \\ w(t) = \Gamma x(t), \end{cases} \quad t \in \mathbb{R}^+,\ x(0) = x_0 \text{ given.} \tag{2.2}$$

As before, the *initial state* x_0 and the *current state* $x(t)$ belong to the Hilbert state space $\mathcal{X}$. The *(interaction) signal* $w(t)$ belongs to the signal space $\mathcal{W}$, which we take to be an arbitrary Kreĭn space (the reason for this will be explained below). We thus no longer assume that $\mathcal{W}$ is of the form $\mathcal{W} = \left[\begin{smallmatrix}\mathcal{U}\\ \mathcal{Y}\end{smallmatrix}\right]$, where $\mathcal{U}$ and $\mathcal{Y}$ are the input and output spaces of a boundary control i/s/o node. The *main operator* L is still an unbounded operator $\mathcal{X} \to \mathcal{X}$ with domain $\operatorname{dom}(L)$, and the *boundary operator* Γ is an unbounded operator $\mathcal{X} \to \mathcal{W}$ with the same domain as L. We denote this system by $\Sigma = (L, \Gamma; \mathcal{X}, \mathcal{W})$.

Note that (2.2) can be written in the *graph form*:

$$\Sigma: \begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix} \in V, \quad t \in \mathbb{R}^+,\ x(0) = x_0,$$

where the *generating subspace* V is the graph of $\left[\begin{smallmatrix}L\\ \Gamma\end{smallmatrix}\right]$:

$$V := \left\{ \left[\begin{smallmatrix} Lx \\ x \\ \Gamma x \end{smallmatrix}\right] \,\middle|\, x \in \operatorname{dom}(L) \right\}. \tag{2.3}$$

The unbounded operator $\left[\begin{smallmatrix}L\\ \Gamma\end{smallmatrix}\right]$ is assumed to be closed, and this is equivalent to assuming that V is a closed subspace of the *node space* $\mathcal{X} \times \mathcal{X} \times \mathcal{W}$. The generating subspace is the key to generalizing the state/signal theory beyond boundary control, as we shall see in Chapter ??. We define the *dynamics* of a state/signal system using the generating subspace V.

Definition 2. Let V be a closed subspace of $\mathcal{X} \times \mathcal{X} \times \mathcal{W}$.

(1) The pair $\left[\begin{smallmatrix}x\\ w\end{smallmatrix}\right]$ is a *classical trajectory* generated by V on $\mathbb{R}^+$ if $x \in C^1(\mathbb{R}^+; \mathcal{X})$, $w \in C(\mathbb{R}^+; \mathcal{W})$, and $\left[\begin{smallmatrix}\dot{x}(t)\\ x(t)\\ w(t)\end{smallmatrix}\right] \in V$ for all $t > 0$.

(2) The pair $\left[\begin{smallmatrix}x\\ w\end{smallmatrix}\right]$ is a *generalized trajectory* generated by V on $\mathbb{R}^+$ if $x \in C(\mathbb{R}^+; \mathcal{X})$, $w \in L^2_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{W})$, and there exists a sequence of classical trajectories $\left[\begin{smallmatrix}x_n\\ w_n\end{smallmatrix}\right]$ such that $x_n \to x$ uniformly on all bounded intervals $[0, T]$ and $w_n \to w$ in $L^2_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{W})$.

Note that $\left[\begin{smallmatrix}\dot{x}(t)\\ x(t)\\ w(t)\end{smallmatrix}\right] \in V$ for all $t > 0$ in item (1) of Definition 2 if and only if $\left[\begin{smallmatrix}\dot{x}(t)\\ x(t)\\ w(t)\end{smallmatrix}\right] \in V$ for all $t \in \mathbb{R}^+$ when we interpret $\dot{x}(0)$ as the right-sided derivative of x at zero. We are now ready to define a boundary control s/s system.

Definition 3. A *boundary control state/signal (s/s) node* is a quadruple $\Sigma = (L, \Gamma; \mathcal{X}, \mathcal{W})$ such that:

(1) The space $\mathcal{X}$ is a Hilbert space and $\mathcal{W}$ is a Kreĭn space.
(2) The operator $\left[\begin{smallmatrix} L \\ \Gamma \end{smallmatrix}\right] : \mathcal{X} \to \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$ is closed and densely defined.
(3) The range of Γ is dense in $\mathcal{W}$.

By the *boundary control state/signal system* induced by a boundary control s/s node $(L, \Gamma; \mathcal{X}, \mathcal{W})$ we mean this node together with the sets of classical and generalized trajectories generated by V in (2.3) on $\mathbb{R}^+$. We denote both the node and the system by $\Sigma = (L, \Gamma; \mathcal{X}, \mathcal{W})$.

In Definition 4 below we will equip the node space $\mathcal{X} \times \mathcal{X} \times \mathcal{W}$ with an indefinite inner product which makes it a Kreĭn space.

3. Conservative state/signal systems in boundary control

In this chapter we shall focus our attention on s/s systems Σ whose classical trajectories on $\mathbb{R}^+$ satisfy the *power equality*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|_{\mathcal{X}}^2 = [w(t), w(t)]_{\mathcal{W}}, \quad t \in \mathbb{R}^+. \tag{3.1}$$

Here $\|x(t)\|_{\mathcal{X}}^2$ stands for (two times) the *internal energy* stored in the state x at time t and $[w(t), w(t)]_{\mathcal{W}}$ represents (two times) the power (energy flow per time unit) entering the system through the signal $w(t)$ at time t. This explains why we need to take $\mathcal{W}$ to be a Kreĭn space: we must allow the inner product $[\cdot,\cdot]_{\mathcal{W}}$ in $\mathcal{W}$ to be indefinite: if the inner product in $\mathcal{W}$ is non-negative, then no energy can leave the system via the (interaction) signal, and if the inner product in $\mathcal{W}$ is non-positive, then no energy can enter the system via the signal.

The equality (3.1) says that the system has *no internal energy sources or sinks*. However, the equality is not enough to make the system Σ conservative: we need an additional *hypermaximality condition*. We give the full definition of a conservative boundary control s/s system in Definition 5 below.

After integration over the interval $[s, t] \subset \mathbb{R}^+$, one can rewrite (3.1) in the equivalent form

$$\|x(t)\|_{\mathcal{X}}^2 - \|x(s)\|_{\mathcal{X}}^2 = \int_s^t [w(v), w(v)]_{\mathcal{W}}\,\mathrm{d}v, \quad s, t \in \mathbb{R}^+,\ s \le t. \tag{3.2}$$

By the continuity of the inner product this inequality remains valid for generalized trajectories as well.

Carrying out the differentiation in (3.1), we get a third equivalent condition in terms of classical trajectories, namely

$$-(\dot{x}(t), x(t))_{\mathcal{X}} - (x(t), \dot{x}(t))_{\mathcal{X}} + [w(t), w(t)]_{\mathcal{W}} = 0, \quad t \in \mathbb{R}^+. \tag{3.3}$$

Using item (1) of Definition 2, we see that (3.3) always holds if

$$-(z, x)_{\mathcal{X}} - (x, z)_{\mathcal{X}} + [w, w]_{\mathcal{W}} = 0, \quad \begin{bmatrix} z \\ x \\ w \end{bmatrix} \in V. \tag{3.4}$$

It is now natural to make the following definition:

Definition 4. Let $\mathcal{X}$ be a Hilbert space and $\mathcal{W}$ a Kreĭn space. The corresponding *node space* is the product space $\mathfrak{K} = \mathcal{X}\times\mathcal{X}\times\mathcal{W}$ equipped with the indefinite inner product induced by the quadratic form in (3.4):

$$\left[\left[\begin{smallmatrix} z_1 \\ x_1 \\ w_1 \end{smallmatrix}\right], \left[\begin{smallmatrix} z_2 \\ x_2 \\ w_2 \end{smallmatrix}\right]\right]_{\mathfrak{K}} = -(z_1, x_2)_{\mathcal{X}} - (x_1, z_2)_{\mathcal{X}} + [w_1, w_2]_{\mathcal{W}}. \tag{3.5}$$

Note that the the quadratic form in (3.4) is strictly indefinite, i.e., it takes both positive and negative values whenever $\mathcal{X} \neq \{0\}$. Furthermore, the inner product in (3.5) makes the node space $\mathfrak{K}$ a Kreĭn space.

The equality (3.4) says that V is a *neutral subspace* of $\mathfrak{K}$ with respect to the inner product (3.5), i.e., that $[v, v]_{\mathfrak{K}} = 0$ for all $v \in V$. The condition that a subspace V is a neutral subspace of $\mathfrak{K}$ can equivalently be written $V \subset V^{[\perp]}$, where

$$V^{[\perp]} := \left\{k \in \mathfrak{K} \mid [k, k']_{\mathfrak{K}} = 0 \text{ for all } k' \in V\right\}. \tag{3.6}$$

If instead $V^{[\perp]} \subset V$, then V is called co-neutral, and if $V^{[\perp]} = V$, then V is called *Lagrangian* or *hypermaximal neutral*.

Definition 5. A boundary control s/s system $\Sigma = (L, \Gamma; \mathcal{X}, \mathcal{W})$ is *conservative* if its generating subspace V in (2.3) is a Lagrangian subspace of the node space $\mathfrak{K}$, i.e., if $V = V^{[\perp]}$.

Since every orthogonal companion is closed, necessarily every Lagrangian subspace is closed. Moreover, in [Kurula et al., 2010, Thm 4.3] it was proved that if V in (2.3) is Lagrangian then $\ker(\Gamma)$ is dense in $\mathcal{X}$ and $\operatorname{ran}(\Gamma)$ is dense in $\mathcal{W}$. Since $\ker(\Gamma) \subset \operatorname{dom}(\Gamma) = \operatorname{dom}(L)$, the operator $\left[\begin{smallmatrix} L \\ \Gamma \end{smallmatrix}\right]$ is closed and automatically densely defined. Thus the conditions in Definition 3 are satisfied for every Lagrangian subspace V of the type (2.3). See also [Derkach et al., 2006, Cor. 2.4].

Remark 6. In the boundary control case the neutrality condition $V \subset V^{[\perp]}$ means that

$$(Lx, x)_{\mathcal{X}} + (x, Lx)_{\mathcal{X}} = [\Gamma x, \Gamma x]_{\mathcal{W}}, \quad x \in \operatorname{dom}(L). \tag{3.7}$$

However, if V is only neutral, then V might for instance be the degenerate trivial system $\{0\}$. This case is excluded by the hypermaximality condition $V \supset V^{[\perp]}$, which in the case of boundary control means that

$$(z_1, x)_{\mathcal{X}} + (x_1, Lx)_{\mathcal{X}} = [w_1, \Gamma x]_{\mathcal{W}}, \ x \in \operatorname{dom}(L) \quad \Longrightarrow \quad \left[\begin{smallmatrix} z_1 \\ x_1 \\ w_1 \end{smallmatrix}\right] \in V. \tag{3.8}$$

Letting $\mathcal{X}$ be a Hilbert space, $\mathcal{W}$ be a Kreĭn space, and $\left[\begin{smallmatrix} L \\ \Gamma \end{smallmatrix}\right] : \mathcal{X} \to \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$, we thus have that $\Sigma = (L, \Gamma; \mathcal{X}, \mathcal{W})$ is a conservative boundary control s/s system if and only if the conditions (3.7) and (3.8) are satisfied.

4. An example: the transmission line

An ideal transmission line of length ℓ can be modeled by the following equations, where $\xi \in [0,\ell]$ and $t \in \mathbb{R}^+$:

$$
\begin{aligned}
\frac{\partial}{\partial t}\begin{bmatrix} i(\xi,t) \\ v(\xi,t)\end{bmatrix} &= \begin{bmatrix} 0 & -\frac{1}{\mathcal{L}(\xi)}\frac{\partial}{\partial \xi} \\ -\frac{1}{\mathcal{C}(\xi)}\frac{\partial}{\partial \xi} & 0 \end{bmatrix}\begin{bmatrix} i(\xi,t) \\ v(\xi,t)\end{bmatrix}, \\
w(t) &= \begin{bmatrix} i(0,t) \\ v(0,t) \\ -i(\ell,t) \\ v(\ell,t)\end{bmatrix}, \quad \begin{bmatrix} i(\xi,0) \\ v(\xi,0)\end{bmatrix} = \begin{bmatrix} i_0(\xi) \\ v_0(\xi)\end{bmatrix}.
\end{aligned}
\tag{4.1}
$$

Here $i(\xi,t)$ and $v(\xi,t)$ are the current and voltage, respectively, at the point $\xi \in [0,\ell]$ at time $t \in \mathbb{R}^+$. The functions $\mathcal{L}(\cdot) > 0$ and $\mathcal{C}(\cdot) > 0$ represent the *distributed inductance and capacitance*, respectively, of the line. For simplicity we assume that $\mathcal{C}(\cdot)$ and $\mathcal{L}(\cdot)$ are continuous on $[0,\ell]$, which implies that $\mathcal{C}$ and $\mathcal{L}$ are both bounded and bounded away from zero. The transmission line is illustrated in Figure 1.

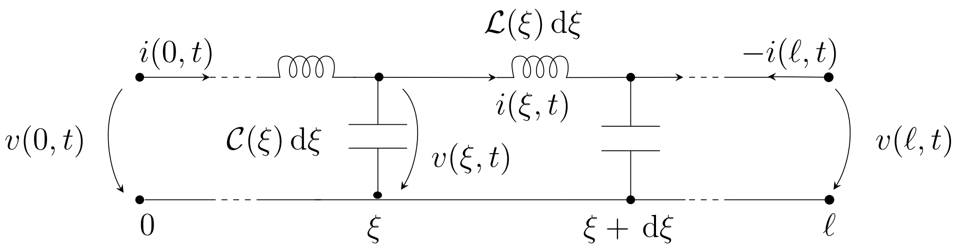

Figure 1. An ideal $\mathcal{LC}$-transmission line of length ℓ with *distributed* inductance $\mathcal{L}$ and capacitance $\mathcal{C}$. Here $i(\xi,t)$ and $v(\xi,t)$ denote the current and the voltage, respectively, at the point $\xi \in [0,\ell]$ at time $t \in \mathbb{R}^+$.

The natural state at time t of this transmission line is the current-voltage vector $x(t) = \left[\begin{smallmatrix} i(\cdot,t) \\ v(\cdot,t)\end{smallmatrix}\right]$, $t \in \mathbb{R}^+$, and the initial state is $x(0) = \left[\begin{smallmatrix} i(\cdot,0) \\ v(\cdot,0)\end{smallmatrix}\right] = \left[\begin{smallmatrix} i_0(\cdot) \\ v_0(\cdot)\end{smallmatrix}\right] =: x_0$. We take the state space $\mathcal{X}$ to be $L^2\big([0,\ell];\mathbb{C}^2\big)$ with inner product $(\cdot,\cdot)_{\mathcal{X}}$ defined by

$$
\left(\left[\begin{smallmatrix} i_1(\cdot) \\ v_1(\cdot)\end{smallmatrix}\right], \left[\begin{smallmatrix} i_2(\cdot) \\ v_2(\cdot)\end{smallmatrix}\right]\right)_{\mathcal{X}} = \int_0^\ell \big(\mathcal{L}(\xi) i_1(\xi)\overline{i_2(\xi)} + \mathcal{C}(\xi) v_1(\xi)\overline{v_2(\xi)}\big)\,\mathrm{d}\xi.
\tag{4.2}
$$

In our setting the corresponding quadratic form $(x(t),x(t))_{\mathcal{X}}$ is equivalent to the standard inner product on $L^2([0,\ell];\mathbb{C}^2)$ and its value is two times the energy stored in the state $x(t)$ of the transmission line at time t.

The operator L is given by

$$
L := \begin{bmatrix} 0 & -\frac{1}{\mathcal{L}(\xi)}\frac{\partial}{\partial \xi} \\ -\frac{1}{\mathcal{C}(\xi)}\frac{\partial}{\partial \xi} & 0 \end{bmatrix}, \quad \operatorname{dom}(L) := W^{1,2}([0,\ell];\mathbb{C}^2),
$$

where $W^{1,2}([0,\ell];\mathbb{C}^2)$ is the Sobolev space of absolutely continuous functions in $L^2([0,\ell];\mathbb{C}^2)$ which have a distribution derivative in $L^2([0,\ell];\mathbb{C}^2)$. The signal space $\mathcal{W}$ is $\mathbb{C}^4$ equipped with the indefinite inner product

(4.3)
$$\left[\begin{bmatrix} i_{01}\\ v_{01}\\ i_{\ell 1}\\ v_{\ell 1}\end{bmatrix},\begin{bmatrix} i_{02}\\ v_{02}\\ i_{\ell 2}\\ v_{\ell 2}\end{bmatrix}\right]_{\mathcal{W}} = \left(\begin{bmatrix} i_{01}\\ v_{01}\\ i_{\ell 1}\\ v_{\ell 1}\end{bmatrix}, J_{\mathcal{W}}\begin{bmatrix} i_{02}\\ v_{02}\\ i_{\ell 2}\\ v_{\ell 2}\end{bmatrix}\right)_{\mathbb{C}^4}, \quad J_{\mathcal{W}} = \begin{bmatrix} \left[\begin{smallmatrix} 0 & 1\\ 1 & 0\end{smallmatrix}\right] & 0\\ 0 & \left[\begin{smallmatrix} 0 & 1\\ 1 & 0\end{smallmatrix}\right]\end{bmatrix}.$$

The boundary operator Γ has the same domain as L, and it is given by

$$\Gamma\begin{bmatrix} i(\cdot)\\ v(\cdot)\end{bmatrix} = \begin{bmatrix} i(0)\\ v(0)\\ -i(\ell)\\ v(\ell)\end{bmatrix}.$$

The operator $\left[\begin{smallmatrix} L\\ \Gamma\end{smallmatrix}\right]$ is closed as an operator from $\mathcal{X}$ to $\left[\begin{smallmatrix} \mathcal{X}\\ \mathcal{W}\end{smallmatrix}\right]$ with domain $\mathrm{dom}\,(\left[\begin{smallmatrix} L\\ \Gamma\end{smallmatrix}\right]) = \mathrm{dom}\,(L) = W^{1,2}([0,\ell];\mathbb{C}^2)$. With these definitions, the transmission line can be modeled as an example of a boundary control s/s system in the sense of Definition 3, as we now show.

We next derive the appropriate Lagrangian identity. Combining $x(t) = \begin{bmatrix} i(\cdot,t)\\ v(\cdot,t)\end{bmatrix}$, (4.1), and (4.2), we make the following computations for $t > 0$:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|^2_{\mathcal{X}} &= 2\mathrm{Re}\,(x(t),\dot{x}(t))_{\mathcal{X}}\\
&= 2\mathrm{Re}\int_0^\ell \left(\mathcal{L}(\xi)i(\xi,t)\overline{\frac{\partial}{\partial t}i(\xi,t)} + \mathcal{C}(\xi)v(\xi,t)\overline{\frac{\partial}{\partial t}v(\xi,t)}\right)\mathrm{d}\xi\\
&= -2\int_0^\ell \mathrm{Re}\left(i(\xi,t)\overline{\frac{\partial}{\partial \xi}v(\xi,t)} + \overline{\frac{\partial}{\partial \xi}i(\xi,t)}v(\xi,t)\right)\mathrm{d}\xi\\
&= -2\int_0^\ell \mathrm{Re}\,\frac{\partial}{\partial \xi}\left(i(\xi,t)\overline{v(\xi,t)}\right)\mathrm{d}\xi\\
&= -2\mathrm{Re}\left[i(\xi,t)\overline{v(\xi,t)}\right]_{\xi=0}^{\ell}\\
&= 2\mathrm{Re}\,i(0,t)\overline{v(0,t)} - 2\mathrm{Re}\,i(\ell,t)\overline{v(\ell,t)}\\
&= \left(\begin{bmatrix} i(0,t)\\ v(0,t)\\ -i(\ell,t)\\ v(\ell,t)\end{bmatrix}, \begin{bmatrix} \left[\begin{smallmatrix} 0 & 1\\ 1 & 0\end{smallmatrix}\right] & 0\\ 0 & \left[\begin{smallmatrix} 0 & 1\\ 1 & 0\end{smallmatrix}\right]\end{bmatrix}\begin{bmatrix} i(0,t)\\ v(0,t)\\ -i(\ell,t)\\ v(\ell,t)\end{bmatrix}\right)_{\mathbb{C}^4}\\
&= [\Gamma x(t), \Gamma x(t)]_{\mathcal{W}},
\end{aligned}$$

where we have used that ($'$ denotes spatial derivative)

$$2\mathrm{Re}\,(i\overline{v'} + \overline{i'}v) = i\overline{v'} + \overline{i'}v + \overline{i}v' + i'\overline{v} = 2\mathrm{Re}\,(i\overline{v})'$$

in the fourth equality. Thus, $[w(t),w(t)]_{\mathcal{W}} = [\Gamma x(t), \Gamma x(t)]_{\mathcal{W}}$ is two times the power entering the transmission line through the terminals at the ends $\xi = 0$ and $\xi = \ell$ of the line at time $t \geq 0$.

These computations tell us that the generating subspace V is a neutral subspace of the node space $\mathfrak{K}$, i.e., that (3.7) holds. It is not difficult to show that this subspace is not only neutral, but even Lagrangian, so that (3.8) also holds; see Example 11 below for the proof idea. Thus, *the transmission line gives rise to a conservative boundary control s/s system.*

Remark 7. Set $\mathcal{U} := \mathbb{C}^2$, $R := iL|_{\ker\Gamma}$, and

$$\Gamma_0 \begin{bmatrix} i(\cdot) \\ v(\cdot) \end{bmatrix} := \begin{bmatrix} i(0) \\ -i(\ell) \end{bmatrix} \quad \text{and} \quad \Gamma_1 \begin{bmatrix} i(\cdot) \\ v(\cdot) \end{bmatrix} := \begin{bmatrix} v(0) \\ v(\ell) \end{bmatrix}. \tag{4.4}$$

Then R is a closed, densely defined and symmetric operator in the Hilbert space $\mathcal{X}$, and the triple $(\Gamma_0, -i\Gamma_1; \mathcal{U})$ is a *boundary triplet* for $R^* = -iL$ in the standard sense; see below. The boundary triplet and its connection to boundary-control state/signal systems is the topic of the last section of this chapter.

Recall that $[w(t), w(t)]_{\mathcal{W}}$ is two times the power entering the transmission line through the terminals at the ends $\xi = 0$ and $\xi = \ell$ of the line at time $t \geq 0$. The decomposition in (4.4) of Γ into an input map Γ_0 and an output map Γ_1 corresponds to choosing the current entering the system at $\xi = 0$ and $\xi = \ell$ as input and the voltages at both ends as output, cf. (2.1). We refer to this as an *impedance decomposition* of the external signal w.

Several other choices of input and output would have been possible, such as for example

$$\begin{aligned} \widetilde{\Gamma}_0 \begin{bmatrix} i(\cdot) \\ v(\cdot) \end{bmatrix} &:= \frac{1}{\sqrt{2}}(\Gamma_1 + \Gamma_0) \begin{bmatrix} i(\cdot) \\ v(\cdot) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} v(0)+i(0) \\ v(\ell)-i(\ell) \end{bmatrix} \quad \text{and} \\ \widetilde{\Gamma}_1 \begin{bmatrix} i(\cdot) \\ v(\cdot) \end{bmatrix} &:= \frac{1}{\sqrt{2}}(\Gamma_1 - \Gamma_0) \begin{bmatrix} i(\cdot) \\ v(\cdot) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} v(0)-i(0) \\ v(\ell)+i(\ell) \end{bmatrix}, \quad \text{or} \end{aligned} \tag{4.5}$$

$$\widehat{\Gamma}_0 \begin{bmatrix} i(\cdot) \\ v(\cdot) \end{bmatrix} := \begin{bmatrix} i(0) \\ v(0) \end{bmatrix} \quad \text{and} \quad \widehat{\Gamma}_1 \begin{bmatrix} i(\cdot) \\ v(\cdot) \end{bmatrix} := \begin{bmatrix} -i(\ell) \\ v(\ell) \end{bmatrix}. \tag{4.6}$$

In (4.5) we have

$$\left\| \widetilde{\Gamma}_0 \begin{bmatrix} i(\cdot) \\ v(\cdot) \end{bmatrix} \right\|_{\mathbb{C}^2}^2 - \left\| \widetilde{\Gamma}_1 \begin{bmatrix} i(\cdot) \\ v(\cdot) \end{bmatrix} \right\|_{\mathbb{C}^2}^2 = \left[\Gamma \begin{bmatrix} i(\cdot) \\ v(\cdot) \end{bmatrix}, \Gamma \begin{bmatrix} i(\cdot) \\ v(\cdot) \end{bmatrix} \right]_{\mathcal{W}},$$

where $[\cdot,\cdot]_{\mathcal{W}}$ still denotes the inner product (4.3). This decomposition is an example of a *scattering decomposition.* In (4.6) we choose voltage and current at $\xi = 0$ as input and the voltage and current at $\xi = \ell$ as output, and this in an example of a *transmission decomposition.*

Remark 8. Making a different choice of input and output signals results in a different map from the input to the output, i.e., a different input/state/output representation, with possibly widely different properties. However, the physical system, i.e., the $\mathcal{LC}$-transmission line with length ℓ, is still the same. This "input/output-free" paradigm is inherent in the state/signal philosophy.

5. The connection to boundary triplets

Boundary triplets originate from the extension theory of symmetrical operators on Hilbert spaces. The following definition is adapted from [Gorbachuk and Gorbachuk, 1991, pp. 154–155], using the more recent terminology and notations from [Derkach et al., 2006, Def. 5.1].

Definition 9. Let R be a closed densely defined symmetric operator on the Hilbert space $\mathcal{X}$ with equal (finite or infinite) defect numbers $n_\pm := \dim\ker(R \mp i)$. Let $\mathcal{U}$ be another Hilbert space, the "external Hilbert space", and let Γ_j, $j = 0, 1$, be linear operators mapping $\operatorname{dom}(R^*)$ into $\mathcal{U}$.

The triplet $(\Gamma_0, \Gamma_1; \mathcal{U})$ is called a *boundary triplet* for the operator R^* if the following two conditions hold:

(1) For all $x_1, x_2 \in \operatorname{dom}(R^*)$ we have

$$(R^* x_1, x_2)_{\mathcal{X}} - (x_1, R^* x_2)_{\mathcal{X}} = (\Gamma_0 x_1, \Gamma_1 x_2)_{\mathcal{U}} - (\Gamma_1 x_1, \Gamma_0 x_2)_{\mathcal{U}}.$$

(2) The range of the combined operator $\Gamma := \left[\begin{smallmatrix}\Gamma_0\\ \Gamma_1\end{smallmatrix}\right]$ is $\left[\begin{smallmatrix}\mathcal{U}\\ \mathcal{U}\end{smallmatrix}\right]$.

Here condition (1) is the *Lagrangian identity* and condition (2) can be interpreted as a *regularity condition* or a *(hyper)maximality condition.*

By a *direct-sum decomposition* $\mathcal{W} = \mathcal{U} \dot{+} \mathcal{Y}$ of a Kreĭn space we mean that $\mathcal{U}$ and $\mathcal{Y}$ are closed subspaces of $\mathcal{W}$, such that $\mathcal{U} + \mathcal{Y} = \mathcal{W}$ and $\mathcal{U} \cap \mathcal{Y} = \{0\}$. This decomposition is *Lagrangian* if $\mathcal{U}$ and $\mathcal{Y}$ are both Lagrangian subspaces: $\mathcal{U} = \mathcal{U}^{[\perp]}$ and $\mathcal{Y} = \mathcal{Y}^{[\perp]}$. For every Hilbert space $\mathcal{U}$, the direct-sum decomposition

$$\mathcal{W} = \widetilde{\mathcal{U}} \dot{+} \widetilde{\mathcal{Y}} := \left[\begin{smallmatrix}\mathcal{U}\\ \{0\}\end{smallmatrix}\right] \dot{+} \left[\begin{smallmatrix}\{0\}\\ \mathcal{U}\end{smallmatrix}\right] \tag{5.1}$$

of $\mathcal{W} = \mathcal{U}^2$ is *Lagrangian* if $\mathcal{W}$ has the inner product

$$[[\left[\begin{smallmatrix}u_1\\ y_1\end{smallmatrix}\right], \left[\begin{smallmatrix}u_2\\ y_2\end{smallmatrix}\right]]]_{\mathcal{W}} = (u_1, y_2)_{\mathcal{U}} + (y_1, u_2)_{\mathcal{U}}\,. \tag{5.2}$$

For instance, the impedance decomposition in the transmission line example, where we take the currents as input and voltages as outputs, is a Lagrangian decomposition.

For a proof of the following result, see [Malinen and Staffans, 2007, Sec. 5]:

Theorem 10. *Let R be a closed and densely defined symmetric operator on $\mathcal{X}$ with equal defect numbers, and let $(\Gamma_0, \Gamma_1; \mathcal{U})$ be a boundary triplet for R^*. Take $\mathcal{W} := \left[\begin{smallmatrix}\mathcal{U}\\ \mathcal{U}\end{smallmatrix}\right]$ with the indefinite inner product* (5.2) *and define $\Gamma := \left[\begin{smallmatrix}\Gamma_0\\ i\Gamma_1\end{smallmatrix}\right]$ with* $\operatorname{dom}(\Gamma) = \operatorname{dom}(R^*)$.

Then $\Sigma = (iR^*, \Gamma; \mathcal{X}, \mathcal{W})$ is a boundary control s/s system *in the sense of Definition 3. The system is moreover* conservative*: $V = V^{[\perp]}$, where V is given by* (2.3).

Consider the conservative boundary control s/s system Σ in Theorem 10. The *input/state/output representation*

$$\Sigma_{i/s/o} = \left(iR^*, \begin{bmatrix}\Gamma_0\\ \{0\}\end{bmatrix}, \begin{bmatrix}\{0\}\\ i\Gamma_1\end{bmatrix}; \mathcal{X}, \left[\begin{smallmatrix}\mathcal{U}\\ \{0\}\end{smallmatrix}\right], \left[\begin{smallmatrix}\{0\}\\ \mathcal{U}\end{smallmatrix}\right]\right)$$

corresponding to the Lagrangian decomposition (5.1) is an example of an *impedance representation* of Σ. We investigate these concepts in more detail in Section ?? of Chapter ??.

The converse of Theorem 10 is not true: *there do exist conservative boundary control s/s systems which are not induced by any boundary triplet of the type in Definition 9.* These examples are of two types:

(1) The signal space $\mathcal{W}$ *need not have a Lagrangian decomposition.* A necessary and sufficient condition for the existence of a Lagrangian decomposition is that $\mathrm{ind}_+\mathcal{W} = \mathrm{ind}_-\mathcal{W}$ $(\leq \infty)$; see Example 11 below. In the case of a boundary triplet we always have at least the Lagrangian decomposition (5.1).

(2) Even if the signal space $\mathcal{W}$ has a Lagrangian decomposition *the main operator L need not be closed*, and we can thus not have $L = iR^*$. Moreover, *the operator* $\Gamma := \left[\begin{smallmatrix}\Gamma_0\\ \Gamma_1\end{smallmatrix}\right]$ *need not be surjective*. See [Malinen and Staffans, 2007] for an example.

More precisely, let $\Sigma = (L, \Gamma; \mathcal{X}, \mathcal{W})$ be a conservative boundary control s/s system. According to [Kurula et al., 2010, Prop. 4.5], *L is closed if and only if the range of Γ is closed.* Combining this with the condition that Γ has dense range, we obtain that L is closed if and only if Γ is surjective. The same conclusion can be made based on Prop. 2.3 and Cor. 2.4 of [Derkach et al., 2006].

We now give an example of a conservative boundary control s/s system that is not induced by a boundary triplet. In a scattering setting this system has no input and a one-dimensional output, and the C_0-semigroup describing the system dynamics is the left shift in $L^2(\mathbb{R}^+;\mathbb{C})$.

Example 11. Choose $\mathcal{X} := L^2(\mathbb{R}^+;\mathbb{C})$ with its standard Hilbert-space inner product, set $\mathcal{W} := -\mathbb{C}$, and define

$$V := \left\{ \begin{bmatrix} \frac{\mathrm{d}x}{\mathrm{d}\xi} \\ x \\ x(0) \end{bmatrix} \middle| \, x \in W^{1,2}(\mathbb{R}^+;\mathbb{C}) \right\} \subset \mathcal{X} \times \mathcal{X} \times \mathcal{W}.$$

It is clear that $\left[\begin{smallmatrix} z \\ 0 \\ 0 \end{smallmatrix}\right] \in V$ implies that $z = 0$, and we will now show that $V = V^{[\perp]}$, i.e., that $(V; \mathcal{X}, \mathcal{W})$ is a conservative boundary control s/s system. Note that the signal space $\mathcal{W}$ has no Lagrangian decompositions.

We first prove that $V^{[\perp]} \subset V$. By definition $\left[\begin{smallmatrix}\widetilde{z}\\ \widetilde{x}\\ \widetilde{w}\end{smallmatrix}\right] \in V^{[\perp]}$ if and only if $\left[\begin{smallmatrix}\widetilde{z}\\ \widetilde{x}\\ \widetilde{w}\end{smallmatrix}\right] \in \mathfrak{K} = L^2(\mathbb{R}^+;\mathbb{C}) \times L^2(\mathbb{R}^+;\mathbb{C}) \times \mathbb{C}$ and for all $x \in W^{1,2}(\mathbb{R}^+;\mathbb{C})$:

$$\left[\!\left[\begin{bmatrix} \widetilde{z} \\ \widetilde{x} \\ \widetilde{w} \end{bmatrix}, \begin{bmatrix} \frac{\mathrm{d}x}{\mathrm{d}\xi} \\ x \\ x(0) \end{bmatrix} \right]\!\right]_{\mathfrak{K}} = -\widetilde{w}\,\overline{x(0)} - \int_0^\infty \left(\widetilde{x}(\xi)\overline{\frac{\mathrm{d}x}{\mathrm{d}\xi}(\xi)} + \widetilde{z}(\xi)\overline{x(\xi)} \right) \mathrm{d}\xi = 0. \tag{5.3}$$

In particular, if we let x vary over the set of test functions in C^∞ with support contained in $(0,\infty)$, then $x(0) = 0$, and we find that $\frac{\mathrm{d}\widetilde{x}}{\mathrm{d}\xi} = \widetilde{z}$ in the distribution sense. Since both $\widetilde{x}$ and $\widetilde{z}$ belong to $L^2(\mathbb{R}^+;\mathbb{C})$, this implies that $\widetilde{x} \in W^{1,2}(\mathbb{R}^+;\mathbb{C})$. This makes it is possible to integrate by parts in (5.3), using that $\widetilde{z}(\xi) = \frac{\mathrm{d}\widetilde{x}}{\mathrm{d}\xi}(\xi)$, in order to get that

$$\widetilde{w}\,\overline{x(0)} = \widetilde{x}(0)\,\overline{x(0)}, \quad x \in W^{1,2}(\mathbb{R}^+;\mathbb{C}).$$

Thus $\widetilde{w} = \widetilde{x}(0)$, and this proves that $V^{[\perp]} \subset V$.

In order to show that $V \subset V^{[\perp]}$, we choose $\widetilde{x} \in W^{1,2}(\mathbb{R}^+;\mathbb{C})$ arbitrarily, and we set $\widetilde{z} := \frac{\mathrm{d}\widetilde{x}}{\mathrm{d}\xi}$ and $\widetilde{w} := \widetilde{x}(0)$. Then (5.3) holds for all $x, \widetilde{x} \in W^{1,2}(\mathbb{R}^+;\mathbb{C})$, i.e., $V \subset V^{[\perp]}$. We are done proving that $V = V^{[\perp]}$, and therefore, that $(V;\mathcal{X},\mathcal{W})$ is a conservative boundary control s/s system whose signal space $\mathcal{W} = -\mathbb{C}$ has no Lagrangian decompositions.

The i/s/o case where $\Gamma = \left[\begin{smallmatrix}\Gamma_0\\ \Gamma_1\end{smallmatrix}\right] : \mathcal{X} \to \mathcal{U}^2$ has dense but non-closed range has been treated using *generalized boundary triplets* in [Derkach and Malamud, 1995] and using *quasi boundary triplets* in [Behrndt and Langer, 2007]. Interconnection of conservative boundary control i/s/o systems with surjective $\left[\begin{smallmatrix}\Gamma_0\\ \Gamma_1\end{smallmatrix}\right]$ was worked out in detail in [Kurula et al., 2010].

A considerably more general notion than that of a boundary triplet is that of a *boundary relation* which was extensively studied in e.g. [Derkach et al., 2006]. The topic of Chapter ??, which is more detailed than he present one, is to show how boundary relations are connected to general (non-boundary control) s/s systems. There the main point is to show that the notion of a boundary relation is connected to the notion of a conservative state/signal system in the same way as the boundary triplet is related to the boundary control s/s system: the former arises as a particular i/s/o impedance representation of the latter.

References

[Behrndt and Langer, 2007] Behrndt, J., and Langer, M. 2007. Boundary value problems for elliptic partial differential operators on bounded domains. *J. Funct. Anal.*, **243**, 536–565.

[Derkach and Malamud, 1995] Derkach, V. A., and Malamud, M. M. 1995. The extension theory of Hermitian operators and the moment problem. *J. Math. Sci.*, **73**, 141–242. Analysis. 3.

[Derkach et al., 2006] Derkach, V. A., Hassi, S., Malamud, M. M., and de Snoo, H. S. V. 2006. Boundary relations and their Weyl families. *Trans. Amer. Math. Soc.*, **358**, 5351–5400 (electronic).

[Gorbachuk and Gorbachuk, 1991] Gorbachuk, V. I., and Gorbachuk, M. L. 1991. *Boundary value problems for operator differential equations*. Mathematics and its Applications (Soviet Series), vol. 48. Dordrecht: Kluwer Academic Publishers Group. translation and revised from the 1984 Russian original.

[Kurula et al., 2010] Kurula, M. M., Zwart, H. J., van der Schaft, A. J., and Behrndt, J. 2010. Dirac structures and their composition on Hilbert spaces. *J. Math. Anal. Appl.*, **372**, 402–422.

[Malinen and Staffans, 2006] Malinen, J., and Staffans, O. J. 2006. Conservative Boundary Control Systems. *J. Differential Equations*, **231**, 290–312.

[Malinen and Staffans, 2007] Malinen, J., and Staffans, O. J. 2007. Impedance passive and conservative boundary control systems. *Complex Anal. Oper. Theory*, **1**, 279–30.

[Salamon, 1987] Salamon, D. 1987. Infinite dimensional linear systems with unbounded control and observation: a functional analytic approach. *Trans. Amer. Math. Soc.*, **300**, 383–431.

[Staffans, 2005] Staffans, O. J. 2005. *Well-Posed Linear Systems*. Cambridge and New York: Cambridge University Press.

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