Dirac structures and their composition on Hilbert spaces

Mikael Kurula*

Åbo Akademi University, Department of Mathematics,
Fänriksgatan 3, FIN-20500 Åbo, Finland

Hans Zwart

University of Twente, Department of Applied Mathematics,
P.O. Box 217, 7500 AE Enschede, The Netherlands

Arjan van der Schaft

University of Groningen, Department of Mathematics and Computer science,
P.O. Box 800, 9700 AV Groningen, The Netherlands

Jussi Behrndt

Technische Universität Berlin, Institut für Mathematik, MA 6-4,
Straße des 17. Juni 136, D-10623 Berlin, Germany

Abstract

Dirac structures appear naturally in the study of certain classes of physical models described by partial differential equations and they can be regarded as the underlying power conserving structures. We study these structures and their properties from an operator-theoretic point of view. In particular, we find necessary and sufficient conditions for the composition of two Dirac structures to be a Dirac structure and we show that they can be seen as Lagrangian (hyper-maximal neutral) subspaces of Kreĭn spaces. Moreover, special emphasis is laid on Dirac structures associated with operator colligations. It turns out that this class of Dirac structures is linked to boundary triplets and that this class is closed under composition.

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*Corresponding author, tel.: +358-50-570 2615, fax.: +358-2-215 4865

Email addresses: mkurula@abo.fi (Mikael Kurula), h.j.zwart@math.utwente.nl (Hans Zwart), a.j.van.der.schaft@math.rug.nl (Arjan van der Schaft), behrndt@math.tu-berlin.de (Jussi Behrndt)

1. Introduction

Consider the following simple partial differential equation (p.d.e.) on the spatial domain $(-\infty, \infty)$:

$$\frac{\partial}{\partial t}x(z,t) = \frac{\partial}{\partial z}\left(\ell(z)x(z,t)\right), \quad z \in (-\infty, \infty),\ t \geq 0. \tag{1.1}$$

This p.d.e. is an example of a *conservation law* (a notion which can be directly extended to non-linear p.d.e.'s, see e.g. [12]). In particular, assuming that ℓx is zero at $z = -\infty$ and $z = \infty$, it is easy to see that $E(t) = \frac{1}{2}\int_{-\infty}^{\infty}\ell(z)x(z,t)^2\,\mathrm{d}z$ is a conserved quantity, that is $\frac{dE}{dt} = 0$. Hence, without knowing ℓ and without knowing existence of a solution of (1.1), we have a conserved quantity. This implies that the existence of a conserved quantity is underlying the partial differential equation. Another way of looking at this is by fixing t and replacing $\frac{\partial}{\partial t}x(z,t)$ by $f(z)$ and $\ell(z)x(z,t)$ by $e(z)$. Hence instead of the partial differential equation (1.1) we then have

$$f(z) = \frac{\partial e}{\partial z}(z), \quad z \in (-\infty, \infty). \tag{1.2}$$

Under the assumption that e is zero in $z = \infty$ and $z = -\infty$, we have that $\int_{-\infty}^{\infty}f(z)e(z)\,\mathrm{d}z = 0$. If $E(t) = \frac{1}{2}\int_{-\infty}^{\infty}\ell(z)x(z,t)^2\,\mathrm{d}z$ can be interpreted as total *energy* of the system (as is the case for many physical systems), then this last equality amounts to the fact that the total *power* is zero. Indeed, since the change of the total energy per unit of time equals the total power, the total energy is conserved if and only if the total power is zero. The power is a bi-linear product of two variables, called the effort and the flow, e and f, respectively.

In many cases of physical interest the spatial domain will have, contrary to the above, a *boundary*, and there will be an energy flow through this boundary. As an example, consider (1.1) on the spatial domain $[0,1]$ with boundary $\{0,1\}$

$$\frac{\partial}{\partial t}x(z,t) = \frac{\partial}{\partial z}\left(\ell(z)x(z,t)\right), \quad z \in [0,1]. \tag{1.3}$$

Defining analogously the internal energy as $E(t) = \frac{1}{2} \int_0^1 \ell(z) x(z,t)^2 \, \mathrm{d}z$, we now find that

$$\frac{d}{dt} E(t) = \frac{1}{2} \left[\ell(z)^2 x(z,t)^2 \right]_0^1. \tag{1.4}$$

So we have to take the energy flow $[\ell(z)^2 x(z,t)^2]_0^1$ through the boundary into account. However, the underlying structure remains very similar to what we have described above; one just defines extra efforts and flow variables, see [13], [19], [27] or [28]. In fact, since we want the product of these extra variables to equal minus the right-hand side of (1.4), a possible choice is

$$f_\partial = \left(-e(1) + e(0) \right)/\sqrt{2}, \quad e_\partial = \left(e(1) + e(0) \right)/\sqrt{2}, \tag{1.5}$$

with $e(z) = \ell(z) x(z,t)$.

The equation (1.2) defines a linear subspace in the effort variable e and flow variable f with the property that for any pair (f, e) in this subspace, the total power $\frac{1}{2} \int_{-\infty}^{\infty} f(z) e(z) \, \mathrm{d}z$ is zero. Spaces with this property are called Dirac structures, see Definition 2.1 for the precise definition. Hence the Dirac structure associated with (1.2) is

$$\left\{ (f, e) \in L^2(\mathbb{R}) \times L^2(\mathbb{R}) \mid e \text{ absolutely continuous, and } f = \frac{\partial}{\partial z} e \right\}.$$

Using (1.3), $e(z) = \ell(z) x(z,t)$, and (1.5), equation (1.4) gives the total power

$$\int_0^1 f(z) e(z) \, \mathrm{d}z + f_\partial e_\partial = 0. \tag{1.6}$$

Thus we can associate to (1.3) and (1.5) the Dirac structure

$$\left\{ (f, e, f_\partial, e_\partial) \mid f, e \in L^2(0,1), e \text{ absolutely continuous, and } f = \frac{\partial}{\partial z} e, \right.$$

$$\left. f_\partial = \left(-e(1) + e(0) \right)/\sqrt{2}, e_\partial = \left(e(1) + e(0) \right)/\sqrt{2} \right\}. \tag{1.7}$$

The above ideas can be used to define Dirac structures on general spaces, see [13], [19], [27], and [28].

The extension to higher-dimensional spatial domains is immediate, see [27]. For example, consider the differential operator associated with the wave equation on a two-dimensional domain. Let Ω be a two-dimensional bounded domain with smooth boundary Γ, and let $H(\mathrm{div}, \Omega) = \{ e \in L^2(\Omega)^2 \mid$

div$(e) \in L^2(\Omega)\}$. By η we denote the outward normal, and by $\cdot$ we denote the standard scalar product in $\mathbb{R}^2$. Consider the subspace

$$\{(f_1, f_2, e_1, e_2, f_\partial, e_\partial) \mid \quad e_1 \in H^1(\Omega), e_2 \in H(\text{div}, \Omega), \tag{1.8}$$
$$f_1 = \text{div}(e_2), f_2 = \text{grad}(e_1), f_\partial \in H^{\frac{1}{2}}(\Gamma),$$
$$e_\partial \in H^{-\frac{1}{2}}(\Gamma), f_\partial = e_1|_\Gamma, e_\partial = \eta \cdot e_2|_\Gamma \Big\}.$$

By Green's identity we have that every element in this subspace satisfies

$$\int_\Omega f_1(z) e_1(z) + f_2(z) \cdot e_2(z) \, \mathrm{d}z - \int_\Gamma f_\partial(\gamma) e_\partial(\gamma) \, \mathrm{d}\gamma = 0. \tag{1.9}$$

Moreover, the subspace (1.8) is a Dirac structure with respect to this balance equation, see Theorem 4.7, Remark 4.4.5 and [20].

Dirac structures are the key to the definition of port-Hamiltonian systems. These are systems which may exchange power with its surrounding via its ports, and have an internal energy function, the Hamiltonian, see [6], [27] or [26]. The notion of infinite-dimensional Dirac structures has been developed before in the study of non-linear partial differential equations on an infinite spatial domain, see in particular [11]. In the examples above the ports are at the boundary of the spatial domain.

Given two, or more, port-Hamiltonian systems, it is natural to connect them to each other, through their ports. For instance, consider a transmission line connected on each side to an electrical device, a multi-body system where some of the masses are connected to each other via flexible beams, or a coupled network of transmission lines. We illustrate this on the physical example of an ideal transmission line, described by the telegrapher's equations.

Consider three transmission lines, $i = 1, 2, 3$, each described by the telegrapher's equations

$$\frac{\partial}{\partial t} Q_i(z, t) = -\frac{\partial}{\partial z}\left(\frac{1}{L_i(z)}\phi_i(z, t)\right)$$
$$\frac{\partial}{\partial t}\phi_i(z, t) = -\frac{\partial}{\partial z}\left(\frac{1}{C_i(z)}Q_i(z, t)\right), \quad z \in [a, b],$$

with $i = 1, 2, 3$, and $L_i(z)$, $C_i(z)$ denoting the distributed inductance and distributed capacitance of the transmission lines, respectively. In this case

the natural flow and effort variables at the boundary $\{a, b\}$ are the voltages $V_{a,i} = \frac{1}{C_i(a)}Q_i(a, t)$, $V_{b,i} = \frac{1}{C_i(b)}Q_i(b, t)$ and the currents $I_{a,i} = L_i(a)\phi_i(a, t)$, $I_{b,i} = L_i(b)\phi_i(b, t)$. We assume that the transmission lines are connected at $z = a$, by putting $V_{a,1} = V_{a,2} = V_{a,3}$ and $I_{a,1} + I_{a,2} + I_{a,3} = 0$.

The coupling of the p.d.e.'s gives naturally an interconnection (composition) of the corresponding Dirac structures. If the Dirac structures are finite-dimensional, then it is well known that the composed structure is again a Dirac structure, see [5], [6] or [25]. However, this result does not hold if all the Dirac structures are infinite-dimensional, see [13, Ex. 5.2.23] for a counterexample. In the above (infinite-dimensional) example it is not hard to show that the composition of the three underlying Dirac structure is again a Dirac structure. However, it is not clear whether this will hold for more complicated p.d.e.'s. Obviously, the problem of interconnecting multiple Dirac structures can be reduced without loss of generality to the problem of the interconnection of two Dirac structures.

Although the examples discussed so far are elementary (for expository reasons), our approach and results are applicable to many physical examples, also for spatial domains of dimension two or higher.

The aim of the present paper is to study Dirac structures and their composition from an operator-theoretic point of view, and the outline is the following. We first define Dirac structures and develop their scattering representations in a Kreĭn-space setting in Section 2. We present necessary and sufficient conditions for the composition of two Dirac structures to be a Dirac structure in terms of scattering representations, after we have introduced the necessary notions in Section 3. Furthermore, we investigate Dirac structures associated to operator colligations or boundary nodes in Section 4. Here we also find necessary and sufficient conditions for the entries in the colligation to induce a Dirac structure. It will also be shown that the composition of Dirac structures associated to strong colligations is again a Dirac structure associated to a strong boundary colligation in Section 5.

We mention that Dirac structures are closely connected to unitary operators and relations acting between Krein spaces, and hence also to the notion of boundary triplets and boundary relations from abstract extension theory of symmetric operators. From this point of view some of the results in Sections 4 and 5 can also be deduced from more general results obtained by Derkach, Hassi, Malamud and de Snoo in [7, 8]. For details see the explanations after Proposition 4.5.

It should also be mentioned that the work towards so-called state/signal systems in continuous time by Ball and Staffans in [2] and that of Kurula and Staffans in [16, 18] is very closely related to the work which we present in this article. The connection is made in [17]. The interconnection results in Section 3 in the present article are expected to be adaptable to interconnection of state/signal systems in discrete time, as developed by Arov and Staffans; see [24] for an overview.

2. Dirac structures, Kreĭn spaces and scattering representations

Let $\mathcal{E}$ and $\mathcal{F}$ be two Hilbert spaces, which we call *the space of efforts* and *the space of flows*, respectively. Assume that there exists a unitary operator $r_{\mathcal{E},\mathcal{F}}$ from $\mathcal{E}$ to $\mathcal{F}$.

By referring to "the Hilbert space $\mathcal{F} \oplus \mathcal{E}$" we mean the product space $\mathcal{F} \times \mathcal{E}$ equipped with the usual Hilbert-space inner product

$$\left\langle \begin{bmatrix} f_1 \\ e_1 \end{bmatrix}, \begin{bmatrix} f_2 \\ e_2 \end{bmatrix} \right\rangle_{\mathcal{F} \oplus \mathcal{E}} = \langle f_1, f_2 \rangle_{\mathcal{F}} + \langle e_1, e_2 \rangle_{\mathcal{E}}, \tag{2.1}$$

where $f_1, f_2 \in \mathcal{F}$, $e_1, e_2 \in \mathcal{E}$. In order to introduce the notions of Dirac and Tellegen structures we first define an indefinite inner product on $\mathcal{F} \times \mathcal{E}$ by

$$\left[\begin{bmatrix} f_1 \\ e_1 \end{bmatrix}, \begin{bmatrix} f_2 \\ e_2 \end{bmatrix} \right]_{\mathcal{B}} := \left\langle \begin{bmatrix} f_1 \\ e_1 \end{bmatrix}, \begin{bmatrix} 0 & r_{\mathcal{E},\mathcal{F}} \\ r_{\mathcal{E},\mathcal{F}}^* & 0 \end{bmatrix} \begin{bmatrix} f_2 \\ e_2 \end{bmatrix} \right\rangle_{\mathcal{F} \oplus \mathcal{E}} \tag{2.2}$$
$$= \langle f_1, r_{\mathcal{E},\mathcal{F}} e_2 \rangle_{\mathcal{F}} + \langle e_1, r_{\mathcal{E},\mathcal{F}}^* f_2 \rangle_{\mathcal{E}}.$$

By the *bond space* $\mathcal{B}$ we mean $\mathcal{F} \times \mathcal{E}$ equipped with the inner product $[\cdot, \cdot]_{\mathcal{B}}$.

In the context of Dirac structures it is common to use real-valued functions, and therefore it is natural to take $\mathcal{E}$ and $\mathcal{F}$ to have real fields. Our definitions and results, however, are equally valid for complex Hilbert spaces. A connection is made in [17, Lem. 4.1], and Example 3.9 below uses complex Dirac structures.

For a linear subspace $\mathcal{C} \subset \mathcal{B}$ the *orthogonal companion* $\mathcal{C}^{[\perp]}$ of $\mathcal{C}$ is defined by

$$\mathcal{C}^{[\perp]} := \left\{ b' \in \mathcal{B} \mid [b, b']_{\mathcal{B}} = 0 \text{ for all } b \in \mathcal{C} \right\}. \tag{2.3}$$

From (2.2) we see that for any linear subspace $\mathcal{C}$ of $\mathcal{B}$ we have that

$$\mathcal{C}^{[\perp]} = \begin{bmatrix} 0 & r_{\mathcal{E},\mathcal{F}} \\ r_{\mathcal{E},\mathcal{F}}^* & 0 \end{bmatrix} (\mathcal{C}^{\perp}),$$

where $\mathcal{C}^\perp$ denotes the orthogonal complement of $\mathcal{C}$ with respect to the scalar product (2.1). Hence any orthogonal companion will be closed, and $\mathcal{B}^{[\perp]} = \{0\}$. This last property is known as the non-degeneration of the bond space.

Definition 2.1. Let $\mathcal{E}$ and $\mathcal{F}$ be the spaces of efforts and flows, respectively, let $\mathcal{B}$ be the associated bond space and let $\mathcal{D}$ be a linear subspace of $\mathcal{B}$. Then $\mathcal{D}$ is called a *Tellegen structure* on $\mathcal{B}$ if $\mathcal{D} \subset \mathcal{D}^{[\perp]}$ and $\mathcal{D}$ is called a *Dirac structure* on $\mathcal{B}$ if $\mathcal{D} = \mathcal{D}^{[\perp]}$. We sometimes omit "on $\mathcal{B}$" if it is clear from the context what the bond space is.

Bond spaces can be viewed as Kreĭn spaces and Dirac structures as hyper-maximal neutral subspaces of these. Let us briefly recall some concepts from the theory of Kreĭn spaces and make this connection explicit. We refer the reader to the monographs [1] and [4] for more details.

Definition 2.2. Let $\mathcal{K}$ be a vector space and let $[\cdot, \cdot]_\mathcal{K}$ be an indefinite inner product on $\mathcal{K}$. Then $(\mathcal{K}, [\cdot, \cdot]_\mathcal{K})$ is said to be a *Kreĭn space* if $\mathcal{K}$ can be decomposed as

$$\mathcal{K} = \mathcal{K}_+ \, [\dot{+}] \, \mathcal{K}_-, \tag{2.4}$$

where $(\mathcal{K}_+, [\cdot, \cdot]_\mathcal{K})$ and $(\mathcal{K}_-, -[\cdot, \cdot]_\mathcal{K})$ are Hilbert spaces and $[\dot{+}]$ stands for the direct $[\cdot, \cdot]_\mathcal{K}$-orthogonal sum. A decomposition of the form (2.4) is called a *fundamental decomposition* of $\mathcal{K}$.

Let $(\mathcal{K}, [\cdot, \cdot]_\mathcal{K})$ be a Kreĭn space. Any fundamental decomposition (2.4) of $\mathcal{K}$ induces a positive definite inner product $\langle \cdot, \cdot \rangle_\mathcal{K}$ on $\mathcal{K}$ via

$$\langle h, k \rangle_\mathcal{K} := [h_+, k_+]_\mathcal{K} - [h_-, k_-]_\mathcal{K}, \quad h = h_+ + h_-, \, k = k_+ + k_-, \, h_\pm, k_\pm \in \mathcal{K}_\pm.$$

With this positive definite inner product $(\mathcal{K}, \langle \cdot, \cdot \rangle_\mathcal{K})$ becomes a Hilbert space. Let P_+ and P_- be the projections in $\mathcal{K}$ defined by $P_+ k := k_+$ and $P_- k := k_-$ for $k = k_+ + k_-$, $k_\pm \in \mathcal{K}_\pm$. The operator $J := P_+ - P_-$ is called *fundamental symmetry* of $\mathcal{K}$ corresponding to the fundamental decomposition (2.4). It is not difficult to see that $J^2 = I$ and $J = J^* = J^{-1}$ holds. Here $*$ denotes the adjoint with respect to the scalar product $\langle \cdot, \cdot \rangle_\mathcal{K}$. Furthermore, the Kreĭn space inner product $[\cdot, \cdot]_\mathcal{K}$ and the Hilbert space inner product $\langle \cdot, \cdot \rangle_\mathcal{K}$ on $\mathcal{K}$ are related by

$$[h, k]_\mathcal{K} = \langle Jh, k \rangle_\mathcal{K} \quad \text{and} \quad \langle h, k \rangle_\mathcal{K} = [Jh, k]_\mathcal{K}, \quad h, k \in \mathcal{K}. \tag{2.5}$$

The orthogonal companion of a subspace $\mathcal{H}$ in the Kreĭn space $(\mathcal{K}, [\cdot, \cdot]_{\mathcal{K}})$ is defined to be the space of all vectors in $\mathcal{K}$ that are $[\cdot, \cdot]_{\mathcal{K}}$-orthogonal to every vector in $\mathcal{H}$ as in (2.3). A linear subspace $\mathcal{H} \subset \mathcal{K}$ is said to be *neutral* if $\mathcal{H} \subset \mathcal{H}^{[\perp]}$ and $\mathcal{H}$ is said to be *Lagrangian*, or *hyper-maximal neutral*, if $\mathcal{H} = \mathcal{H}^{[\perp]}$.

The statements in the following two propositions are now immediate translations of the notions of bond space, Tellegen and Dirac structure into the language of Kreĭn space theory.

Proposition 2.3. Let $\mathcal{B} = \mathcal{F} \times \mathcal{E}$ be the bond space equipped with the power product $[\cdot, \cdot]_{\mathcal{B}}$ from (2.2). Then $(\mathcal{B}, [\cdot, \cdot]_{\mathcal{B}})$ is a Kreĭn space and

$$\mathcal{B} = \mathcal{B}_+ \, [\dot{+}] \, \mathcal{B}_-, \quad \text{where} \quad \mathcal{B}_\pm = \begin{bmatrix} \pm r_{\mathcal{E},\mathcal{F}} \\ I \end{bmatrix} \mathcal{E}, \tag{2.6}$$

is a fundamental decomposition of $\mathcal{B}$ with $(\mathcal{B}_+, [\cdot, \cdot]_{\mathcal{B}})$ and $(\mathcal{B}_-, -[\cdot, \cdot]_{\mathcal{B}})$ Hilbert spaces. The corresponding fundamental symmetry is $J = \begin{bmatrix} 0 & r_{\mathcal{E},\mathcal{F}} \\ r_{\mathcal{E},\mathcal{F}}^* & 0 \end{bmatrix}$ and the projections onto $\mathcal{B}_+$ and $\mathcal{B}_-$ are given by

$$P_+ = \frac{1}{2} \begin{bmatrix} I_{\mathcal{F}} & r_{\mathcal{E},\mathcal{F}} \\ r_{\mathcal{E},\mathcal{F}}^* & I_{\mathcal{E}} \end{bmatrix} \quad \text{and} \quad P_- = \frac{1}{2} \begin{bmatrix} I_{\mathcal{F}} & -r_{\mathcal{E},\mathcal{F}} \\ -r_{\mathcal{E},\mathcal{F}}^* & I_{\mathcal{E}} \end{bmatrix}. \tag{2.7}$$

Proof. Note that P_+ and P_- are orthogonal projections of the Hilbert space $\mathcal{F} \oplus \mathcal{E}$ onto P_+ and P_-, respectively, and that $J = P_+ - P_-$ holds. Furthermore, $\pm[b_\pm, b_\pm]_{\mathcal{B}} = \pm\langle Jb_\pm, b_\pm \rangle_{\mathcal{F} \oplus \mathcal{E}} = \langle b_\pm, b_\pm \rangle_{\mathcal{F} \oplus \mathcal{E}}$, $b_\pm \in \mathcal{B}_\pm$, which shows that $(\mathcal{B}_\pm, \pm[\cdot, \cdot]_{\mathcal{B}})$ are Hilbert spaces. $\square$

Observe that from (2.6) we have

$$\mathcal{E} = \{e \in \mathcal{E} \mid \text{there exists a } f \in \mathcal{F} \text{ such that } [\begin{smallmatrix} f \\ e \end{smallmatrix}] \in \mathcal{B}_+\}$$
$$= \{e \in \mathcal{E} \mid \text{there exists a } f \in \mathcal{F} \text{ such that } [\begin{smallmatrix} f \\ e \end{smallmatrix}] \in \mathcal{B}_-\}$$

and that a similar representation holds for $\mathcal{F}$.

Proposition 2.4. Let $\mathcal{B} = \mathcal{F} \times \mathcal{E}$ be the bond space equipped with the power product $[\cdot, \cdot]_{\mathcal{B}}$ in (2.2). Then $\mathcal{D}$ is a Tellegen structure on $\mathcal{B}$ if and only if $\mathcal{D}$ is a neutral subspace of the Kreĭn space $(\mathcal{B}, [\cdot, \cdot]_{\mathcal{B}})$ and $\mathcal{D}$ is a Dirac structure on $\mathcal{B}$ if and only if $\mathcal{D}$ is a hyper-maximal neutral subspace of the Kreĭn space $(\mathcal{B}, [\cdot, \cdot]_{\mathcal{B}})$.

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In order to show that a subspace is Dirac structure, one normally begins by showing that it is a Tellegen structure. The following lemma gives an easily checkable condition for this. A proof can be found e.g. in [1, Statement 4.17, p. 29].

Lemma 2.5. *Let $\mathcal{D}$ be a subspace of $\mathcal{B}$. The following conditions are equivalent.*

1. *$\mathcal{D}$ is a Tellegen structure.*
2. *$d \in \mathcal{D}$ implies that $[d, d']_{\mathcal{B}} = 0$ for all $d' \in \mathcal{D}$.*
3. *$d \in \mathcal{D}$ implies that $[d, d]_{\mathcal{B}} = 0$.*

In the following theorem we describe the concept of a *scattering representation* of a Dirac structure. Roughly speaking, we show that a Dirac structure can be represented by a unitary operator $\mathcal{O}$, a so-called *scattering operator*, which connects the *scattering variables* $e - r_{\mathcal{E},\mathcal{F}}^* f$ and $e + r_{\mathcal{E},\mathcal{F}}^* f$. In the case of a Tellegen structure, $\mathcal{O}$ is in general only a *partial isometry*, i.e., it is isometric from its domain but neither its domain nor its range needs to be the full space. Besides the spaces of efforts $\mathcal{E}$ and flows $\mathcal{F}$, we make use of a Hilbert space $\mathcal{G}$ and a unitary map $r_{\mathcal{E},\mathcal{G}}$ from $\mathcal{E}$ to $\mathcal{G}$.

The theorem is known from [13, Sect. 5.2], but for the convenience of the reader we present a short proof which fits into the Kreĭn-space theory and makes use of Propositions 2.3 and 2.4.

Theorem 2.6. *Assume that $\mathcal{D}$ is a Dirac structure on the bond space $\mathcal{B} = \mathcal{F} \times \mathcal{E}$. Then there exists a Hilbert space $\mathcal{G}$, a unitary operator $r_{\mathcal{E},\mathcal{G}}$ from $\mathcal{E}$ to $\mathcal{G}$, and a unitary operator $\mathcal{O}$ on $\mathcal{G}$ such that*

$$\begin{bmatrix} f \\ e \end{bmatrix} \in \mathcal{D} \quad \Longleftrightarrow \quad (e + r_{\mathcal{E},\mathcal{F}}^* f) = r_{\mathcal{E},\mathcal{G}}^* \mathcal{O} r_{\mathcal{E},\mathcal{G}}(e - r_{\mathcal{E},\mathcal{F}}^* f). \qquad (2.8)$$

On the other hand, if $\mathcal{O}$ is a unitary operator on a Hilbert space $\mathcal{G}$ and $r_{\mathcal{E},\mathcal{G}} : \mathcal{E} \to \mathcal{G}$ is unitary, then

$$\mathcal{D} := \left\{ \begin{bmatrix} r_{\mathcal{E},\mathcal{F}} r_{\mathcal{E},\mathcal{G}}^* (\mathcal{O}g - g) \\ r_{\mathcal{E},\mathcal{G}}^* (\mathcal{O}g + g) \end{bmatrix} \mid g \in \mathrm{dom}\,(\mathcal{O}) \right\} \qquad (2.9)$$

defines a Dirac structure on $\mathcal{B} = \mathcal{F} \times \mathcal{E}$ for which (2.8) holds.

The claims remain valid for Tellegen structures $\mathcal{D}$, but then $\mathcal{O}$ is in general only a partial isometry. Moreover, we need to add the condition $r_{\mathcal{E},\mathcal{G}}(e - r_{\mathcal{E},\mathcal{F}}^* f) \in \mathrm{dom}\,(\mathcal{O})$ to the right-hand side of (2.8) in order for the equality to make sense in the Tellegen-structure case.

Proof. Let $\mathcal{B}_\pm$ and $P_\pm$ be given by (2.6) and (2.7), respectively. Assume that $\mathcal{D}$ is a Tellegen structure, i.e., that $\mathcal{D}$ is a neutral subspace of the Kreĭn space $\mathcal{B}$. Then it is well-known, see e.g. [1, Thm 8.10], that there exists a partial isometry U_-, partially defined on the Hilbert space $(\mathcal{B}_+, [\cdot, \cdot]_\mathcal{B})$, mapping into the Hilbert space $(\mathcal{B}_-, -[\cdot, \cdot]_\mathcal{B})$, such that

$$d \in \mathcal{D} \quad \Longleftrightarrow \quad P_-d \in \mathrm{dom}\,(U_-) \quad \text{and} \quad P_+d = U_-P_-d. \tag{2.10}$$

Now note that the operators

$$r_{\mathcal{E},\mathcal{B}_+} := \frac{1}{\sqrt{2}} \begin{bmatrix} r_{\mathcal{E},\mathcal{F}} \\ 1 \end{bmatrix} \quad \text{and} \quad r_{\mathcal{E},\mathcal{B}_-} := \frac{1}{\sqrt{2}} \begin{bmatrix} -r_{\mathcal{E},\mathcal{F}} \\ 1 \end{bmatrix}$$

are unitary from $\mathcal{E}$ to the Hilbert spaces $(\mathcal{B}_+, [\cdot, \cdot]_\mathcal{B})$ and $(\mathcal{B}_-, -[\cdot, \cdot]_\mathcal{B})$, respectively. Moreover, we observe that $P_+ = r_{\mathcal{E},\mathcal{B}_+}r_{\mathcal{E},\mathcal{B}_+}^*$ and $P_- = r_{\mathcal{E},\mathcal{B}_-}r_{\mathcal{E},\mathcal{B}_-}^*$, and substituting this into (2.10), we obtain for $d = \begin{bmatrix} f \\ e \end{bmatrix}$ that

$$\begin{bmatrix} f \\ e \end{bmatrix} \in \mathcal{D} \quad \Longleftrightarrow \quad r_{\mathcal{E},\mathcal{B}_-}r_{\mathcal{E},\mathcal{B}_-}^* \begin{bmatrix} f \\ e \end{bmatrix} \in \mathrm{dom}\,(U_-) \quad \text{and}$$

$$r_{\mathcal{E},\mathcal{B}_+}r_{\mathcal{E},\mathcal{B}_+}^* \begin{bmatrix} f \\ e \end{bmatrix} = U_-r_{\mathcal{E},\mathcal{B}_-}r_{\mathcal{E},\mathcal{B}_-}^* \begin{bmatrix} f \\ e \end{bmatrix}. \tag{2.11}$$

Now let $\mathcal{G}$ be any Hilbert space, such that there exists a unitary operator $r_{\mathcal{E},\mathcal{G}} : \mathcal{E} \to \mathcal{G}$, for instance, but not necessarily, $\mathcal{G} = \mathcal{E}$ with $r_{\mathcal{E},\mathcal{G}} = I$. Setting

$$\mathcal{O} := r_{\mathcal{E},\mathcal{G}}r_{\mathcal{E},\mathcal{B}_+}^* U_- r_{\mathcal{E},\mathcal{B}_-}r_{\mathcal{E},\mathcal{G}}^* \tag{2.12}$$

in (2.11), we obtain (2.8) with both sides of the equality pre-multiplied by $1/\sqrt{2}$. Moreover, $\mathcal{O}$ is a partial isometry or unitary if and only if U_- is a partial isometry or unitary, respectively, because $r_{\mathcal{E},\mathcal{G}}$, $r_{\mathcal{E},\mathcal{B}_+}$ and $r_{\mathcal{E},\mathcal{B}_-}$ in (2.12) are all unitary. According to [1, Thm 8.10], $\mathcal{D}$ is a Dirac structure if and only if U_- is unitary. We have now proved the first part of the theorem.

We now prove the second claim, and therefore assume that $\mathcal{D}$ is given by (2.9), where $\mathcal{O}$ is a partial isometry on $\mathcal{G}$. Then $\begin{bmatrix} f \\ e \end{bmatrix} \in \mathcal{D}$ if and only if there exists a $g \in \mathrm{dom}\,(\mathcal{O})$, such that

$$\begin{bmatrix} f \\ e \end{bmatrix} = \begin{bmatrix} r_{\mathcal{E},\mathcal{F}}r_{\mathcal{E},\mathcal{G}}^*(\mathcal{O}g - g) \\ r_{\mathcal{E},\mathcal{G}}^*(\mathcal{O}g + g) \end{bmatrix}.$$

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Pre-multiplying this equality by the boundedly invertible bounded operator $\begin{bmatrix} r^*_{\mathcal{E},\mathcal{F}} & 1 \\ -r^*_{\mathcal{E},\mathcal{F}} & 1 \end{bmatrix}$, we obtain that $\begin{bmatrix} f \\ e \end{bmatrix} \in \mathcal{D}$ if and only if

$$\begin{bmatrix} r^*_{\mathcal{E},\mathcal{F}} f + e \\ -r^*_{\mathcal{E},\mathcal{F}} f + e \end{bmatrix} = \begin{bmatrix} r^*_{\mathcal{E},\mathcal{G}} \mathcal{O} 2g \\ r^*_{\mathcal{E},\mathcal{G}} 2g \end{bmatrix}$$

for some $g \in \mathrm{dom}\,(\mathcal{O})$. Eliminating g, we obtain that this is equivalent to (2.8) with the extra condition that $r_{\mathcal{E},\mathcal{G}}(e - r^*_{\mathcal{E},\mathcal{F}} f) \in \mathrm{dom}\,(\mathcal{O})$.

Letting U_- be the unique operator which satisfies (2.12), we obtain (2.11), and therefore (2.10). Since U_- is a partial isometry or unitary if and only if $\mathcal{O}$ is a partial isometry, or unitary, respectively, [1, Thm 8.10] yields that $\mathcal{D}$ is a Tellegen structure, and moreover, that this Tellegen structure is a Dirac structure if and only if $\mathcal{O}$ is unitary. The proof is done. $\square$

Note that we made no claims on uniqueness of the scattering representation (2.8) in Theorem 2.6. The following remark, whose proof is based directly on (2.8), elaborates on this issue.

Remark 2.7. The Hilbert space $\mathcal{G}$ and the unitary operator (partial isometry) $\mathcal{O}$ in Theorem 2.6 are unique in the following sense: Assume that $\mathcal{H}$ is another Hilbert space and that $r_{\mathcal{E},\mathcal{H}} : \mathcal{E} \to \mathcal{H}$ is unitary. If $\mathcal{Q}$ is a unitary operator (partial isometry) in $\mathcal{H}$ such that (2.8) holds with $r_{\mathcal{E},\mathcal{G}}$ and $\mathcal{O}$ replaced by $r_{\mathcal{E},\mathcal{H}}$ and $\mathcal{Q}$, respectively, then it immediately follows from (2.8) that

$$r^*_{\mathcal{E},\mathcal{G}} \,\mathrm{dom}\,(\mathcal{O}) = \mathrm{dom}\,(\mathcal{O} r_{\mathcal{E},\mathcal{G}}) = r^*_{\mathcal{E},\mathcal{H}} \,\mathrm{dom}\,(\mathcal{Q}) = \mathrm{dom}\,(\mathcal{Q} r_{\mathcal{E},\mathcal{H}})$$

$$= \left\{ e - r^*_{\mathcal{E},\mathcal{F}} f \;\middle|\; \begin{bmatrix} f \\ e \end{bmatrix} \in \mathcal{D} \right\} \tag{2.13}$$

$$\text{and that} \quad r_{\mathcal{E},\mathcal{H}} r^*_{\mathcal{E},\mathcal{G}} \mathcal{O} = \mathcal{Q} r_{\mathcal{E},\mathcal{H}} r^*_{\mathcal{E},\mathcal{G}}.$$

In particular, the scattering operators $\mathcal{O}$ and $\mathcal{Q}$ are unitarily equivalent.

In many situations it is convenient to choose the auxiliary Hilbert space $\mathcal{G}$ in Theorem 2.6 to be $\mathcal{E}$ and take $r_{\mathcal{E},\mathcal{G}} = I$. In this case the scattering representation is unique and Theorem 2.6 reduces to the following corollary.

Corollary 2.8. If $\mathcal{D}$ is a Dirac structure (Tellegen structure) on the bond space $\mathcal{B} = \mathcal{F} \times \mathcal{E}$, then there exists a unique unitary operator (partial isometry) $\mathcal{O}$ on $\mathcal{E}$ such that

$$\begin{bmatrix} f \\ e \end{bmatrix} \in \mathcal{D} \iff (e + r^*_{\mathcal{E},\mathcal{F}} f) = \mathcal{O}(e - r^*_{\mathcal{E},\mathcal{F}} f). \tag{2.14}$$

On the other hand, if $\mathcal{O}$ is a unitary operator (partial isometry) on $\mathcal{E}$, then

$$\mathcal{D} := \left\{ \left[\begin{array}{c} r_{\mathcal{E},\mathcal{F}}(\mathcal{O}e - e) \\ \mathcal{O}e + e \end{array} \right] \mid e \in \mathrm{dom}\,(\mathcal{O}) \right\}$$

defines a Dirac structure (Tellegen structure) on $\mathcal{B} = \mathcal{F} \times \mathcal{E}$ such that (2.14) holds. Furthermore, we have

$$\mathrm{dom}\,(\mathcal{O}) = \left\{ \tilde{e} \in \mathcal{E} \mid \text{there exists } \left[\begin{smallmatrix} f \\ e \end{smallmatrix} \right] \in \mathcal{D} \text{ such that } \tilde{e} = e - r_{\mathcal{E},\mathcal{F}}^{*}f \right\},$$
$$\mathrm{ran}\,(\mathcal{O}) = \left\{ \tilde{e} \in \mathcal{E} \mid \text{there exists } \left[\begin{smallmatrix} f \\ e \end{smallmatrix} \right] \in \mathcal{D} \text{ such that } \tilde{e} = e + r_{\mathcal{E},\mathcal{F}}^{*}f \right\}.$$

We are now ready to study the composition of Dirac and Tellegen structures. This is the subject of the following section.

3. Composition of Dirac structures

In this section we study the composition (interconnection) of two Dirac structures. In order to define composition, both Dirac structures need to have a joint pair of variables that can be used for interconnection. Hence we assume that the efforts and flows of both Dirac structures can be split into an "own" pair and a "joint" pair, and that the power product splits accordingly. This is formalised in the following definition.

Definition 3.1. Assume that the spaces of efforts and flows are decomposed as $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2$ and $\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2$, and that $r_{\mathcal{E}_i,\mathcal{F}_i}$ are unitary mappings from $\mathcal{E}_i$ onto $\mathcal{F}_i$, $i = 1, 2$. A subspace $\mathcal{D} \subset \mathcal{B} = (\mathcal{F}_1 \oplus \mathcal{F}_2) \times (\mathcal{E}_1 \oplus \mathcal{E}_2)$ is called a *split Tellegen structure (split Dirac structure)* if it is a Tellegen structure (Dirac structure, respectively) in the sense of Definition 2.1, with $r_{\mathcal{E},\mathcal{F}} = \left[\begin{smallmatrix} r_{\mathcal{E}_1,\mathcal{F}_1} & 0 \\ 0 & r_{\mathcal{E}_2,\mathcal{F}_2} \end{smallmatrix} \right]$.

The composition of two split Dirac structures is defined as follows.

Definition 3.2. Let $\mathcal{F}_i$ and $\mathcal{E}_i$, $i = 1, 2, 3$, be Hilbert spaces and let

$$\mathcal{D}^A \subset (\mathcal{F}_1 \oplus \mathcal{F}_2) \times (\mathcal{E}_1 \oplus \mathcal{E}_2) \quad \text{and} \quad \mathcal{D}^B \subset (\mathcal{F}_3 \oplus \mathcal{F}_2) \times (\mathcal{E}_3 \oplus \mathcal{E}_2) \quad (3.1)$$

be split Tellegen or Dirac structures. Then the *composition* $\mathcal{D}^A \circ \mathcal{D}^B$ of $\mathcal{D}^A$ and $\mathcal{D}^B$ (through $\mathcal{F}_2 \times \mathcal{E}_2$) is defined as

$$\mathcal{D}^A \circ \mathcal{D}^B = \left\{ \left[\begin{array}{c} f_1 \\ f_3 \\ e_1 \\ e_3 \end{array} \right] \,\middle|\, \exists \left[\begin{array}{c} f_1 \\ f_2 \\ e_1 \\ e_2 \end{array} \right] \in \mathcal{D}^A \text{ and } \left[\begin{array}{c} f_3 \\ -f_2 \\ e_3 \\ e_2 \end{array} \right] \in \mathcal{D}^B \right\}. \quad (3.2)$$
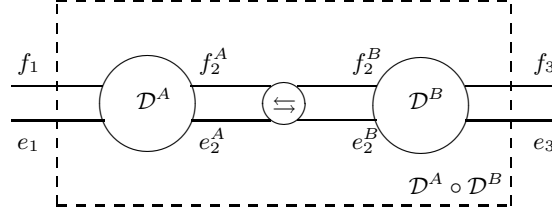
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Figure 1: A graphical interpretation of composition. We compose the structures $\mathcal{D}^A$ and $\mathcal{D}^B$ by making the power-conserving connection "$\leftrightarrows$", i.e. by setting $e_2^A = e_2^B$ and $f_2^A = -f_2^B$

Composition of two Dirac structures is illustrated graphically in Figure 1.

In the following we find necessary and sufficient conditions for the composition to be a split Dirac structure. We start with the following simple proposition on split Tellegen structures. The straightforward proof is left to the reader. It makes use of (3.2) and Lemma 2.5.

Proposition 3.3. Assume that $\mathcal{D}^A$ and $\mathcal{D}^B$ in Definition 3.2 are split Tellegen structures. Then the composition

$$\mathcal{D}^A \circ \mathcal{D}^B \subset \left(\mathcal{F}_1 \oplus \mathcal{F}_3 \right) \times \left(\mathcal{E}_1 \oplus \mathcal{E}_3 \right)$$

is a split Tellegen structure with $r_{\mathcal{E},\mathcal{F}} = \begin{bmatrix} r_{\mathcal{E}_1,\mathcal{F}_1} & 0 \\ 0 & r_{\mathcal{E}_3,\mathcal{F}_3} \end{bmatrix}$.

From now on let $\mathcal{D}^A$ and $\mathcal{D}^B$ in (3.1) be split Dirac structures. According to Corollary 2.8 there exist unique unitary operators

$$\mathcal{O}^A = \begin{bmatrix} \mathcal{O}_{11}^A & \mathcal{O}_{12}^A \\ \mathcal{O}_{21}^A & \mathcal{O}_{22}^A \end{bmatrix} : \begin{bmatrix} \mathcal{E}_1 \\ \mathcal{E}_2 \end{bmatrix} \to \begin{bmatrix} \mathcal{E}_1 \\ \mathcal{E}_2 \end{bmatrix}, \quad \mathcal{O}^B = \begin{bmatrix} \mathcal{O}_{22}^B & \mathcal{O}_{23}^B \\ \mathcal{O}_{32}^B & \mathcal{O}_{33}^B \end{bmatrix} : \begin{bmatrix} \mathcal{E}_2 \\ \mathcal{E}_3 \end{bmatrix} \to \begin{bmatrix} \mathcal{E}_2 \\ \mathcal{E}_3 \end{bmatrix}$$

such that

$$
\begin{aligned}
\begin{bmatrix} e_1 + r_1 f_1 \\ e_2^A + r_2 f_2^A \end{bmatrix} &= \begin{bmatrix} \mathcal{O}_{11}^A & \mathcal{O}_{12}^A \\ \mathcal{O}_{21}^A & \mathcal{O}_{22}^A \end{bmatrix} \begin{bmatrix} e_1 - r_1 f_1 \\ e_2^A - r_2 f_2^A \end{bmatrix}, \\
\begin{bmatrix} e_2^B + r_2 f_2^B \\ e_3 + r_3 f_3 \end{bmatrix} &= \begin{bmatrix} \mathcal{O}_{22}^B & \mathcal{O}_{23}^B \\ \mathcal{O}_{32}^B & \mathcal{O}_{33}^B \end{bmatrix} \begin{bmatrix} e_2^B - r_2 f_2^B \\ e_3 - r_3 f_3 \end{bmatrix}
\end{aligned}
\tag{3.3}
$$

hold for all $(f_1, f_2^A, e_1, e_2^A) \in \mathcal{D}^A$ and $(f_3, f_2^B, e_3, e_3^B) \in \mathcal{D}^B$, respectively. Here and in the following we use the abbreviations $r_i = r_{\mathcal{E}_i,\mathcal{F}_i}^*$, $i = 1, 2, 3$.

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Now compose the Dirac structures $\mathcal{D}^A$ and $\mathcal{D}^B$ by setting $e_2^A = e_2^B$ and $f_2^A = -f_2^B$, or equivalently:

$$e_2^A - r_2 f_2^A = e_2^B + r_2 f_2^B \quad \text{and} \quad e_2^A + r_2 f_2^A = e_2^B - r_2 f_2^B.$$

From Proposition 3.3 we know that $\mathcal{D}_A \circ \mathcal{D}_B$ is a Tellegen structure and hence by Corollary 2.8 there exists a unique partial isometry $\mathcal{O}^{AB}$ on $\mathcal{E}_1 \oplus \mathcal{E}_3$, which connects the scattering variables as

$$\begin{bmatrix} e_1 + r_1 f_1 \\ e_3 + r_3 f_3 \end{bmatrix} = \mathcal{O}^{AB} \begin{bmatrix} e_1 - r_1 f_1 \\ e_3 - r_3 f_3 \end{bmatrix}, \tag{3.4}$$

with

$$\text{dom}\,(\mathcal{O}^{AB}) = \left\{ \begin{bmatrix} e_1 - r_1 f_1 \\ e_3 - r_3 f_3 \end{bmatrix} \Big| (3.3) \text{ holds for some } e_2^A = e_2^B,\, f_2^A = -f_2^B \right\}$$

and

$$\text{ran}\,(\mathcal{O}^{AB}) = \left\{ \begin{bmatrix} e_1 + r_1 f_1 \\ e_3 + r_3 f_3 \end{bmatrix} \Big| (3.3) \text{ holds for some } e_2^A = e_2^B,\, f_2^A = -f_2^B \right\}.$$

The mapping $\mathcal{O}^{AB}$ is depicted in Figure 2 in the case $\mathcal{E}_k = \mathcal{F}_k$, $r_k = I$. For clarity we have abbreviated $f_1 = f_1^A$, $e_1 = e_1^A$, $f_2 = f_2^A$, $e_2 = e_2^A$, $f_3 = f_3^B$ and $e_3 = e_3^B$ in the picture.

In a composed Dirac structure, the scattering operator $\mathcal{O}^{AB}$ is called the *Redheffer star product* of the scattering operators $\mathcal{O}^A$ and $\mathcal{O}^B$. We refer the reader to [29, Chapter 10] and [22] for further information on the Redheffer star product.
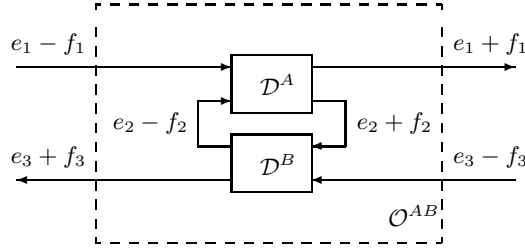

Figure 2: Composition considered from a scattering point of view

Let $\mathcal{D}^A$ and $\mathcal{D}^B$ be split Dirac structures with scattering operators $\mathcal{O}^A$ and $\mathcal{O}^B$, respectively, cf. (3.3). It follows from (3.15) and (3.16) in the proof of Theorem 3.7 below that the following claims are true:

(i) $\operatorname{ran}\left(\begin{bmatrix} \mathcal{O}_{21}^A & \mathcal{O}_{22}^A \mathcal{O}_{23}^B \end{bmatrix}\right) \subset \overline{\operatorname{ran}(\mathcal{O}_{22}^A \mathcal{O}_{22}^B - I)}$, where the bar denotes closure (in $\mathcal{E}_2$), and

(ii) $\operatorname{ran}\left(\begin{bmatrix} \mathcal{O}_{22}^{B*} \mathcal{O}_{12}^{A*} & \mathcal{O}_{32}^{B*} \end{bmatrix}\right) \subset \overline{\operatorname{ran}(\mathcal{O}_{22}^{B*} \mathcal{O}_{22}^{A*} - I)}$.

Compare these range inclusions to the following theorem, where we give necessary and sufficient conditions for the partial isometry $\mathcal{O}^{AB}$ to be unitary, that is, we characterise the case when $\mathcal{D}^A \circ \mathcal{D}^B$ is a Dirac structure.

Theorem 3.4. *Let $\mathcal{D}^A$ and $\mathcal{D}^B$ be split Dirac structures on $(\mathcal{F}_1 \oplus \mathcal{F}_2) \times (\mathcal{E}_1 \oplus \mathcal{E}_2)$ and $(\mathcal{F}_3 \oplus \mathcal{F}_2) \times (\mathcal{E}_3 \oplus \mathcal{E}_2)$, respectively. Let $\mathcal{O}^A$ and $\mathcal{O}^B$ be corresponding scattering operators in (3.3) and let $\mathcal{O}^{AB}$ be the unique partial isometry in (3.4). Then the following claims are valid:*

(i) $\operatorname{dom}\left(\mathcal{O}^{AB}\right) = \mathcal{E}_1 \oplus \mathcal{E}_3$ *if and only if*

$$\operatorname{ran}\left(\begin{bmatrix} \mathcal{O}_{21}^A & \mathcal{O}_{22}^A \mathcal{O}_{23}^B \end{bmatrix}\right) \subset \operatorname{ran}(\mathcal{O}_{22}^A \mathcal{O}_{22}^B - I). \tag{3.5}$$

(ii) $\operatorname{ran}\left(\mathcal{O}^{AB}\right) = \mathcal{E}_1 \oplus \mathcal{E}_3$ *if and only if*

$$\operatorname{ran}\left(\begin{bmatrix} \mathcal{O}_{22}^{B*} \mathcal{O}_{12}^{A*} & \mathcal{O}_{32}^{B*} \end{bmatrix}\right) \subset \operatorname{ran}(\mathcal{O}_{22}^{B*} \mathcal{O}_{22}^{A*} - I). \tag{3.6}$$

(iii) $\mathcal{D}^A \circ \mathcal{D}^B$ *is a split Dirac structure on $(\mathcal{F}_1 \oplus \mathcal{F}_3) \times (\mathcal{E}_1 \oplus \mathcal{E}_3)$ if and only if the (non-equivalent) conditions (3.5) and (3.6) both hold.*

Proof. Step 1. Observe first that by the definition of $\mathcal{O}^{AB}$ we have

$$\begin{bmatrix} e_1 - r_1 f_1 \\ e_3 - r_3 f_3 \end{bmatrix} \in \operatorname{dom}\left(\mathcal{O}^{AB}\right)$$

if and only if there exists some (composition) flow-effort pair $\begin{bmatrix} f_2 \\ e_2 \end{bmatrix}$ and corresponding scattering output

$$\begin{bmatrix} e_1 + r_1 f_1 \\ e_3 + r_3 f_3 \end{bmatrix} \in \operatorname{ran}\left(\mathcal{O}^{AB}\right), \quad \text{such that} \quad \begin{bmatrix} f_1 \\ f_2 \\ e_1 \\ e_2 \end{bmatrix} \in \mathcal{D}^A, \quad \text{and} \quad \begin{bmatrix} f_3 \\ -f_2 \\ e_3 \\ e_2 \end{bmatrix} \in \mathcal{D}^B.$$

Analogously we have

$$\begin{bmatrix} e_1 + r_1 f_1 \\ e_3 + r_3 f_3 \end{bmatrix} \in \operatorname{ran}\left(\mathcal{O}^{AB}\right)$$

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if and only if there exists some (composition) flow-effort pair $\left[\begin{smallmatrix} f_2 \\ e_2 \end{smallmatrix}\right]$ and corresponding scattering input

$$\begin{bmatrix} e_1 - r_1 f_1 \\ e_3 - r_3 f_3 \end{bmatrix} \in \operatorname{dom}\left(\mathcal{O}^{AB}\right), \quad \text{such that} \quad \begin{bmatrix} f_1 \\ f_2 \\ e_1 \\ e_2 \end{bmatrix} \in \mathcal{D}^A, \qquad \begin{bmatrix} f_3 \\ -f_2 \\ e_3 \\ e_2 \end{bmatrix} \in \mathcal{D}^B.$$

From the scattering representations (3.3) of $\mathcal{D}^A$ and $\mathcal{D}^B$ it follows that an element $(f_1,\, f_3,\, e_1,\, e_3)$ belongs to the composition $\mathcal{D}^A \circ \mathcal{D}^B$ if and only if there exist $e_2 \in \mathcal{E}_2$ and $f_2 \in \mathcal{F}_2$ such that

$$\begin{bmatrix} e_1 + r_1 f_1 \\ e_2 + r_2 f_2 \\ e_3 + r_3 f_3 \end{bmatrix} = \begin{bmatrix} \mathcal{O}_{11}^A & \mathcal{O}_{12}^A & 0 \\ \mathcal{O}_{21}^A & \mathcal{O}_{22}^A & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} e_1 - r_1 f_1 \\ e_2 - r_2 f_2 \\ e_3 + r_3 f_3 \end{bmatrix} \tag{3.7}$$

and

$$\begin{bmatrix} e_1 - r_1 f_1 \\ e_2 - r_2 f_2 \\ e_3 + r_3 f_3 \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & \mathcal{O}_{22}^B & \mathcal{O}_{23}^B \\ 0 & \mathcal{O}_{32}^B & \mathcal{O}_{33}^B \end{bmatrix} \begin{bmatrix} e_1 - r_1 f_1 \\ e_2 + r_2 f_2 \\ e_3 - r_3 f_3 \end{bmatrix}. \tag{3.8}$$

By multiplication it follows that

$$\begin{bmatrix} e_1 + r_1 f_1 \\ e_2 + r_2 f_2 \\ e_3 + r_3 f_3 \end{bmatrix} = \begin{bmatrix} \mathcal{O}_{11}^A & \mathcal{O}_{12}^A \mathcal{O}_{22}^B & \mathcal{O}_{12}^A \mathcal{O}_{23}^B \\ \mathcal{O}_{21}^A & \mathcal{O}_{22}^A \mathcal{O}_{22}^B & \mathcal{O}_{22}^A \mathcal{O}_{23}^B \\ 0 & \mathcal{O}_{32}^B & \mathcal{O}_{33}^B \end{bmatrix} \begin{bmatrix} e_1 - r_1 f_1 \\ e_2 + r_2 f_2 \\ e_3 - r_3 f_3 \end{bmatrix}. \tag{3.9}$$

We denote the 3×3 block operator matrix on $\mathcal{E}_1 \oplus \mathcal{E}_2 \oplus \mathcal{E}_3$ in (3.9) by $\widetilde{\mathcal{O}}$ and remark that $\widetilde{\mathcal{O}}$ as a product of two unitary operators is also unitary. Pre-multiplication of (3.9) with the adjoint of $\widetilde{\mathcal{O}}$ yields

$$\begin{bmatrix} e_1 - r_1 f_1 \\ e_2 + r_2 f_2 \\ e_3 - r_3 f_3 \end{bmatrix} = \begin{bmatrix} \mathcal{O}_{11}^{A*} & \mathcal{O}_{21}^{A*} & 0 \\ \mathcal{O}_{22}^{B*}\mathcal{O}_{12}^{A*} & \mathcal{O}_{22}^{B*}\mathcal{O}_{22}^{A*} & \mathcal{O}_{32}^{B*} \\ \mathcal{O}_{23}^{B*}\mathcal{O}_{12}^{A*} & \mathcal{O}_{23}^{B*}\mathcal{O}_{22}^{A*} & \mathcal{O}_{33}^{B*} \end{bmatrix} \begin{bmatrix} e_1 + r_1 f_1 \\ e_2 + r_2 f_2 \\ e_3 + r_3 f_3 \end{bmatrix}. \tag{3.10}$$

Step 2. We verify assertion (i). Suppose first that $\operatorname{dom}\left(\mathcal{O}^{AB}\right) = \mathcal{E}_1 \oplus \mathcal{E}_3$ holds. This implies that for all $\left[\begin{smallmatrix} e_1 - r_1 f_1 \\ e_3 - r_3 f_3 \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{E}_1 \\ \mathcal{E}_3 \end{smallmatrix}\right]$ there exist $e_1 + r_1 f_1 \in \mathcal{E}_1$, $e_2 + r_2 f_2, e_2 - r_2 f_2 \in \mathcal{E}_2$ and $e_3 + r_3 f_3 \in \mathcal{E}_3$, such that (3.7) and (3.8) hold. The second row of (3.9) then implies that for all $\left[\begin{smallmatrix} e_1 - r_1 f_1 \\ e_3 - r_3 f_3 \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{E}_1 \\ \mathcal{E}_3 \end{smallmatrix}\right]$ there exists $e_2 + r_2 f_2 \in \mathcal{E}_2$ such that

$$-(\mathcal{O}_{22}^A \mathcal{O}_{22}^B - I)(e_2 + r_2 f_2) = \begin{bmatrix} \mathcal{O}_{21}^A & \mathcal{O}_{22}^A \mathcal{O}_{23}^B \end{bmatrix} \begin{bmatrix} e_1 - r_1 f_1 \\ e_3 - r_3 f_3 \end{bmatrix}, \tag{3.11}$$

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i.e., (3.5) holds.

Assume now that (3.5) holds. Then for an arbitrary $\left[\begin{smallmatrix} e_1 - r_1 f_1 \\ e_3 - r_3 f_3 \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{E}_1 \\ \mathcal{E}_3 \end{smallmatrix}\right]$, choose $e_2 + r_2 f_2 \in \mathcal{E}_2$ such that (3.11) holds and define $e_2 - r_2 f_2$, $e_3 + r_3 f_3$ by (3.8) and $e_1 + r_1 f_1$ by (3.7). We claim that then also the second row in (3.7) holds. In fact, since $\mathcal{O}^B$ is unitary, (3.11) and (3.8) yield

$$
\begin{aligned}
e_2 + r_2 f_2 &= \begin{bmatrix} \mathcal{O}_{21}^A & \mathcal{O}_{22}^A \mathcal{O}_{22}^B & \mathcal{O}_{22}^A \mathcal{O}_{23}^B \end{bmatrix} \begin{bmatrix} e_1 - r_1 f_1 \\ e_2 + r_2 f_2 \\ e_3 - r_3 f_3 \end{bmatrix} \\
&= \begin{bmatrix} \mathcal{O}_{21}^A & \mathcal{O}_{22}^A \mathcal{O}_{22}^B & \mathcal{O}_{22}^A \mathcal{O}_{23}^B \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & \mathcal{O}_{22}^{B*} & \mathcal{O}_{32}^{B*} \\ 0 & \mathcal{O}_{23}^{B*} & \mathcal{O}_{33}^{B*} \end{bmatrix} \begin{bmatrix} e_1 - r_1 f_1 \\ e_2 - r_2 f_2 \\ e_3 + r_3 f_3 \end{bmatrix} \\
&= \begin{bmatrix} \mathcal{O}_{21}^A & \mathcal{O}_{22}^A & 0 \end{bmatrix} \begin{bmatrix} e_1 - r_1 f_1 \\ e_2 - r_2 f_2 \\ e_3 + r_3 f_3 \end{bmatrix}.
\end{aligned}
$$

As (3.7) and (3.8) both hold we have $(f_1, f_3, e_1, e_3) \in \mathcal{D}^A \circ \mathcal{D}^B$, and so any choice of $\left[\begin{smallmatrix} e_1 - r_1 f_1 \\ e_3 - r_3 f_3 \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{E}_1 \\ \mathcal{E}_3 \end{smallmatrix}\right]$ lies in the domain of $\mathcal{O}^{AB}$.

Step 3. In order to verify (ii) one has to study which $\left[\begin{smallmatrix} e_1 + r_1 f_1 \\ e_3 + r_3 f_3 \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{E}_1 \\ \mathcal{E}_3 \end{smallmatrix}\right]$ lie in the range of $\mathcal{O}^{AB}$. Instead of (3.9) one makes use of (3.10) and obtains as the counterpart of (3.11) that

$$
-(\mathcal{O}_{22}^{B*} \mathcal{O}_{22}^{A*} - I)(e_2 + r_2 f_2) = \begin{bmatrix} \mathcal{O}_{22}^{B*} \mathcal{O}_{12}^{A*} & \mathcal{O}_{32}^{B*} \end{bmatrix} \begin{bmatrix} e_1 + r_1 f_1 \\ e_3 + r_3 f_3 \end{bmatrix}.
$$

The proof then continues with an argument similar to Step 2 above.

Step 4. We prove assertion (iii). As $\mathcal{D}^A \circ \mathcal{D}^B$ is a Tellegen structure the scattering operator $\mathcal{O}^{AB}$ in (3.4) is a partial isometry. We have $\mathrm{dom}\left(\mathcal{O}^{AB}\right) = \mathrm{ran}\left(\mathcal{O}^{AB}\right) = \mathcal{E}_1 \oplus \mathcal{E}_3$ if and only if $\mathcal{O}^{AB}$ is unitary. By Corollary 2.8 this holds if and only if $\mathcal{D}^A \circ \mathcal{D}^B$ is a Dirac structure. $\square$

For the case $\mathcal{O}^B = -I$ Theorem 3.4 can be found in [13]. In his thesis Golo gives an example which shows the non-equivalence of conditions (3.5) and (3.6) in Theorem 3.4.

Remark 3.5. Trivially, $\mathcal{D}^A \circ \mathcal{D}^B$ is a Dirac structure on $(\mathcal{F}_1 \oplus \mathcal{F}_3) \times (\mathcal{E}_1 \oplus \mathcal{E}_3)$ with $r_{\mathcal{E},\mathcal{F}} = \begin{bmatrix} r_{\mathcal{E}_1,\mathcal{F}_1} & 0 \\ 0 & r_{\mathcal{E}_3,\mathcal{F}_3} \end{bmatrix}$ if and only if $\mathcal{D}^B \circ \mathcal{D}^A$ is a Dirac structure on $(\mathcal{F}_3 \oplus \mathcal{F}_1) \times (\mathcal{E}_3 \oplus \mathcal{E}_1)$ with $r_{\mathcal{E},\mathcal{F}} = \begin{bmatrix} r_{\mathcal{E}_3,\mathcal{F}_3} & 0 \\ 0 & r_{\mathcal{E}_1,\mathcal{F}_1} \end{bmatrix}$. Swapping places of $\mathcal{D}^A$ and

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$\mathcal{D}^B$, i.e. $A \leftrightarrow B$ and the indices $1 \leftrightarrow 3$, in Theorem 3.4 turns conditions (3.5) and (3.6) into the respective equivalent conditions

$$\operatorname{ran}\left(\left[\begin{array}{cc} \mathcal{O}_{22}^B \mathcal{O}_{21}^A & \mathcal{O}_{23}^B \end{array}\right]\right) \subset \operatorname{ran}(\mathcal{O}_{22}^B \mathcal{O}_{22}^A - I) \quad \text{and}$$
$$\operatorname{ran}\left(\left[\begin{array}{cc} \mathcal{O}_{12}^{A*} & \mathcal{O}_{22}^{A*} \mathcal{O}_{32}^{B*} \end{array}\right]\right) \subset \operatorname{ran}(\mathcal{O}_{22}^{A*} \mathcal{O}_{22}^{B*} - I).$$

Let again $\mathcal{D}^A$ and $\mathcal{D}^B$ in (3.1) be split Dirac structures and let $\mathcal{O}^A$ and $\mathcal{O}^B$ be corresponding scattering operators as in (3.3). Since $\mathcal{O}^A$ and $\mathcal{O}^B$ are unitary it follows, in particular, that $\mathcal{O}_{22}^A$, $\mathcal{O}_{22}^B$ and $\mathcal{O}_{22}^A \mathcal{O}_{22}^B$ are contractive operators on $\mathcal{E}_2$, i.e., for instance $\|\mathcal{O}_{22}^A e_2\|_{\mathcal{E}_2} \le \|e_2\|_{\mathcal{E}_2}$ for all $e_2 \in \mathcal{E}_2$.

We formulate the following lemma for a general contraction T on the Hilbert space E for simplicity of notation. Later we will apply the lemma in the case $T = \mathcal{O}_{22}^A \mathcal{O}_{22}^B$ and $E = \mathcal{E}_2$, where the operators $\mathcal{O}_{22}^A$ and $\mathcal{O}_{22}^B$ arise from scattering representations of split Dirac structures.

Lemma 3.6. *Let T be a contraction on the Hilbert space E, and decompose E into*

$$E = \left(\ker(T - I)\right)^\perp \oplus \ker(T - I). \tag{3.12}$$

Denote the orthogonal projection in E onto $(\ker(T - I))^\perp$ by $\mathcal{P}$ and the canonical embedding of $(\ker(T - I))^\perp$ into E by $\mathcal{I}$. Then the following holds:

(i) $\ker(T - I) = \ker(T^* - I)$,

(ii) $\overline{\operatorname{ran}(T - I)} = \left(\ker(T - I)\right)^\perp = \overline{\operatorname{ran}(T^* - I)}$,

(iii) *with respect to the decomposition (3.12) we have*

$$T = \begin{bmatrix} \mathcal{P}T\mathcal{I} & 0 \\ 0 & I \end{bmatrix} \quad \text{and } T - I = \begin{bmatrix} \mathcal{P}(T - I)\mathcal{I} & 0 \\ 0 & 0 \end{bmatrix},$$

(iv) *and $\mathcal{P}(T - I)\mathcal{I}$ is an injective operator on $\left(\ker(T - I)\right)^\perp$ with a (possibly unbounded) inverse which we denote by $\left(\mathcal{P}(T - I)\mathcal{I}\right)^{-1}$.*

Proof. (i) Let $e \in \ker(T - I)$. Since T is a contraction we have $\|T^*\| = \|T\| \le 1$ and from $Te = e$ we obtain

$$0 \le \|(T^* - I)e\|^2 = \|T^*e\|^2 - \langle e, Te \rangle - \langle Te, e \rangle + \|e\|^2$$
$$= \|T^*e\|^2 - \|e\|^2 \le 0.$$

Therefore $e \in \ker(T^* - I)$. The converse inclusion in (i) follows by interchanging T and T^*.

(ii) This assertion follows immediately from (i).

(iii) With respect to the decomposition (3.12) it is clear that

$$T - I = \begin{bmatrix} \mathcal{P}(T - I)\mathcal{I} & 0 \\ (I - \mathcal{P})(T - I)\mathcal{I} & 0 \end{bmatrix}. \tag{3.13}$$

By the definition of $\mathcal{P}$ and claim (ii), $\ker(I - \mathcal{P}) = \overline{\operatorname{ran}(T - I)}$ and therefore $(I - \mathcal{P})(T - I)\mathcal{I} = 0$. This gives the second statement in (iii) and then the expression for T follows immediately.

(iv) The validity of this claim is obvious. $\qquad\square$

From now on we always denote the orthogonal projection in $\mathcal{E}_2$ onto $(\ker(\mathcal{O}_{22}^A \mathcal{O}_{22}^B - I))^\perp$ by $\mathcal{P}$ and the canonical embedding of $(\ker(\mathcal{O}_{22}^A \mathcal{O}_{22}^B - I))^\perp$ into $\mathcal{E}_2$ by $\mathcal{I}$.

In the next theorem we give a sufficient criterion for $\mathcal{D}^A \circ \mathcal{D}^B$ to be a Dirac structure and an explicit expression for one of its scattering operators $\mathcal{O}^{AB}$ in terms of the entries in the block matrix representations of $\mathcal{O}^A$ and $\mathcal{O}^B$ in (3.3).

Theorem 3.7. *Let $\mathcal{D}^A$ and $\mathcal{D}^B$ be split Dirac structures with scattering operators $\mathcal{O}^A$ and $\mathcal{O}^B$ as in (3.3). Then the scattering operator $\mathcal{O}^{AB}$ in (3.4) corresponding to the Tellegen structure $\mathcal{D}^A \circ \mathcal{D}^B$ is*

$$\mathcal{O}^{AB} = \begin{bmatrix} \mathcal{O}_{11}^A & \mathcal{O}_{12}^A \mathcal{O}_{23}^B \\ 0 & \mathcal{O}_{33}^B \end{bmatrix}$$
$$- \begin{bmatrix} \mathcal{O}_{12}^A \mathcal{O}_{22}^B \\ \mathcal{O}_{32}^B \end{bmatrix} \mathcal{I}\big(\mathcal{P}(\mathcal{O}_{22}^A \mathcal{O}_{22}^B - I)\mathcal{I}\big)^{-1} \mathcal{P} \begin{bmatrix} \mathcal{O}_{21}^A & \mathcal{O}_{22}^A \mathcal{O}_{23}^B \end{bmatrix} \tag{3.14}$$

with domain given by

$$\left\{ \begin{bmatrix} g_1 \\ g_3 \end{bmatrix} \in \mathcal{E}_1 \oplus \mathcal{E}_3 \;\middle|\; \mathcal{P} \begin{bmatrix} \mathcal{O}_{21}^A & \mathcal{O}_{22}^A \mathcal{O}_{23}^B \end{bmatrix} \begin{bmatrix} g_1 \\ g_3 \end{bmatrix} \in \operatorname{ran}(\mathcal{O}_{22}^A \mathcal{O}_{22}^B - I) \right\}.$$

Furthermore, if $\operatorname{ran}(\mathcal{O}_{22}^A \mathcal{O}_{22}^B - I)$ *is closed, or equivalently,* $\operatorname{ran}(\mathcal{O}_{22}^{B*} \mathcal{O}_{22}^{A*} - I)$ *is closed, then* $\mathcal{D}^A \circ \mathcal{D}^B$ *is a Dirac structure and the scattering operator* $\mathcal{O}^{AB}$ *is unitary.*

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Proof. We first verify the representation (3.14) of the scattering operator $\mathcal{O}^{AB}$ of the split Tellegen structure $\mathcal{D}^A \circ \mathcal{D}^B$. For this decompose $E = \mathcal{E}_2$ as in (3.12), with $T = \mathcal{O}_{22}^A \mathcal{O}_{22}^B$, and rewrite the block operator matrix $\widetilde{\mathcal{O}}$ in (3.9) with the help of Lemma 3.6 (iii) accordingly, in order to obtain

$$
\widetilde{\mathcal{O}} = \begin{bmatrix} \mathcal{O}_{11}^A & \mathcal{O}_{12}^A\mathcal{O}_{22}^B\mathcal{I} & \mathcal{O}_{12}^A\mathcal{O}_{22}^B(I-\mathcal{I}) & \mathcal{O}_{12}^A\mathcal{O}_{23}^B \\ \mathcal{P}\mathcal{O}_{21}^A & \mathcal{P}\mathcal{O}_{22}^A\mathcal{O}_{22}^B\mathcal{I} & 0 & \mathcal{P}\mathcal{O}_{22}^A\mathcal{O}_{23}^B \\ (I-\mathcal{P})\mathcal{O}_{21}^A & 0 & I & (I-\mathcal{P})\mathcal{O}_{22}^A\mathcal{O}_{23}^B \\ 0 & \mathcal{O}_{32}^B\mathcal{I} & \mathcal{O}_{32}^B(I-\mathcal{I}) & \mathcal{O}_{33}^B \end{bmatrix}.
$$

Since $\mathcal{D}^A$ and $\mathcal{D}^B$ are Dirac structures the operator $\widetilde{\mathcal{O}}$ is unitary (see Step 1 in the proof of Theorem 3.4) and therefore we have

$$(I - \mathcal{P})\mathcal{O}_{21}^A = 0, \quad (I - \mathcal{P})\mathcal{O}_{22}^A\mathcal{O}_{23}^B = 0, \tag{3.15}$$

$$\mathcal{O}_{12}^A\mathcal{O}_{22}^B(I - \mathcal{I}) = 0 \quad \text{and} \quad \mathcal{O}_{32}^B(I - \mathcal{I}) = 0. \tag{3.16}$$

Hence (3.9) becomes

$$
\begin{bmatrix} e_1 + r_1 f_1 \\ \mathcal{P}(e_2 + r_2 f_2) \\ (I-\mathcal{P})(e_2 + r_2 f_2) \\ e_3 + r_3 f_3 \end{bmatrix} = \\ \begin{bmatrix} \mathcal{O}_{11}^A & \mathcal{O}_{12}^A\mathcal{O}_{22}^B\mathcal{I} & 0 & \mathcal{O}_{12}^A\mathcal{O}_{23}^B \\ \mathcal{P}\mathcal{O}_{21}^A & \mathcal{P}\mathcal{O}_{22}^A\mathcal{O}_{22}^B\mathcal{I} & 0 & \mathcal{P}\mathcal{O}_{22}^A\mathcal{O}_{23}^B \\ 0 & 0 & I & 0 \\ 0 & \mathcal{O}_{32}^B\mathcal{I} & 0 & \mathcal{O}_{33}^B \end{bmatrix} \begin{bmatrix} e_1 - r_1 f_1 \\ \mathcal{P}(e_2 + r_2 f_2) \\ (I-\mathcal{P})(e_2 + r_2 f_2) \\ e_3 - r_3 f_3 \end{bmatrix} \tag{3.17}
$$

and we obtain

$$
\mathcal{P}(e_2 + r_2 f_2) = -\bigl(\mathcal{P}(\mathcal{O}_{22}^A\mathcal{O}_{22}^B - I)\mathcal{I}\bigr)^{-1}\mathcal{P}\begin{bmatrix} \mathcal{O}_{21}^A & \mathcal{O}_{22}^A\mathcal{O}_{23}^B \end{bmatrix}\begin{bmatrix} e_1 - r_1 f_1 \\ e_3 - r_3 f_3 \end{bmatrix},
$$

whenever $\left[\begin{smallmatrix} e_1 - r_1 f_1 \\ e_3 - r_3 f_3 \end{smallmatrix}\right] \in \mathrm{dom}\,(\mathcal{O}^{AB})$. Substituting this back into (3.17) we have eliminated $e_2 + r_2 f_2$ and the representation of $\mathcal{O}^{AB}$ follows without difficulties.

Next we will use Theorem 3.4 to show that $\mathcal{D}^A \circ \mathcal{D}^B$ is a Dirac structure if the range of $\mathcal{O}_{22}^A\mathcal{O}_{22}^B - I$ is closed. By the closed range theorem, $\mathcal{O}_{22}^{B*}\mathcal{O}_{22}^{A*} - I$ has closed range if and only if $\mathcal{O}_{22}^A\mathcal{O}_{22}^B - I$ has closed range. In that case Lemma 3.6 yields that

$$\mathrm{ran}(\mathcal{O}_{22}^A\mathcal{O}_{22}^B - I) = \mathrm{ran}\bigl(\mathcal{P}(\mathcal{O}_{22}^A\mathcal{O}_{22}^B - I)\mathcal{I}\bigr) = \mathrm{ker}\bigl((\mathcal{O}_{22}^A\mathcal{O}_{22}^B - I)\bigr)^{\perp}$$

holds. As a consequence of (3.15) we have

$$\operatorname{ran}\left(\begin{bmatrix} \mathcal{O}_{21}^A & \mathcal{O}_{22}^A \mathcal{O}_{23}^B \end{bmatrix}\right) = \operatorname{ran}\left(\begin{bmatrix} \mathcal{P}\mathcal{O}_{21}^A & \mathcal{P}\mathcal{O}_{22}^A \mathcal{O}_{23}^B \end{bmatrix}\right) \tag{3.18}$$

and hence $\operatorname{ran}\left([\mathcal{O}_{21}^A \ \mathcal{O}_{22}^A \mathcal{O}_{23}^B]\right) \subset \ker((\mathcal{O}_{22}^A \mathcal{O}_{22}^B - I))^\perp = \operatorname{ran}(\mathcal{O}_{22}^A \mathcal{O}_{22}^B - I)$, i.e., condition (3.5) in Theorem 3.4 holds. By taking adjoints in (3.16), we obtain that $(I - \mathcal{P})\mathcal{O}_{22}^{B*}\mathcal{O}_{12}^{A*} = 0$ and $(I - \mathcal{P})\mathcal{O}_{32}^{B*} = 0$. Hence by Lemma 3.6,

$$\operatorname{ran}\left(\begin{bmatrix} \mathcal{O}_{22}^{B*}\mathcal{O}_{12}^{A*} & \mathcal{O}_{32}^{B*} \end{bmatrix}\right) = \operatorname{ran}\left(\begin{bmatrix} \mathcal{P}\mathcal{O}_{22}^{B*}\mathcal{O}_{12}^{A*} & \mathcal{P}\mathcal{O}_{32}^{B*} \end{bmatrix}\right)$$
$$\subset \ker\left((\mathcal{O}_{22}^A \mathcal{O}_{22}^B - I)\right)^\perp = \operatorname{ran}\left(\mathcal{O}_{22}^{B*}\mathcal{O}_{22}^{A*} - I\right),$$

so that also condition (3.6) in Theorem 3.4 is satisfied. Therefore $\mathcal{D}^A \circ \mathcal{D}^B$ is a split Dirac structure on $(\mathcal{F}_1 \oplus \mathcal{F}_3) \times (\mathcal{E}_1 \oplus \mathcal{E}_3)$. $\qquad\square$

The following corollary highlights two useful consequences of Theorem 3.7.

Corollary 3.8. Let $\mathcal{D}^A$ and $\mathcal{D}^B$ be split Dirac structures with scattering operators $\mathcal{O}^A$ and $\mathcal{O}^B$, respectively, as in (3.3). Then the following hold:

(i) If $\|\mathcal{O}_{22}^A\| < 1$ or $\|\mathcal{O}_{22}^B\| < 1$, then $\mathcal{D}^A \circ \mathcal{D}^B$ is a split Dirac structure;

(ii) If $\mathcal{F}_2 \times \mathcal{E}_2$ is finite-dimensional, then $\mathcal{D}^A \circ \mathcal{D}^B$ is a split Dirac structure.

Proof. Assertion (i) holds since $\|\mathcal{O}_{22}^A \mathcal{O}_{22}^B\| < 1$ implies that $\mathcal{O}_{22}^A \mathcal{O}_{22}^B - I$ is boundedly invertible and, in particular, has closed range. Assertion (ii) follows from the fact that the range of the finite-rank operator $\mathcal{O}_{22}^A \mathcal{O}_{22}^B - I$ is closed. $\qquad\square$

We now conclude the section with an example that illustrates how Theorem 3.7 can be applied. In the example, the Dirac structures are interconnected through an infinite-dimensional channel. Note that this example uses a complex bond space.

Example 3.9. We set $\mathcal{E}_1 = \mathcal{F}_1 = L^2(0, \infty) \oplus \mathbb{C}$, $\mathcal{E}_2 = \mathcal{F}_2 = L^2(0, \infty)$ and $\mathcal{E}_3 = \mathcal{F}_3 = \{0\}$, and we take r_1, r_2 and r_3 equal to the identity. The first

Dirac structure is defined as

$$\mathcal{D}^A = \Big\{ (f_1, f_2, e_1, e_2) \in (\mathcal{F}_1 \oplus \mathcal{F}_2) \times (\mathcal{E}_1 \oplus \mathcal{E}_2) \mid$$

$$e_1 = (e_{1,1}, \, e_{1,\partial}), \, f_1 = (f_{1,1}, \, f_{1,\partial}),$$

$$e_{1,1} \text{ and } e_2 \text{ absolutely continuous, and } f_{1,1} = \frac{\partial}{\partial z} e_2,$$

$$f_2 = \frac{\partial}{\partial z} e_{1,1}, \, f_{1,\partial} = e_2(0), \, e_{1,\partial} = e_{1,1}(0) \Big\}. \qquad (3.19)$$

A slight adaptation of the argument in [19, Section 3] can be used to prove that $\mathcal{D}^A$ is a Dirac structure. That $\mathcal{D}^A$ is a Dirac structure can also be seen using Theorem 4.3 by taking $L = \begin{bmatrix} 0 & \partial/\partial z \\ \partial/\partial z & 0 \end{bmatrix}$, $G = \begin{bmatrix} \delta_0 & 0 \end{bmatrix}$ and $K = \begin{bmatrix} 0 & \delta_0 \end{bmatrix}$, where δ_0 is the operator that evaluates its continuous argument function at zero. We may see $\mathcal{D}^A$ as the Dirac structure associated with the wave equation on the half-line $\mathbb{R}^+$.

The second Dirac structure is given by

$$\mathcal{D}^B = \{ (f_2, e_2) \in \mathcal{F}_2 \times \mathcal{E}_2 \mid f_2 = ie_2 \}.$$

The unitary operator which maps the scattering variable $e_2 - f_2$ into $e_2 + f_2$ is for the Dirac structure $\mathcal{D}^B$ clearly

$$\mathcal{O}^B = \mathcal{O}_{22}^B = \frac{1+i}{1-i} = i.$$

Thus it remains to determine the lower-right block $\mathcal{O}_{22}^A$ of $\mathcal{O}^A$. For this we define the two operators

$$A \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} e_{1,1} - \frac{\partial e_2}{\partial z} \\ e_{1,1}(0) - e_2(0) \\ e_2 - \frac{\partial e_{1,1}}{\partial z} \end{bmatrix} \text{ and} \qquad (3.20)$$

$$B \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} e_{1,1} + \frac{\partial e_2}{\partial z} \\ e_{1,1}(0) + e_2(0) \\ e_2 + \frac{\partial e_{1,1}}{\partial z} \end{bmatrix},$$

where we have used the splitting of e_1 as given in (3.19). From (2.14) it follows that $\mathcal{O}^A = BA^{-1}$ is the scattering representation of $\mathcal{D}^A$ corresponding

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to $\mathcal{G} = \mathcal{E}$ and $r_{\mathcal{E},\mathcal{G}} = I$. We begin by calculating the inverse of A. For this we introduce

$$A \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} g_{1,1} \\ g_{1,\partial} \\ g_2 \end{bmatrix}, \tag{3.21}$$

and since we are interested in the lower-right block of $\mathcal{O}^A$, we may take $g_1 = 0$. Combining (3.21) with (3.20) gives an ordinary differential equation which we can solve for e. The solution is given by

$$e_{1,1}(z) = \frac{1}{2} \int_z^\infty e^{z-\tau} g_2(\tau) \, \mathrm{d}\tau - \frac{1}{2} \int_0^z e^{-(z-\tau)} g_2(\tau) \, \mathrm{d}\tau$$

$$e_{1,\partial} = e_{1,1}(0) = e_2(0)$$

$$e_2(z) = \frac{1}{2} \int_z^\infty e^{z-\tau} g_2(\tau) \, \mathrm{d}\tau + \frac{1}{2} \int_0^z e^{-(z-\tau)} g_2(\tau) \, \mathrm{d}\tau.$$

Letting B operate on this, we find

$$B \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} \int_z^\infty e^{z-\tau} g_2(\tau) \, \mathrm{d}\tau - \int_0^z e^{-(z-\tau)} g_2(\tau) \, \mathrm{d}\tau \\ \int_0^\infty e^{-\tau} g_2(\tau) \, \mathrm{d}\tau \\ \int_z^\infty e^{z-\tau} g_2(\tau) \, \mathrm{d}\tau + \int_0^z e^{-(z-\tau)} g_2(\tau) \, \mathrm{d}\tau - g_2(z) \end{bmatrix}$$

By the definition of A and B, we know that this equals $\mathcal{O}^A \begin{bmatrix} 0 \\ g_2 \end{bmatrix}$. Concluding, we find

$$\left(\mathcal{O}_{22}^A g_2\right)(z) = \int_z^\infty e^{z-\tau} g_2(\tau) \, \mathrm{d}\tau + \int_0^z e^{-(z-\tau)} g_2(\tau) \, \mathrm{d}\tau - g_2(z). \tag{3.22}$$

The first two terms on the right-hand side of (3.22) can be combined into

$$\int_0^\infty h(z,\tau) g_2(\tau) \, \mathrm{d}\tau =: \left(Q g_2\right)(z),$$

where $h(z,\tau) = e^{(z-\tau)} H(\tau - z) + e^{-(z-\tau)} H(z - \tau)$ and H denotes the Heaviside step function, i.e., $H(x)$ is one for positive x and zero otherwise. Since $h(z,\tau) = h(\tau, z)$ with values in $\mathbb{R}$ whenever $z, \tau \in \mathbb{R}$, we have that Q is self-adjoint, see, e.g. [15, Ex. III.3.17]. This in turn implies that the spectrum of iQ lies on the imaginary axis.

The operator $\mathcal{O}_{22}^A \mathcal{O}_{22}^B - I$ equals

$$(Q - I)i - I = iQ - (1+i)I,$$

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the point $1 + i$ lies outside the spectrum of iQ, and we conclude that the range of $\mathcal{O}_{22}^A \mathcal{O}_{22}^B - I$ is the whole space $L^2(0, \infty)$. In particular, the range is closed and Theorem 3.7 yields that $\mathcal{D}^A \circ \mathcal{D}^B$ is a Dirac structure.

We finish this example by determining

$$
\mathcal{D}^A \circ \mathcal{D}^B = \left\{ \begin{bmatrix} f_1 \\ e_1 \end{bmatrix} \middle| \text{ there exists } \begin{bmatrix} f_1 \\ f_2 \\ e_1 \\ e_2 \end{bmatrix} \in \mathcal{D}^A \text{ and } \begin{bmatrix} -f_2 \\ e_2 \end{bmatrix} \in \mathcal{D}^B \right\}.
$$

Using the definitions of $\mathcal{D}^A$ and $\mathcal{D}^B$, we find that an element $(f_1, e_1) \in \mathcal{D}^A \circ \mathcal{D}^B$ satisfies

$$
f_1 = \begin{bmatrix} f_{1,1} \\ f_{1,\partial} \end{bmatrix} = \begin{bmatrix} \frac{\partial e_2}{\partial z} \\ e_2(0) \end{bmatrix} = \begin{bmatrix} i\frac{\partial f_2}{\partial z} \\ if_2(0) \end{bmatrix} = \begin{bmatrix} i\frac{\partial^2 e_{1,1}}{\partial z^2} \\ i\frac{\partial e_{1,1}}{\partial z}(0) \end{bmatrix}.
$$

Thus

$$
\mathcal{D}^A \circ \mathcal{D}^B = \{ (f_1, e_1) \in \mathcal{F}_1 \times \mathcal{E}_1 \mid e_1 = (e_{1,1}, e_{1,\partial}), f_1 = (f_{1,1}, f_{1,\partial}),
$$
$$
e_{1,1} \text{ and } \frac{\partial e_{1,1}}{\partial z} \text{ are absolutely continuous, } f_{1,1} = i\frac{\partial^2 e_{1,1}}{\partial z^2},
$$
$$
f_{1,\partial} = i\frac{\partial e_{1,1}}{\partial z}(0), \text{ and } e_{1,\partial} = e_{1,1}(0) \}.
$$

This is the Dirac structure associated with the Schrödinger equation/operator with zero potential on the half-line $\mathbb{R}^+$; see [21, Sect. 7.5.2].

If we were working on a compact subinterval of $\mathbb{R}$ in Example 3.9, instead of $[0, \infty)$, then the unitary operator $\mathcal{O}^A$ would be a Fredholm operator, and so the closedness of the range of $\mathcal{O}_{22}^A \mathcal{O}_{22}^B - I$ would be immediate.

4. Dirac structures defined by boundary colligations

In this section we introduce a class of Dirac structures to which, e.g., the Dirac structure $\mathcal{D}^A$ in Example 3.9 and the examples in the introduction belong. The Dirac structures studied here are obtained from operator colligations associated with boundary control. They can alternatively be viewed as unitary operators with respect to a particular indefinite structure. This point of view and the connection to abstract notions from extension theory of

symmetric operators, as, e.g., boundary triplets and the more general recent concept of boundary relations, is explained after Proposition 4.5.

The following definition is compiled from Definitions 2.1 and 4.4, and the introduction to Section 5 in [20]. See also [2] for similar ideas.

Definition 4.1. Let U, X and Y be Hilbert spaces, and let G, L, and K be linear operators, with common domain in X, that map into U, X, and Y, respectively.

1. The pair $\left(\begin{bmatrix} G \\ L \\ K \end{bmatrix}, \begin{bmatrix} U \\ X \\ Y \end{bmatrix} \right)$ is called an *operator colligation* or *colligation*.

2. The colligation is said to be *strong* if $\Xi := \begin{bmatrix} G \\ L \\ K \end{bmatrix}$ and L are both closed operators (with domain $\operatorname{dom}(L) = \operatorname{dom}(\Xi)$).

3. The *minimal (interior) operator* of Ξ is defined as

$$L_0 := L|_{\{x \in \operatorname{dom}(L) \mid Kx=0,\, Gx=0\}}.$$

We will often call Ξ the *colligation*, when the spaces are clear.

Now we want to associate a Dirac structure $\mathcal{D}$ to a colligation. Therefore we assume that the Hilbert spaces U and Y have the same cardinality, and we fix a unitary map $r_{U,Y}$ between U and Y. Furthermore, we introduce the effort and flow spaces as

$$\mathcal{E} := X \oplus U \quad \text{and} \quad \mathcal{F} := X \oplus Y, \tag{4.1}$$

respectively. As our unitary mapping from $\mathcal{E}$ to $\mathcal{F}$ we take

$$r_{\mathcal{E},\mathcal{F}} := \begin{bmatrix} I & 0 \\ 0 & -r_{U,Y} \end{bmatrix}. \tag{4.2}$$

Observe that according to (2.2) the indefinite power product on the bond space $\mathcal{B} = \mathcal{F} \times \mathcal{E}$ is then given by

$$\left[\begin{bmatrix} z_1 \\ y_1 \\ x_1 \\ u_1 \end{bmatrix}, \begin{bmatrix} z_2 \\ y_2 \\ x_2 \\ u_2 \end{bmatrix} \right]_{\mathcal{B}} = \langle z_1, x_2 \rangle_X - \langle y_1, r_{U,Y} u_2 \rangle_Y + \langle x_1, z_2 \rangle_X - \langle u_1, r_{U,Y}^* y_2 \rangle_U, \tag{4.3}$$

where $x_1, z_1, x_2, z_2 \in X$, $y_1, y_2 \in Y$ and $u_1, u_2 \in U$. Let $\left(\begin{bmatrix} G \\ L \\ K \end{bmatrix}, \begin{bmatrix} U \\ X \\ Y \end{bmatrix} \right)$ be a colligation defined on $\operatorname{dom}(\Xi)$ as given in Definition 4.1. In the following we

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study the space $\mathcal{D}$ defined by

$$\mathcal{D} := \begin{bmatrix} L \\ K \\ I \\ G \end{bmatrix} \operatorname{dom}(\Xi) \subset \mathcal{F} \times \mathcal{E} = (X \oplus Y) \times (X \oplus U). \qquad (4.4)$$

We find necessary and sufficient criteria for the operators L, K, and G for $\mathcal{D}$ to be a Dirac structure on $\mathcal{B} = \mathcal{F} \times \mathcal{E}$ with respect to $r_{\mathcal{E},\mathcal{F}}$ in (4.2).

First, however, we give a characterisation of Tellegen structures defined by colligations. The proof follows directly from Lemma 2.5, and it is left to the reader.

Proposition 4.2. Let the bond space $\mathcal{B} = \mathcal{F} \times \mathcal{E}$ with its power product be as in (4.1)–(4.3), and let $\mathcal{D}$ be defined by (4.4). Then $\mathcal{D}$ is a Tellegen structure on $\mathcal{B}$ if and only if

$$\operatorname{Re}\langle Lx, x\rangle_X = \operatorname{Re}\langle Kx, r_{U,Y}Gx\rangle_Y, \quad x \in \operatorname{dom}(\Xi). \qquad (4.5)$$

We now characterise a class of Dirac structures originating from colligations. Since the graph of the colligation Ξ and the linear subspace $\mathcal{D}$ in (4.4) are unitarily equivalent the following result gives, roughly speaking, necessary and sufficient conditions for the graph of a colligation to be a Dirac structure.

Theorem 4.3. *Let the bond space $\mathcal{B} = \mathcal{F} \times \mathcal{E}$ with its power product be as in (4.1)–(4.4), let $\mathcal{D}$ be defined as in (4.4), and assume that the operator L is closed. Then $\mathcal{D}$ is a Dirac structure on $\mathcal{B}$ if and only if the following conditions hold:*

1. *Equation (4.5) is satisfied.*
2. *The minimal operator L_0 is densely defined and $L_0^* = -L$ holds.*
3. *The range of the operator $\begin{bmatrix} G \\ K \end{bmatrix}$ is dense in $U \oplus Y$.*

Proof. Assume first that $\mathcal{D}$ in (4.4) is a Dirac structure. Then, in particular, $\mathcal{D}$ is a Tellegen structure and hence (1) is satisfied. Next it will be shown that $\operatorname{dom}(\Xi)$ is dense. Let $z \in X$ be such that $\langle z, x_1\rangle = 0$ for all $x_1 \in \operatorname{dom}(\Xi)$. Then

$$0 = \langle x_1, z\rangle_X = \left[\left[\begin{bmatrix} Lx_1 \\ Kx_1 \\ x_1 \\ Gx_1 \end{bmatrix}, \begin{bmatrix} z \\ 0 \\ 0 \\ 0 \end{bmatrix} \right]\right]_{\mathcal{B}}$$

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for all $x_1 \in \mathrm{dom}(\Xi)$. Thus $(z, 0, 0, 0) \in \mathcal{D}^{[\perp]}$, and since $\mathcal{D}$ is a Dirac structure, we conclude that $(z, 0, 0, 0) \in \mathcal{D}$. In particular, $z = L0 = 0$ and hence $\mathrm{dom}\,(\Xi) = \mathrm{dom}\,(L)$ is dense. In particular, the (possibly unbounded) adjoint L^* of L is well-defined. An element $x_2 \in X$ lies in $\mathrm{dom}(L^*)$ if and only if there exists some $z_2 \in X$ such that for all $x_1 \in \mathrm{dom}(L)$ we have $\langle Lx_1, x_2 \rangle_X = \langle x_1, z_2 \rangle_X$, that is,

$$0 = \langle Lx_1, x_2 \rangle_X + \langle x_1, -z_2 \rangle_X = \left[\!\!\left[\begin{bmatrix} Lx_1 \\ Kx_1 \\ x_1 \\ Gx_1 \end{bmatrix}, \begin{bmatrix} -z_2 \\ 0 \\ x_2 \\ 0 \end{bmatrix} \right]\!\!\right]_{\mathcal{B}} \quad \text{for all } x_1 \in \mathrm{dom}(L).$$

Since $\mathrm{dom}(L) = \mathrm{dom}(\Xi)$, we see that $(-z_2, 0, x_2, 0) \in \mathcal{D}^{[\perp]}$. Using the fact that $\mathcal{D}$ is a Dirac structure, we conclude that $x_2 \in \mathrm{dom}(\Xi)$, $Lx_2 = -z_2$, and $Gx_2 = Kx_2 = 0$. Thus $x_2 \in \mathrm{dom}(L_0)$ and $L_0x_2 = -z_2 = -L^*x_2$, i.e., $L^* \subset -L_0$. By reading the above reasoning backwards, we see that if $x_2 \in \mathrm{dom}(L_0)$, then $x_2 \in \mathrm{dom}(L^*)$ and $L_0x_2 = -L^*x_2$. Hence $L^* = -L_0$ and since L is a closed operator we conclude $L = -L_0^*$, i.e., (2) holds. In order show (3) suppose that $(u, y) \in U \oplus Y$ is orthogonal to $\mathrm{ran}\,(\left[\begin{smallmatrix} G \\ K \end{smallmatrix}\right])$. This implies

$$\left[\!\!\left[\begin{bmatrix} Lx_1 \\ Kx_1 \\ x_1 \\ Gx_1 \end{bmatrix}, \begin{bmatrix} 0 \\ r_{U,Y}u \\ 0 \\ r_{U,Y}^*y \end{bmatrix} \right]\!\!\right]_{\mathcal{B}} = 0 \quad \text{for all } x_1 \in \mathrm{dom}(\Xi)$$

and hence $(0, r_{U,Y}u, 0, r_{U,Y}^*y) \in \mathcal{D}^{[\perp]}$. Since $\mathcal{D}$ is a Dirac structure we conclude that $r_{U,Y}^*y = G0 = 0$, and $r_{U,Y}u = K0 = 0$, and hence $u = 0$, $y = 0$.

Let us now prove the converse direction. Condition (1) and Proposition 4.2 imply $\mathcal{D} \subset \mathcal{D}^{[\perp]}$ and so we only have to show $\mathcal{D}^{[\perp]} \subset \mathcal{D}$. For this let $(z_2, y_2, x_2, u_2) \in \mathcal{D}^{[\perp]}$. For any $x_1 \in \mathrm{dom}(L_0)$, we see that

$$0 = \left[\!\!\left[\begin{bmatrix} Lx_1 \\ 0 \\ x_1 \\ 0 \end{bmatrix}, \begin{bmatrix} z_2 \\ y_2 \\ x_2 \\ u_2 \end{bmatrix} \right]\!\!\right]_{\mathcal{B}} = \langle L_0x_1, x_2 \rangle_X + \langle x_1, z_2 \rangle_X.$$

This implies $x_2 \in \mathrm{dom}(L_0^*)$ and $L_0^*x_2 = -z_2$. Hence by item 2 we have $x_2 \in \mathrm{dom}(\Xi) = \mathrm{dom}\,(L)$ and $Lx_2 = z_2$. Now let $x_1 \in \mathrm{dom}(\Xi)$ be arbitrary.

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Since $(z_2,\, y_2,\, x_2,\, u_2) \in \mathcal{D}^{[\perp]}$ and $\mathcal{D}$ is a Tellegen structure we can compute

$$
\begin{aligned}
0 &= \left[\!\!\left[\begin{bmatrix} Lx_1 \\ Kx_1 \\ x_1 \\ Gx_1 \end{bmatrix}, \begin{bmatrix} z_2 \\ y_2 \\ x_2 \\ u_2 \end{bmatrix} \right]\!\!\right]_{\mathcal{B}} = \left[\!\!\left[\begin{bmatrix} Lx_1 \\ Kx_1 \\ x_1 \\ Gx_1 \end{bmatrix}, \begin{bmatrix} Lx_2 \\ y_2 \\ x_2 \\ u_2 \end{bmatrix} \right]\!\!\right]_{\mathcal{B}} \\
&= \langle Lx_1, x_2 \rangle_X + \langle x_1, Lx_2 \rangle_X - \langle r_{U,Y} Gx_1, y_2 \rangle_Y - \langle Kx_1, r_{U,Y} u_2 \rangle_Y \\
&= \langle r_{U,Y} Gx_1, Kx_2 \rangle_Y + \langle Kx_1, r_{U,Y} Gx_2 \rangle_Y \\
&\quad - \langle r_{U,Y} Gx_1, y_2 \rangle_Y - \langle Kx_1, r_{U,Y} u_2 \rangle_Y \\
&= \langle r_{U,Y} Gx_1, (Kx_2 - y_2) \rangle_Y + \langle Kx_1, r_{U,Y}(Gx_2 - u_2) \rangle_Y.
\end{aligned}
$$

Using the denseness of the range of $\begin{bmatrix} G \\ K \end{bmatrix}$, we conclude $u_2 = Gx_2$, $y_2 = Kx_2$ and hence $(z_2,\, y_2,\, x_2,\, u_2) = (Lx_2, Kx_2, x_2, Gx_2) \in \mathcal{D}$. $\qquad\square$

We note that the minimal operator L_0 in Theorem 4.3 is skew-symmetric, i.e., $L_0 \subset L = -L_0^*$ and that in the proof of Theorem 4.3 we have shown $L^* = -L_0$, even when L is not closed and not even closable. We also mention that strong colligations whose graph form a Dirac structure are the same as so-called impedance conservative internally well-posed boundary nodes; cf. [20, Theorem 5.2].

Remark 4.4. If Ξ is a colligation and $\mathcal{D}$ in (4.4) is a Dirac structure, then Ξ must be a closed operator. It thus follows from assumptions $1-3$ in Theorem 4.3 that $\mathcal{D}$ is closed. Hence, if $\mathcal{D}$ is a Dirac structure and L is closed, then the colligation Ξ is automatically strong. According to [20, Lemma 4.5] this holds if and only if L is closed and G and K are continuous with respect to the graph norm of L.

Condition (3) in Theorem 4.3 can be strengthened. This is done in the following result which is inspired by [7, Proposition 2.3]. The result can also be deduced from [23]. For the convenience of the reader we give a short direct proof.

Proposition 4.5. Let the bond space $\mathcal{B} = \mathcal{F} \times \mathcal{E}$ and its power product be as in $(4.1)-(4.3)$, and assume that $\mathcal{D}$ in (4.4) is a Dirac structure on $\mathcal{B} = \mathcal{F} \oplus \mathcal{E}$. Then the operator L is closed if and only if the operator $\begin{bmatrix} G \\ K \end{bmatrix}$ has closed range.

Proof. Let $\mathcal{M} \subset X \oplus X \oplus U \oplus Y$ be the subspace

$$\mathcal{M} := \begin{bmatrix} L \\ I \\ G \\ K \end{bmatrix} \mathrm{dom}(\Xi).$$

Since $\mathcal{D}$ is a Dirac structure, and thus a closed linear subspace, we have that $\mathcal{M}$ is a closed linear subspace. As $\mathcal{N}$ we define $X \oplus X \oplus \{0\} \oplus \{0\}$. The following relation is immediate:

$$\mathcal{M} + \mathcal{N} = X \oplus X \oplus \mathrm{ran}\left(\begin{bmatrix} G \\ K \end{bmatrix}\right). \tag{4.6}$$

Next we calculate $\mathcal{M}^\perp$. Let $(z, x, u, y) \in \mathcal{M}^\perp$, then for all $x_1 \in \mathrm{dom}(\Xi)$

$$0 = \langle z, Lx_1 \rangle_X + \langle x, x_1 \rangle_X + \langle u, Gx_1 \rangle_U + \langle y, Kx_1 \rangle_Y = \left[\begin{bmatrix} x \\ -r_{U,Y}u \\ z \\ -r_{U,Y}^*y \end{bmatrix}, \begin{bmatrix} Lx_1 \\ Kx_1 \\ x_1 \\ Gx_1 \end{bmatrix}\right]_{\mathcal{B}}.$$

Thus $(x, -r_{U,Y}u, z, -r_{U,Y}^*y) \in \mathcal{D}^{[\perp]}$ and since $\mathcal{D}$ is assumed to be a Dirac structure this element belongs to $\mathcal{D}$. By the definition of $\mathcal{D}$ this implies that $x = Lz$, $-r_{U,Y}u = Kz$, and $-r_{U,Y}^*y = Gz$. So we find that

$$\mathcal{M}^\perp \subset \begin{bmatrix} I \\ L \\ -r_{U,Y}^*K \\ -r_{U,Y}G \end{bmatrix} \mathrm{dom}(\Xi).$$

The other inclusion is shown similarly. Since $\mathcal{N}^\perp = \{0\} \oplus \{0\} \oplus U \oplus Y$, we find that

$$\mathcal{M}^\perp + \mathcal{N}^\perp = \mathrm{ran}\left(\begin{bmatrix} I \\ L \end{bmatrix}\right) \oplus U \oplus Y. \tag{4.7}$$

By Theorem IV.4.8 of [15] we have that $\mathcal{M} + \mathcal{N}$ is closed if and only if $\mathcal{M}^\perp + \mathcal{N}^\perp$ is closed. Using (4.6) and (4.7) we see that this implies that $\mathrm{ran}\left(\begin{bmatrix} G \\ K \end{bmatrix}\right)$ is closed if and only if $\mathrm{ran}\left(\begin{bmatrix} I \\ L \end{bmatrix}\right)$ is closed. The latter is closed if and only if L is a closed operator. $\square$

Next we will explain how Dirac structures defined by colligations are related to linear operators which are unitary with respect to certain Kreĭn space inner products. The following (more abstract) considerations are of auxiliary nature and will not be used further in the present paper.

Let Ξ be a colligation as in Definition 4.1 and let $\mathcal{D}$ be as in (4.4). We associate to $\mathcal{D}$ a linear mapping $\mathfrak{D}$ from $X \times X$ into $U \times U$ which is defined on the graph of the operator $-iL$ by

$$\mathfrak{D} : X \times X \supset \mathrm{dom}\,(\mathfrak{D}) \to U \times U, \quad \begin{bmatrix} x \\ -iLx \end{bmatrix} \mapsto \begin{bmatrix} Gx \\ -ir_{U,Y}^* Kx \end{bmatrix}. \qquad (4.8)$$

Observe that $\mathfrak{D}$ is a well defined linear operator mapping a closed subspace of $X \times X$ into $U \times U$. The space $X \times X$ will be equipped with the Kreĭn space inner product

$$\left[\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \right]_{X \times X} := i\big(\langle x_2, z_1 \rangle_X - \langle x_1, z_2 \rangle_X \big), \quad x_1, x_2, z_1, z_2 \in X,$$

and the Kreĭn space inner product $[\cdot, \cdot]_{U \times U}$ on $U \times U$ is defined in the same way. The adjoint $\mathfrak{D}^{[*]}$ of $\mathfrak{D}$ with respect to $(X \times X, [\cdot, \cdot]_{X \times X})$ and $(U \times U, [\cdot, \cdot]_{U \times U})$ is defined in the sense of linear relations:

$$\mathfrak{D}^{[*]} = \left\{ \left\{ \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\} \mid \left[\begin{bmatrix} Gx \\ -ir_{U,Y}^* Kx \end{bmatrix}, \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right]_{U \times U} \right.$$
$$\left. = \left[\begin{bmatrix} x \\ -iLx \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right]_{X \times X} \right\},$$

where the equality of the indefinite inner products holds for all $x \in \mathrm{dom}\,(\Xi)$.

The operator $\mathfrak{D}$ is said to be *isometric* (*unitary*) with respect to the inner products $[\cdot, \cdot]_{X \times X}$ and $[\cdot, \cdot]_{U \times U}$ if $\mathfrak{D}^{-1} \subset \mathfrak{D}^{[*]}$ ($\mathfrak{D}^{-1} = \mathfrak{D}^{[*]}$, respectively), where $\mathfrak{D}^{-1}$ denotes the inverse of $\mathfrak{D}$ in the sense of linear relations. The next proposition which connects Dirac and Tellegen structures defined by colligations with isometric and unitary operators acting between the Kreĭn spaces $(X \times X, [\cdot, \cdot]_{X \times X})$ and $(U \times U, [\cdot, \cdot]_{U \times U})$ is now immediate.

Proposition 4.6. Let Ξ, $\mathcal{D}$ and $\mathfrak{D}$ be as above. Then the following holds:

1. $\mathcal{D}$ is a Tellegen structure if and only if $\mathfrak{D}^{-1} \subset \mathfrak{D}^{[*]}$ and
2. $\mathcal{D}$ is a Dirac structure if and only if $\mathfrak{D}^{-1} = \mathfrak{D}^{[*]}$.

The fact that Dirac structures defined by colligations can be regarded as unitary operators in Kreĭn spaces also provides a direct connection to the concepts of boundary triplets, generalised boundary triplets, quasi boundary triplets and boundary relations used in the extension theory of symmetric operators; cf. [3, 7, 8, 9, 10, 14]. With the help of these connections also Theorem 4.3 can be proved. Without going into further details on boundary relations we mention for completeness that by Proposition 4.6 (ii) the operator $\mathfrak{D}$ in (4.8) is a boundary relation for $-iL$ if and only if $\mathcal{D}$ in (4.4) is a Dirac structure.

For brevity we only recall here the notion of boundary triplets, i.e., surjective boundary relations, and we point out only a few facts that are of interest to us.

Let A be a densely defined, closed and symmetric operator in the Hilbert space X. A triplet $(\mathcal{G}, \Gamma_0, \Gamma_1)$ is said to be a *boundary triplet* or *boundary value space* for the adjoint operator A^*, if $\mathcal{G}$ is a Hilbert space and $\Gamma_0, \Gamma_1 : \mathrm{dom}\,(A^*) \to \mathcal{G}$ are linear mappings such that the abstract Green's identity

$$\langle A^* x, z \rangle_X - \langle x, A^* z \rangle_X = \langle \Gamma_1 x, \Gamma_0 z \rangle_{\mathcal{G}} - \langle \Gamma_0 x, \Gamma_1 z \rangle_{\mathcal{G}}$$

holds for all $x, z \in \mathrm{dom}\,(A^*)$ and the mapping $\left[\begin{smallmatrix} \Gamma_0 \\ \Gamma_1 \end{smallmatrix}\right] : \mathrm{dom}\,(A^*) \to \mathcal{G} \oplus \mathcal{G}$ is surjective.

It can be shown that a boundary triplet for A^* exists if and only if the symmetric operator A has equal (possibly infinite) deficiency indices $n_{\pm}(A) := \dim\,(\ker(A \mp i))$. Then necessarily $\dim(\mathcal{G}) = n_{\pm}(A)$ holds and Γ_0 and Γ_1 are continuous with respect to the graph norm of A^*. We note that a boundary triplet (if it exists) is never unique. Furthermore, it follows that

$$\mathrm{dom}\,(A) = \{x \in \mathrm{dom}\,(A^*) \mid \Gamma_0 x = \Gamma_1 x = 0\}$$

and hence $A = A^*|_{\ker(\Gamma_0) \cap \ker(\Gamma_1)}$. The following result is a consequence of (4.8) and Proposition 4.6.

Theorem 4.7. *Let A be a densely defined closed symmetric operator in X with equal deficiency indices and let (U, Γ_0, Γ_1) be a boundary triplet for A^*. Then the subspace $\mathcal{D}$ in (4.4) associated with the strong colligation*

$$\Xi = \begin{bmatrix} \Gamma_0 \\ iA^* \\ ir_{U,Y} \Gamma_1 \end{bmatrix}$$

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is a Dirac structure of the type described in Theorem 4.3.

Conversely, if L is a closed operator in X and $\mathcal{D}$ is a Dirac structure as in Theorem 4.3, then iL_0 is a densely defined closed symmetric operator with equal deficiency indices $n_\pm(iL_0) = \dim(U)$ and $(U, G, -ir^*_{U,Y}K)$ is a boundary triplet for $-iL$.

Finally we consider Dirac structures associated to colligations which are not necessarily strong. In particular, the operator L is not assumed to be closed. Instead of the minimal operator L_0 we now make use of the restrictions

$$L_K = L|_{\{x \in \mathrm{dom}(L) \mid Kx=0\}} \quad \text{and} \quad L_G = L|_{\{x \in \mathrm{dom}(L) \mid Gx=0\}}$$

of the operator L.

The following two propositions can be deduced from Proposition 4.6 together with abstract results on special subclasses of boundary relations in [7, Sect. 5], and the results can also be proved directly. Since these results are not used further in this paper we leave the proofs to the reader.

Proposition 4.8. Let the bond space $\mathcal{B} = \mathcal{F} \times \mathcal{E}$ with its power product be as in $(4.1)-(4.3)$, let $\mathcal{D}$ in (4.4) be a Tellegen structure, and assume that the operators L_K and L_G are densely defined. Then the following claims hold:

1. $L_K \subset -L_K^*$ and $L_G \subset -L_G^*$.
2. If the closure $\overline{L_K}$ of the operator L_K satisfies $\overline{L_K} = -L_K^*$ and $\mathrm{ran}(K) = Y$, then $\mathcal{D}$ is a Dirac structure on $\mathcal{B}$.
3. If $\overline{L_G} = -L_G^*$ and $\mathrm{ran}(G) = U$, then $\mathcal{D}$ is a Dirac structure on $\mathcal{B}$.

We now finish this section with a partial converse to Proposition 4.8.

Proposition 4.9. Let the bond space $\mathcal{B} = \mathcal{F} \times \mathcal{E}$ with its power product be as in $(4.1)-(4.3)$, let $\mathcal{D}$ in (4.4) be a Dirac structure on $\mathcal{B}$, and assume that L is a closed operator. Then we have that

$$L_K = -L_K^* \quad \text{and} \quad L_G = -L_G^*.$$

In particular, L_K and L_G are closed.

5. Composition of boundary colligation Dirac structures

In Section 4 we studied Dirac structures associated with strong colligations and in this section we study the composition of two of these Dirac structures. When we study this composition it is good to recall what the meaning of the different operators in a colligation are.

From our examples in the introduction, we see that Ξ is normally an operator acting on functions defined on some spatial domain Ω. In this situation, the action of L results in another function defined on the same spatial domain Ω, whereas G and K are maps to functions defined on the boundary of Ω.

In this section, we assume that the composition is made only via a part of the boundary ∂ of Ω, as depicted in Figure 3. Thus, Example 3.9 is not covered by the theory in this section. In the figure, the domain on which $\mathcal{D}^A$ is defined is Ω^A with boundary $\partial_b^A \cup \partial_c$, while the domain, on which $\mathcal{D}^B$ is defined, is Ω^B with boundary $\partial_b^B \cup \partial_c$. The composition $\mathcal{D}^A \circ \mathcal{D}^B$ is then defined on $\Omega^A \cup \Omega^B \cup \partial_c$ with boundary $\partial_b^A \cup \partial_b^B$.
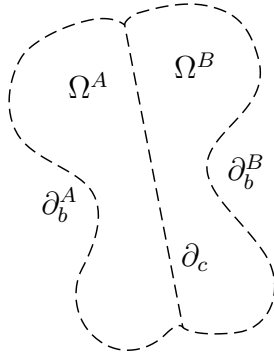

Figure 3: Composition of Dirac structures from the perspective of the associated boundaries

Let $j \in \{A, B\}$, let U^j, X^j, and Y^j be Hilbert spaces and assume that U^j and Y^j split into $U^j = U_b^j \oplus U_c$ and $Y^j = Y_b^j \oplus Y_c$. Let G^j, L^j, and K^j be linear operators with common domain $\mathrm{dom}\,(\Xi^j)$ dense in X^j that map into U^j, X^j, and Y^j, respectively. Split G^j and K^j according to the decomposition of U^j and Y^j into

$$G^j = \begin{bmatrix} G_b^j \\ G_c^j \end{bmatrix} \quad \text{and} \quad K^j = \begin{bmatrix} K_b^j \\ K_c^j \end{bmatrix}, \tag{5.1}$$

respectively. Furthermore, assume that there exist unitary operators

$$r_{U^j,Y^j} = \begin{bmatrix} r_{U_b^j,Y_b^j} & 0 \\ 0 & r_{U_c,Y_c} \end{bmatrix} : \begin{bmatrix} U_b^j \\ U_c \end{bmatrix} \to \begin{bmatrix} Y_b^j \\ Y_c \end{bmatrix}, \quad j = A, B.$$

Thus we obtain the colligations

$$\Xi^j = \begin{bmatrix} G_b^j \\ G_c^j \\ L^j \\ K_b^j \\ K_c^j \end{bmatrix}, \quad j = A, B \tag{5.2}$$

defined on the dense subspaces $\mathrm{dom}\,(\Xi^j)$ of X^j.

This leads naturally to the following set-up for split Dirac structures, see Definition 3.1 and formula (4.4): Let

$$\mathcal{E}^j = \begin{bmatrix} X^j \\ U_b^j \end{bmatrix}, \quad \mathcal{F}^j = \begin{bmatrix} X^j \\ Y_b^j \end{bmatrix}, \quad \mathcal{E}_2 = U_c, \quad \mathcal{F}_2 = Y_c, \quad j = A, B,$$

let $\mathcal{B}^j := \mathcal{F}^j \oplus \mathcal{F}_2 \times \mathcal{E}^j \oplus \mathcal{E}_2$, and define the subspace $\mathcal{D}^j \subset \mathcal{B}^j$ by

$$\mathcal{D}^j = \begin{bmatrix} L^j \\ K_b^j \\ \hline K_c^j \\ \hline I \\ \hline G_b^j \\ G_c^j \end{bmatrix} \mathrm{dom}\,(\Xi^j), \quad j = A, B. \tag{5.3}$$

Following Definition 3.2, we have that the composition of Tellegen or Dirac structures $\mathcal{D}^A$ and $\mathcal{D}^B$ is done via $f_2^A = -f_2^B$ and $e_2^A = e_2^B$. Hence if $\mathcal{D}^A$ and $\mathcal{D}^B$ in (5.3) are Tellegen structures this becomes

$$K_c^A x^A + K_c^B x^B = 0 \quad \text{and} \quad G_c^A x^A = G_c^B x^B \tag{5.4}$$

for $x^A \in \mathrm{dom}\,(\Xi^A)$ and $x^B \in \mathrm{dom}\,(\Xi^B)$. Let us now introduce the subspace

$$\mathrm{dom}\,(\Xi^{AB}) = \left\{ \begin{bmatrix} x^A \\ x^B \end{bmatrix} \mid x^j \in \mathrm{dom}\,(\Xi^j) \text{ and } (5.4) \text{ holds} \right\} \tag{5.5}$$

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of $X^A \oplus X^B$, the operators

$$G^{AB} = \begin{bmatrix} G_b^A & 0 \\ 0 & G_b^B \end{bmatrix} : \mathrm{dom}\,(\Xi^{AB}) \to U_b^A \oplus U_b^B,$$

$$L^{AB} = \begin{bmatrix} L^A & 0 \\ 0 & L^B \end{bmatrix} : \mathrm{dom}\,(\Xi^{AB}) \to X^A \oplus X^B,$$

$$K^{AB} = \begin{bmatrix} K_b^A & 0 \\ 0 & K_b^B \end{bmatrix} : \mathrm{dom}\,(\Xi^{AB}) \to Y_b^A \oplus Y_b^B$$

and the colligation $\Xi^{AB} = \begin{bmatrix} G^{AB} \\ L^{AB} \\ K^{AB} \end{bmatrix}$.

Proposition 5.1. If the colligations Ξ^A and Ξ^B are strong, then the colligation Ξ^{AB} is strong. Furthermore, if $\mathcal{D}^A$ and $\mathcal{D}^B$ in (5.3) are Tellegen structures, then

$$\mathcal{D}^A \circ \mathcal{D}^B = \begin{bmatrix} L^{AB} \\ K^{AB} \\ I \\ G^{AB} \end{bmatrix} \mathrm{dom}\,(\Xi^{AB}) \qquad (5.6)$$

is a Tellegen structure associated with the colligation Ξ^{AB}, on the bond space $\begin{bmatrix} X^A \oplus X^B \\ Y_b^A \oplus Y_b^B \end{bmatrix} \times \begin{bmatrix} X^A \oplus X^B \\ U_b^A \oplus U_b^B \end{bmatrix}$, with power product given by

$$r_{\mathcal{E},\mathcal{F}} = \begin{bmatrix} I_{X^A} & 0 & 0 & 0 \\ 0 & I_{X^B} & 0 & 0 \\ 0 & 0 & -r_{U_b^A, Y_b^A} & 0 \\ 0 & 0 & 0 & -r_{U_b^B, Y_b^B} \end{bmatrix}. \qquad (5.7)$$

Proof. By Proposition 3.3 we know that $\mathcal{D}^A \circ \mathcal{D}^B$ is a Tellegen structure and with the help of Definition 3.2 one easily verifies that $\mathcal{D}^A \circ \mathcal{D}^B$ is given by (5.6).

We check that L^{AB} is closed. Let $x_n := (x_n^A, x_n^B) \in \mathrm{dom}\,(\Xi^{AB})$ be a converging sequence in $X^A \oplus X^B$ such that $L^{AB} x_n$ converges to some $z := (z^A, z^B)$. By the definition of L^{AB} it is clear that $L^A x_n^A$ and $L^B x_n^B$ are both converging sequences. Since L^A and L^B are (by assumption) closed operators, we conclude that

$$x^A := \lim_{n \to \infty} x_n^A \in \mathrm{dom}\,(\Xi^A), \quad x^B := \lim_{n \to \infty} x_n^B \in \mathrm{dom}\,(\Xi^B), \qquad (5.8)$$

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and

$$z^A = \lim_{n \to \infty} L^A x_n^A \quad \text{and} \quad z^B = \lim_{n \to \infty} L^B x_n^B. \tag{5.9}$$

Thus if we can show that $(z^A, z^B) \in \mathrm{dom}\,(\Xi^{AB})$, then we have shown that L^{AB} is a closed operator. Since the colligation Ξ^A is strong, we conclude from Lemma 4.5 of [20] that the operators K^A and G^A in (5.1) are bounded with respect to the graph norm of L^A. Combining this with (5.8) and (5.9), we find $\lim_n K_c^A x_n^A = K_c^A x^A$ and $\lim_n G_c^A x_n^A = G_c^A x^A$. Similarly, we obtain that $\lim_n K_c^B x_n^B = K_c^B x^B$ and $\lim_n G_c^B x_n^B = G_c^B x^B$. In particular, since $(x_n^A, x_n^B) \in \mathrm{dom}\,(\Xi^{AB})$ this implies that

$$K_c^A x^A + K_c^B x^B = \lim_{n \to \infty} \left(K_c^A x_n^A + K_c^B x_n^B \right) = 0,$$
$$G_c^A x^A - G_c^B x^B = \lim_{n \to \infty} \left(G_c^A x_n^A - G_c^B x_n^B \right) = 0.$$

Thus $x = (x^A, x^B) \in \mathrm{dom}\,(\Xi^{AB})$ and $L^{AB} x = z$, i.e. L^{AB} is closed.

The closedness of Ξ^{AB} follows from the closedness of L^{AB}, the boundedness of G^{AB} and K^{AB} with respect to the graph norm, and Lemma 4.5 of [20]. □

Based on Theorem 4.3 and Propositions 4.5 and 5.1 we show that $\mathcal{D}^A \circ \mathcal{D}^B$ is a Dirac structure if $\mathcal{D}^A$ and $\mathcal{D}^B$ are Dirac structures.

Theorem 5.2. *Assume that the colligations Ξ^A and Ξ^B defined by (5.2) are strong and that $\mathcal{D}^A$ and $\mathcal{D}^B$ in (5.3) are Dirac structures. Then $\mathcal{D}^A \circ \mathcal{D}^B$ in (5.6) is a Dirac structure with (5.7), which is associated with the strong colligation Ξ^{AB}.*

Proof. By Propositions 5.1 and 4.2 we conclude that the first item of Theorem 4.3 holds.

It follows from Theorem 4.3 and Proposition 4.5 that the operators $\left[\begin{smallmatrix} G^A \\ K^A \end{smallmatrix} \right]$ and $\left[\begin{smallmatrix} G^B \\ K^B \end{smallmatrix} \right]$ are surjective. Now it is easy to see that $\left[\begin{smallmatrix} G^{AB} \\ K^{AB} \end{smallmatrix} \right]$ is surjective and hence item 3 of Theorem 4.3 is satisfied.

It remains to show that the minimal operator

$$L_0^{AB} = L^{AB}\big|_{\left\{ \left[\begin{smallmatrix} x^A \\ x^B \end{smallmatrix} \right] \in \mathrm{dom}\,(L^{AB}) \,\middle|\, G^{AB} \left[\begin{smallmatrix} x^A \\ x^B \end{smallmatrix} \right] = 0,\, K^{AB} \left[\begin{smallmatrix} x^A \\ x^B \end{smallmatrix} \right] = 0 \right\}}$$

of the colligation Ξ^{AB} is densely defined and has the property $(L_0^{AB})^* = -L^{AB}$. Therefore we recall the minimal operators of Ξ^A and Ξ^B

$$L_0^A = L^A\big|_{\{ x^A \in \mathrm{dom}\,(L^A) \,|\, K^A x^A = 0,\, G^A x^A = 0 \}},$$
$$L_0^B = L^B\big|_{\{ x^B \in \mathrm{dom}\,(L^B) \,|\, K^B x^B = 0,\, G^B x^B = 0 \}}.$$

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If we restrict the operator L_0^{AB} to $\mathrm{dom}\,(L_0^A) \oplus \mathrm{dom}\,(L_0^B)$, then we obtain that

$$\begin{bmatrix} L_0^A & 0 \\ 0 & L_0^B \end{bmatrix} \subset L_0^{AB}.$$

Since L_0^A and L_0^B are densely defined, we see that this implies that L_0^{AB} is densely defined. Furthermore, this relation implies that

$$\left(L_0^{AB}\right)^* \subset \begin{bmatrix} \left(L_0^A\right)^* & 0 \\ 0 & \left(L_0^B\right)^* \end{bmatrix} = \begin{bmatrix} -L^A & 0 \\ 0 & -L^B \end{bmatrix}, \tag{5.10}$$

where we have used Theorem 4.3 for L^A and L^B. In particular, we have that

$$\mathrm{dom}\left(\left(L_0^{AB}\right)^*\right) \subset \mathrm{dom}\,(\Xi^A) \oplus \mathrm{dom}\,(\Xi^B). \tag{5.11}$$

Let us verify the inclusion $(L_0^{AB})^* \subset -L^{AB}$. For $\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} \in \mathrm{dom}\left(\left(L_0^{AB}\right)^*\right)$ and $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathrm{dom}\,(L_0^{AB})$, we find by (5.10) that

$$0 = \left\langle L_0^{AB} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} \right\rangle_{X^A \oplus X^B} + \left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} L^A & 0 \\ 0 & L^B \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} \right\rangle_{X^A \oplus X^B}.$$

Combining (4.5) and Lemma 2.5 for L^A and L^B, we find that

$$0 = \left\langle L_0^{AB} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} \right\rangle_{X^A \oplus X^B} + \left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} L^A & 0 \\ 0 & L^B \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} \right\rangle_{X^A \oplus X^B}$$

$$= \left\langle \begin{bmatrix} L^A & 0 \\ 0 & L^B \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} \right\rangle_{X^A \oplus X^B}$$

$$\quad + \left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} L^A & 0 \\ 0 & L^B \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} \right\rangle_{X^A \oplus X^B}$$

$$= \left\langle r_{U^A, Y^A} \begin{bmatrix} G_b^A \\ G_c^A \end{bmatrix} x_1, \begin{bmatrix} K_b^A \\ K_c^A \end{bmatrix} \tilde{x}_1 \right\rangle_{Y^A}$$

$$\quad + \left\langle \begin{bmatrix} K_b^A \\ K_c^A \end{bmatrix} x_1, r_{U^A, Y^A} \begin{bmatrix} G_b^A \\ G_c^A \end{bmatrix} \tilde{x}_1 \right\rangle_{Y^A} +$$

$$\left\langle r_{U^B, Y^B} \begin{bmatrix} G_b^B \\ G_c^B \end{bmatrix} x_2, \begin{bmatrix} K_b^B \\ K_c^B \end{bmatrix} \tilde{x}_2 \right\rangle_{Y^B}$$

$$\quad + \left\langle \begin{bmatrix} K_b^B \\ K_c^B \end{bmatrix} x_2, r_{U^B, Y^B} \begin{bmatrix} G_b^B \\ G_c^B \end{bmatrix} \tilde{x}_2 \right\rangle_{Y^B}$$

$$= \left\langle r_{U_c, Y_c} G_c^A x_1, K_c^A \tilde{x}_1 \right\rangle_{Y_c} + \left\langle K_c^A x_1, r_{U_c, Y_c} G_c^A \tilde{x}_1 \right\rangle_{Y_c}$$

$$\quad + \left\langle r_{U_c, Y_c} G_c^B x_2, K_c^B \tilde{x}_2 \right\rangle_{Y_c} + \left\langle K_c^B x_2, r_{U_c, Y_c} G_c^B \tilde{x}_2 \right\rangle_{Y_c},$$

where we have used that $\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \in \mathrm{dom}\,(L_0^{AB})$. Using the composition relations (5.4), we find that for $\left[\begin{smallmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{smallmatrix}\right] \in \mathrm{dom}\,((L_0^{AB})^*)$ and $\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \in \mathrm{dom}\,(L_0^{AB})$ the relation

$$0 = \left\langle r_{U_c,Y_c} G_c^A x_1, K_c^A \tilde{x}_1 + K_c^B \tilde{x}_2 \right\rangle_{Y_c} + \left\langle r_{Y_c,U_c} K_c^A x_1, G_c^A \tilde{x}_1 - G_c^B \tilde{x}_2 \right\rangle_{Y_c} \quad (5.12)$$

holds. Since the operators $\left[\begin{smallmatrix} G^A \\ K^A \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} G^B \\ K^B \end{smallmatrix}\right]$ are surjective, it follows that $\left[\begin{smallmatrix} G_c^A \\ K_c^A \end{smallmatrix}\right]$ restricted to $\mathrm{dom}\,(L_0^{AB})$ is surjective. Combining this with (5.12), we conclude

$$K_c^A \tilde{x}_1 + K_c^B \tilde{x}_2 = 0 \quad \text{and} \quad G_c^A \tilde{x}_1 = G_c^B \tilde{x}_2,$$

and hence

$$\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} \in \mathrm{dom}\,(\Xi^{AB}) = \mathrm{dom}\,(L^{AB}) \quad \text{and} \quad L^{AB} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = -(L_0^{AB})^* \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}.$$

Therefore $(L_0^{AB})^* \subset -L^{AB}$. If $\left[\begin{smallmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{smallmatrix}\right] \in \mathrm{dom}\,(L^{AB})$, then we can read the above equation backwards, and we find that $\left[\begin{smallmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{smallmatrix}\right] \in \mathrm{dom}\,((L_0^{AB})^*)$. In other words, the domains are the same. By equation (5.10) we see that item 2 of Theorem 4.3 holds and therefore $\mathcal{D}^A \circ \mathcal{D}^B$ is a Dirac structure. $\square$

Theorem 5.2 also follows from [8, Thm 2.10(iv)] with the appropriate identifications.

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References

[1] T. Y. Azizov and I. S. Iokhvidov. *Linear Operators in Spaces with an Indefinite Metric*. Pure and Applied Mathematics (New York). John Wiley & Sons Ltd., Chichester, 1989. Translated from the Russian by E. R. Dawson.

[2] J. A. Ball and O. J. Staffans. Conservative state-space realizations of dissipative system behaviors. *Integral Equations Operator Theory*, 54(2):151–213, 2006.

[3] J. Behrndt and M. Langer. Boundary value problems for elliptic partial differential operators on bounded domains. *J. Funct. Anal.*, 243(2):536–565, 2007.

[4] J. Bognár. *Indefinite inner product spaces.* Springer-Verlag, New York, 1974. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 78.

[5] J. Cervera, A. J. van der Schaft, and A. Baños. Interconnection of port-Hamiltonian systems and composition of Dirac structures. *Automatica J. IFAC*, 43(2):212–225, 2007.

[6] M. Dalsmo and A. J. van der Schaft. On representations and integrability of mathematical structures in energy-conserving physical systems. *SIAM J. Control Optim.*, 37(1):54–91 (electronic), 1999.

[7] V. A. Derkach, S. Hassi, M. M. Malamud, and H. de Snoo. Boundary relations and their Weyl families. *Trans. Amer. Math. Soc.*, 358(12):5351–5400 (electronic), 2006.

[8] V. A. Derkach, S. Hassi, M. M. Malamud, and H. de Snoo. Boundary relations and generalized resolvents of symmetric operators. *Russ. J. Math. Phys.*, 16(1):17–60, 2009.

[9] V. A. Derkach and M. M. Malamud. Generalized resolvents and the boundary value problems for Hermitian operators with gaps. *J. Funct. Anal.*, 95(1):1–95, 1991.

[10] V. A. Derkach and M. M. Malamud. The extension theory of Hermitian operators and the moment problem. *J. Math. Sci.*, 73(2):141–242, 1995.

[11] I. Dorfman. *Dirac Structures and Integrability of Nonlinear Evolution Equations.* Nonlinear Science: Theory and Applications. John Wiley & Sons Ltd., Chichester, 1993.

[12] E. Godlewski and P.-A. Raviart. *Numerical Approximation of Hyperbolic Systems of Conservation Laws*, volume 118 of *Applied Mathematical Sciences.* Springer-Verlag, New York, 1996.

[13] G. Golo. *Interconnection Structures in Port-Based Modelling: Tools for Analysis and Simulation.* Twente University Press, 2002. Ph.D. Thesis. Available at `http://doc.utwente.nl/`.

[14] V. I. Gorbachuk and M. L. Gorbachuk. *Boundary value problems for operator differential equations*, volume 48 of *Mathematics and its Applications (Soviet Series)*. Kluwer Academic Publishers Group, Dordrecht, 1991. translated and revised from the 1984 Russian original.

[15] T. Kato. *Perturbation theory for linear operators.* Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition.

[16] M. Kurula. On passive and conservative state/signal systems in continuous time. To appear in Integral Equations Operator Theory, SpringerLink Online First version available with DOI 10.1007/s00020-010-1787-6, 2010.

[17] M. Kurula. *Towards Input/Output-Free Modelling of Linear Infinite-Dimensional Systems in Continuous Time.* PhD thesis, 2010. Available at `http://users.abo.fi/mkurula/`.

[18] M. Kurula and O. J. Staffans. Well-posed state/signal systems in continuous time. 2009. To appear in Complex Analysis and Operator Theory, SpringerLink Online First version available with DOI 10.1007/s11785-009-0021-5.

[19] Y. Le Gorrec, H. Zwart, and B. Maschke. Dirac structures and boundary control systems associated with skew-symmetric differential operators. *SIAM J. Control Optim.*, 44(5):1864–1892 (electronic), 2005.

[20] J. Malinen and O. J. Staffans. Impedance passive and conservative boundary control systems. *Complex Anal. Oper. Theory*, 1(2):279–300, 2007.

[21] A. Pazy. *Semi-Groups of Linear Operators and Applications to Partial Differential Equations.* Springer-Verlag, Berlin, 1983.

[22] R. M. Redheffer. On a certain linear fractional transformation. *J. Math. and Phys.*, 39:269–286, 1960.

[23] J. L. Šmul′jan. Theory of linear relations, and spaces with indefinite metric. *Funkcional. Anal. i Priložen.*, 10(1):67–72, 1976.

[24] O. J. Staffans. Passive linear discrete time-invariant systems. In *International Congress of Mathematicians. Vol. III*, pages 1367–1388. Eur. Math. Soc., Zürich, 2006.

[25] A. J. van der Schaft. Interconnection and geometry. In *The Mathematics of Systems and Control: From Intelligent Control to Behavior Systems*. University of Groningen, 1999.

[26] A. J. van der Schaft. L_2-*Gain and Passivity Techniques in Nonlinear Control*, volume 218 of *Springer Communications and Control Engineering series*. Springer-Verlag, London, 2000. 2nd revised and enlarged edition.

[27] A. J. van der Schaft and B. M. Maschke. Hamiltonian formulation of distributed-parameter systems with boundary energy flow. *J. Geom. Phys.*, 42(1-2):166–194, 2002.

[28] J. A. Villegas. *A port-Hamiltonian approach to distributed parameter systems*. PhD thesis, May 2007. Available at `http://doc.utwente.nl/`.

[29] K. Zhou, J. C. Doyle, and K. Glover. *Robust and Optimal Control*. Prentice–Hall, Englewood Cliffs, New Jersey, 1996.