Canonical State/Signal Shift Realizations of Passive Continuous Time Behaviors

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Abstract

This work is devoted to the construction of canonical passive and conservative state/signal shift realizations of arbitrary passive continuous time behaviors. By definition, a passive future continuous time behavior is a maximal nonnegative right-shift invariant subspace of the Kreĭn space $L^2([0,\infty);\mathcal{W})$, where $\mathcal{W}$ is a Kreĭn space, and the inner product in $L^2([0,\infty);\mathcal{W})$ is the one inherited from $\mathcal{W}$. A state/signal system $\Sigma = (V;\mathcal{X},\mathcal{W})$, with a Hilbert state space $\mathcal{X}$ and a Kreĭn signal space $\mathcal{W}$, is a dynamical system whose classical trajectories (x,w) on $[0,\infty)$ satisfy $x \in C^1([0,\infty);\mathcal{X})$, $w \in C([0,\infty);\mathcal{W})$, and

$$(\dot{x}(t), x(t), w(t)) \in V, \qquad t \in [0,\infty),$$

where the generating subspace V is a given subspace of the node space $\mathfrak{K} := \mathcal{X} \times \mathcal{X} \times \mathcal{W}$. Passivity of this systems means that V is maximal nonnegative with respect to a certain Kreĭn space inner product on $\mathfrak{K}$, and that $(z,0,0) \in V$ implies $z = 0$.

We present three canonical passive shift models: a) an observable and co-energy preserving model, b) a controllable and energy preserving model, and c) a simple conservative model. In order to construct these models we first introduce the notions of the input map, the output map, and the past/future map of a passive state/signal system. Our canonical passive state/signal shift realizations are analogous to the corresponding de Branges–Rovnyak type input/state/output realizations of a given Schur function.

Keywords

Passive, conservative, behavior, state/signal system, de Branges–Rovnyak model, input/state/output system, input map, output map, past/future map, transfer function, Krein space.

Contents

1 Introduction

A linear continuous time invariant s/s (state/signal) system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ has a Hilbert (state) space, a Kreĭn (signal) space, and a closed (generating) subspace V of the (node) space $\mathfrak{K} = \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix}$ that satisfies some additional conditions, among them the condition

$$\begin{bmatrix} z \\ 0 \\ 0 \end{bmatrix} \in V \Rightarrow z = 0. \tag{1.1}$$

Condition (1.1) means that V is the graph of some linear operator $G\colon \begin{bmatrix} \mathcal{X} \\ \mathcal{W} \end{bmatrix} \to \mathcal{X}$ with domain $\operatorname{dom}(G) \subset \begin{bmatrix} \mathcal{X} \\ \mathcal{W} \end{bmatrix}$. Since V is assumed to be closed, the operator G is closed. The reason for taking $\mathcal{W}$ to be a Kreĭn space instead of a Hilbert space will be explained later when we come to the notion of a *passive* s/s system.

Let $I \subset \mathbb{R}$ be a time interval with positive length. By a *classical trajectory* of Σ on I we mean a pair of functions $\begin{bmatrix} x \\ w \end{bmatrix} \in \begin{bmatrix} C^1(I;\mathcal{X}) \\ C(I;\mathcal{W}) \end{bmatrix}$ satisfying

$$\Sigma : \begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix} \in V, \qquad t \in I, \tag{1.2}$$

or equivalently,

$$\Sigma : \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \in \operatorname{dom}(G) \text{ and } \dot{x}(t) = G \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}, \qquad t \in I. \tag{1.3}$$

By a *generalized trajectory* of Σ on I we mean a pair of functions $\begin{bmatrix} x \\ w \end{bmatrix} \in \begin{bmatrix} C(I;\mathcal{X}) \\ L^2_{\mathrm{loc}}(I;\mathcal{W}) \end{bmatrix}$ which is the limit in this space of a sequence $\begin{bmatrix} x_n \\ w_n \end{bmatrix}$ of classical trajectories of Σ on I.

We call Σ *passive* if its generating subspace V satisfies two additional conditions. The first condition is related to the fact that its classical trajectories should satisfy the (power) inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} \|x(t)\|^2_{\mathcal{X}} \le [w(t), w(t)]_{\mathcal{W}}, \qquad t \in I, \tag{1.4}$$

or equivalently,

$$-(\dot{x}(t), x(t))_{\mathcal{X}} - (\dot{x}(t), x(t))_{\mathcal{X}} + [w(t), w(t)]_{\mathcal{W}} \ge 0, \qquad t \in I. \tag{1.5}$$

Above $\|x(t)\|^2_{\mathcal{X}}$ stands for the *internal energy* stored in the state x at time t and $[w(t), w(t)]_{\mathcal{W}}$ represents the power flow (energy flow per time unit) into the system at time t through the signal $w(t)$. Incidentally, this explains why we need to allow the inner product in $\mathcal{W}$ to be indefinite: If the inner product in $\mathcal{W}$ is positive, then no energy can leave the system through the signal, and if the inner product in $\mathcal{W}$ is negative, then no energy can enter the system. The inequality (1.4) says that the system has no internal energy sources.

By (1.2), a sufficient condition for (1.4) and (1.5) to hold is that

$$-(z,x)_{\mathcal{X}} - (z,x)_{\mathcal{X}} + [w,w]_{\mathcal{W}} \geq 0, \qquad \begin{bmatrix} z \\ x \\ w \end{bmatrix} \in V. \tag{1.6}$$

This makes it natural to introduce the following (strictly indefinite) Kreĭn space inner product in the node space $\mathfrak{K}$:

$$\left[\begin{bmatrix} z_1 \\ x_1 \\ w_1 \end{bmatrix}, \begin{bmatrix} z_2 \\ x_2 \\ w_2 \end{bmatrix}\right]_{\mathfrak{K}} = -(z_1,x_2) - (x_1,z_2) + [w_1,w_2]_{\mathcal{W}}. \tag{1.7}$$

Then (1.6) says that V is a nonnegative subspace of $\mathfrak{K}$ with respect to the inner product (1.7), and (1.5) can be rewritten in the form

$$\left[\begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix}, \begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix}\right]_{\mathfrak{K}} = -\frac{\mathrm{d}}{\mathrm{d}t}\,\|x(t)\|^2_{\mathcal{X}} + [w(t),w(t)]_{\mathcal{W}} \geq 0, \qquad t \in I. \tag{1.8}$$

The first condition that we require of the system $\Sigma = (V;\mathcal{W},\mathcal{W})$, in addition to (1.1), in order to call it passive is that V *is a nonnegative subspace of the node space* $\mathfrak{K}$ with respect to the inner product in (1.7).

The second condition that we require of Σ in order to be passive is a maximality condition. It is not enough to require V to be nonnegative in $\mathfrak{K}$, but it should be even *maximal nonnegative*, i.e., it should not be properly contained in any other nonnegative subspace of $\mathfrak{K}$. This condition is analogous to the condition that one needs to impose on an operator A in order for A to generate a C_0 contraction semigroup on a Hilbert space $\mathcal{X}$: It is not enough that A is dissipative, but it must, in fact, be *maximal dissipative.*

Thus, summarizing the preceding discussion, $\Sigma = (V;\mathcal{X},\mathcal{W})$ *is a passive s/s system if* V *is a maximal nonnegative subspace of* $\mathfrak{K}$ *with respect to the inner product in* (1.7) *and* (1.1) *holds.* Note, in particular, that the maximality of V implies that V is closed.

One often encounters passive s/s systems where (1.4)–(1.6) hold as equalities instead of inequalities. Such systems are called *energy preserving.* Thus, a passive energy preserving system is characterized by the fact that V is maximal nonnegative and neutral, i.e., $V \subset V^{[\perp]}$, where $V^{[\perp]}$ is the orthogonal companion to V in $\mathfrak{K}$. If instead V is maximal nonnegative and co-neutral, i.e., $V^{[\perp]} \subset V$, then Σ is called *co-energy preserving.* Finally, if V is Lagrangian, i.e., if $V = V^{[\perp]}$, then V is called *conservative.*

By integrating (1.4) over the interval $[s,t]$ one can rewrite (1.4) in the equivalent form

$$\|x(t)\|_{\mathcal{X}}^2 - \|x(s)\|_{\mathcal{X}}^2 \le \int_s^t [w(v),w(v)]_{\mathcal{W}}\,\mathrm{d}v, \qquad s,t\in I,\ s\le t. \tag{1.9}$$

By continuity of the integral, (1.9) remains valid for all generalized trajectories of Σ on I. The right-hand side of (1.9) can be interpreted as the inner product of the function w restricted to the interval $[s,t]$ with itself in a certain Kreĭn space. For each interval I with positive length we define the Kreĭn space $K^2(I;\mathcal{W})$ to be the space which coincides with $L^2(I;\mathcal{W})$ as a topological vector space, equipped with the inner product

$$[w_1,w_2]_{K^2(I;\mathcal{W})} := \int_I [w_1(s),w_2(s)]_{\mathcal{W}}\,\mathrm{d}s. \tag{1.10}$$

The quadratic form $[w,w]_{K^2(I;\mathcal{W})}$ measures the amount of energy that enters the s/s system $\Sigma=(V;\mathcal{X},\mathcal{W})$ through the signal w during the time interval I when its evolution is described by a generalized trajectory $\left[\begin{smallmatrix} x\\ w\end{smallmatrix}\right]$ with $w\in L^2(I;\mathcal{W})$. We shall be especially interested in the cases where $I=\mathbb{R}^+$, $I=\mathbb{R}$, or $I=\mathbb{R}^-$, and denote

$$K^2(\mathcal{W}) := K^2(\mathbb{R};\mathcal{W}), \quad K^2_{\pm}(\mathcal{W}) := K^2(\mathbb{R}^{\pm};\mathcal{W}). \tag{1.11}$$

In view of (1.9), the family $\mathfrak{K}_{0,t} := \left[\begin{smallmatrix} \mathcal{X}\\ \mathcal{X}\\ K^2([0,t];\mathcal{W})\end{smallmatrix}\right]$, $t\in\mathbb{R}^+$, of Kreĭn spaces which the indefinite inner products

$$\left[\left[\begin{smallmatrix} z_1\\ x_1\\ w_1\end{smallmatrix}\right], \left[\begin{smallmatrix} z_2\\ x_2\\ w_2\end{smallmatrix}\right]\right]_{\mathfrak{K}_{0,t}} = -(z_1,z_2)+(x_1,x_2)+[w_1,w_2]_{K^2([0,t];\mathcal{W})} \tag{1.12}$$

enters naturally into the theory of passive s/s systems. Indeed, inequality (1.9) says that if we denote

$$\mathcal{T}_{0,t} := \left\{ \left[\begin{smallmatrix} x(t)\\ x(0)\\ \pi_{[0,t]}w\end{smallmatrix}\right] \,\middle|\, \left[\begin{smallmatrix} x\\ w\end{smallmatrix}\right] \text{ is a generalized trajectory of } \Sigma \text{ on } [0,t] \right\},$$

then $\mathcal{T}_{0,t}$ is a nonnegative subspace of $\mathfrak{K}_{0,t}$ for all $t\in\mathbb{R}^+$. As will be shown in Theorem 3.5 below, it is even maximal nonnegative, and this fact is an important ingredient in the complete characterization given in that theorem of a passive s/s system in terms of its trajectories $\left[\begin{smallmatrix} x\\ w\end{smallmatrix}\right]$ on $\mathbb{R}^+$ satisfying $w\in K^2_+(\mathcal{W})$. Moreover, the generating subspace V can be recovered from the family $\mathcal{T}_{0,t}$ in a way that is analogous to the definition of the generator of a C_0 semigroup.

Intuitively, the state space $\mathcal{X}$ of Σ plays the role of an internal memory, and at each time t the state vector $x(t)$ contains the part of the past history of the system which may have some influence on the future dynamics. All the exchange of information with the environment takes place via the signal part w of a trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$. Two s/s systems Σ_1 and Σ_2 with the same signal space $\mathcal{W}$ are *externally equivalent* if they cannot be distinguished from each other by observing only the signal parts w of the trajectories $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ of the two systems on the time interval $I = \mathbb{R}^+ := [0, \infty)$ whose initial states are zero (i.e., the two systems start "from rest" at time $t = 0$). Trajectories of this type are called *externally generated* on $\mathbb{R}^+$. We call an externally generated trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ on $\mathbb{R}^+$ *stable* if $w \in L^2(\mathbb{R}^+; \mathcal{W})$, and by the (stable) *future behavior* $\mathfrak{W}_+^\Sigma$ of Σ we mean the set of all the signal parts w of the externally generated stable trajectories $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ of Σ on $\mathbb{R}^+$.

The future behavior $\mathfrak{W}_+^\Sigma$ of a passive s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ has two characteristic properties. One of them is fairly obvious, namely that $\mathfrak{W}_+^\Sigma$ is *right-shift invariant* in the Kreĭn space $K_+^2(\mathcal{W})$. This follows from the fact that if $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is an externally generated trajectory of Σ on $\mathbb{R}^+$, and if we shift this trajectory to the right by the amount t and define $\left[\begin{smallmatrix} x(s) \\ w(s) \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$ for $0 \le s < t$, then we obtain another externally generated trajectory on $\mathbb{R}^+$. This means that if we denote the right-shift semigroup in $K_+^2(\mathcal{W})$ by τ_+^* (i.e., $(\tau_+^{*t} w)(s) = w(s - t)$ if $s \ge t$ and $(\tau_+^{*t} w)(s) = 0$ otherwise), then $\tau_+^{*t} \mathfrak{W}_+^\Sigma \subset \mathfrak{W}_+^\Sigma$ for all $t \in \mathbb{R}^+$.

According to the above discussion, $\mathfrak{W}_\Sigma^+$ is also a nonnegative subspace of $K_+^2(\mathcal{W})$. As we shall show in Section 3, $\mathfrak{W}_+^\Sigma$ is even *maximal nonnegative*. This is the second characteristic property of $\mathfrak{W}_+^\Sigma$ that we mentioned above. In particular, $\mathfrak{W}_+^\Sigma$ is closed in $K_+^2(\mathcal{W})$.

The above facts lead us to the following definition. By a *passive future behavior* on the Kreĭn signal space $\mathcal{W}$ we mean a maximal nonnegative, right-shift invariant subspace of $K_+^2(\mathcal{W})$. According to the above discussion, the future behavior of a passive s/s system with signal space $\mathcal{W}$ is a *passive* future behavior on $\mathcal{W}$.

This paper ends with a study of the inverse problem, in which we construct three different *canonical shift realizations of an arbitrary passive future behavior* $\mathfrak{W}_+$. By this we mean the following. A passive s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is a *realization* of $\mathfrak{W}_+$ if the future behavior $\mathfrak{W}_+^\Sigma$ of Σ is equal to $\mathfrak{W}_+$. By a *canonical realization* we mean a realization that is completely determined by the given data $\mathfrak{W}_+$. By a *shift realization* we mean a realization whose dynamics can be interpreted (in a generalized sense) as a compressed shift in some space of functions with values in the given signal space $\mathcal{W}$.

Our three canonical realizations have the following characteristic properties: (a) the first realization is observable and co-energy preserving, (b) the second is controllable and energy preserving, and (c) the third is simple and conservative (see Section 3 for the precise definitions). It will be shown that every other passive s/s realization that has one of the properties (a), (b), or (c) is unitarily similar to the corresponding canonical realization that we have constructed, and for this reason we also call them *canonical models* of passive s/s systems with one of the properties (a), (b), or (c).

The canonical models mentioned above are analogous to the three canonical de Branges–Rovnyak shift models of types (a), (b), and (c) with a given scattering matrix that belongs to the Schur class over the right half-plane, i.e., it is an analytic function whose values are contractive operators from one Hilbert space $\mathcal{U}$ to another Hilbert space $\mathcal{Y}$. They were originally presented in [dBR66a, dBR66b] (in discrete time), and they can also be found in, e.g., [ADRdS97] and in [NV89, NV98]. Indeed, there is a two-sided connection between our canonical models and the de Branges–Rovnyak models analogous to the one described in [AS10] in the discrete time case. Since the continuous time result looks more or less the same as the discrete time results (with the unit disc replaced by the right half-plane), and since this article is already quite long, we have chosen to here give only a very short outline of this connection. A future passive behavior $\mathfrak{W}_+$ can be mapped into the frequency domain by use of the Laplace transform, and we denote the image by $\widehat{\mathfrak{W}}_+$. This is a maximal nonnegative shift-invariant subspace of the Kreĭn–Hardy space $H^2_+(\mathcal{W})$ over the right-half plane with values in $\mathcal{W}$ (the inner product in $H^2_+(\mathcal{W})$ is the one inherited from $\mathcal{W}$). Each fundamental decomposition $\mathcal{W} = -\mathcal{W}_- \boxplus \mathcal{W}_+$ gives rise to a fundamental decomposition $H^2_+(\mathcal{W}) = -H^2_+(\mathcal{W}_-) \boxplus H^2_+(\mathcal{W}_+)$, and with respect to this decomposition $\widehat{\mathfrak{W}}_+$ has a graph representation

$$\widehat{\mathfrak{W}}_+ = \left\{ \left[\begin{smallmatrix} \widehat{S}\widehat{w}_+ \\ \widehat{w}_+ \end{smallmatrix} \right] \middle| \, \widehat{w}_+ \in H^2_+(\mathcal{W}_+) \right\},$$

where $\widehat{S}$ is a multiplication operator whose symbol φ is a Schur function mapping $\mathcal{W}_+$ into $\mathcal{W}_-$. This symbol φ is called a scattering matrix. It is uniquely determined by $\widehat{\mathfrak{W}}_+$ and the decomposition $\mathcal{W} = -\mathcal{W}_- \boxplus \mathcal{W}_+$, but of course, different fundamental decompositions of $\mathcal{W}$ lead to different scattering matrices. From our canonical s/s models of a passive s/s system with a given future behavior $\mathfrak{W}_+$ we can derive the corresponding de Branges–Rovnyak canonical models with the given scattering matrix φ by first passing to the i/s/o representation of the given s/s system corresponding to the fundamental decomposition $\mathcal{W} = -\mathcal{W}_- \boxplus \mathcal{W}_+$, and then applying certain unitary maps. It is also in principle possible to proceed in the opposite direction, i.e., to start with one of the de Branges–Rovnyak models and to define a state signal system in terms of an i/s/o scattering representation, but this does not lead to a canonical s/s model, since the result depends on some arbitrarily chosen fundamental decomposition of the signal space. See [AS10] for details. By replacing the fundamental decompositions of $\mathcal{W}$ used above by Lagrangian decompositions or orthogonal decompositions (these are defined in Section 2 below) one can also derive canonical models whose transfer functions belong to certain subclasses of Nevanlinna functions or Potapov functions. We shall return to this elsewhere.

We end this introduction with a short overview of the remaining sections. In Section 2 we review the notion of a Kreĭn space and present some Kreĭn space results that will be needed later. Some background on passive s/s systems is presented in Section 3. In addition to the future behaviors $\mathfrak{W}_+^\Sigma$ of a passive s/s system Σ we also introduce the *past behavior* $\mathfrak{W}_-^\Sigma$ and the *full behavior* $\mathfrak{W}^\Sigma$ of Σ. They are in principle constructed in the same way as $\Sigma_+^{\mathfrak{W}}$, but with $I = \mathbb{R}^+$ replaced by $I = \mathbb{R}^- = (-\infty, 0]$ or $I = \mathbb{R}$, respectively. We also introduce the general notions of a *passive past behaviors* and a *passive full behaviors* on a given signal space $\mathcal{W}$ (without reference to any s/s system). It will always be true that if $\mathfrak{W}$ is a passive full behavior on $\mathcal{W}$, then

$$\mathfrak{W}_- := \{w \in K_-^2(\mathcal{W}) \mid w \text{ is the restriction to } \mathbb{R}^- \text{ of a function in } \mathfrak{W}\}$$

is a passive past behavior on $\mathcal{W}$, and

$$\mathfrak{W}_+ := \{w \in \mathfrak{W} \mid w(t) = 0 \text{ for } t < 0\}.$$

is a passive future behavior on $\mathcal{W}$. As shown in Proposition 3.15 below, either one of $\mathfrak{W}_-$ and $\mathfrak{W}_+$ determines $\mathfrak{W}$ uniquely. It is also true that the past and full behaviors of a passive s/s system Σ are *passive* past and full behaviors.

Many of the results in Section 3 are either taken from [KS09] and [Kur10] or are straightforward extensions of results in [AS09b]. However, it also contains some significant new results, the most important of which is Theorem 3.5 which gives necessary and sufficient conditions on a subspace $\mathcal{T}_+$ of $\begin{bmatrix} BUC(\mathbb{R}^+;\mathcal{X}) \\ K_+^2(\mathcal{W}) \end{bmatrix}$ to be the family all stable future generalized trajectories of some passive s/s system. Adjoint systems and behaviors, as well as anti-passive time reflected s/s systems are studied in Section 4.

In Sections 5.2 we present two Hilbert spaces $\mathcal{H}(\mathfrak{W}_+)$ and $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$ that play fundamental roles in the rest of this article. Here $\mathcal{H}(\mathfrak{W}_+)$ is the subspace of the quotient $K_+^2(\mathcal{W})/\mathfrak{W}_+$ consisting of all those equivalence classes whose $\mathcal{H}(\mathfrak{W}_+)$-norm, defined in (5.8) below, is finite. The Hilbert space $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$ is constructed in a similar way, with $\mathfrak{W}_+$ replaced by the orthogonal companion to a passive past behavior $\mathfrak{W}_-$, interpreted as a maximal nonnegative subspace of $-K^2_-(\mathcal{W})$. Both of these spaces are special cases of the spaces $\mathcal{H}(\mathcal{Z})$ introduced and studied in [AS09a], where $\mathcal{Z}$ is a maximal nonnegative subspace of a Kreĭn space $\mathcal{X}$. A short review of the spaces $\mathcal{H}(\mathcal{Z})$ is included in Section 2.

In Section 5.3 we introduce the past/future map $\Gamma_{\mathfrak{W}}$ of a passive full behavior $\mathfrak{W}$. This map plays a decisive role in our study of the construction of our three canonical realizations. It is a contraction from $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$ to $\mathcal{H}(\mathfrak{W}_+)$, and it is uniquely determined by the property that if $w \in \mathfrak{W}$ and if w_- and w_+ are the restrictions of w to $\mathbb{R}^-$ and $\mathbb{R}^+$, respectively, then the image under $\Gamma_{\mathfrak{W}}$ of the equivalence class in $K^2_-(\mathcal{W})/\mathfrak{W}_-^{[\perp]}$ containing w_- is the equivalence class in $K^2_+(\mathcal{W})/\mathfrak{W}_+$ containing w_+. This map is used to construct a third Hilbert space $\mathcal{D}(\mathfrak{W})$, which is continuously contained in $\begin{bmatrix} \mathcal{H}(\mathfrak{W}_-^{[\perp]}) \\ \mathcal{H}(\mathfrak{W}_+) \end{bmatrix}$. The space $\mathcal{D}(\mathfrak{W})$ has the property that the operators $T_- := \begin{bmatrix} \Gamma_{\mathfrak{W}} \\ 1_{\mathcal{H}(\mathfrak{W}_-^{[\perp]})} \end{bmatrix}$ and $T_+ := \begin{bmatrix} 1_{\mathcal{H}(\mathfrak{W}_+)} \\ \Gamma_{\mathfrak{W}}^* \end{bmatrix}$ map $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$ and $\mathcal{H}(\mathfrak{W}_+)$ isometrically onto subspaces $\mathcal{L}_-$ and $\mathcal{L}_+$ of $\mathcal{D}(\mathfrak{W})$, respectively, where $\mathcal{L}_- + \mathcal{L}_+$ is dense in $\mathcal{D}(\mathfrak{W})$ and the angle operator $P_{\mathcal{L}_+}|_{\mathcal{L}_-}$ (the orthogonal projection onto $\mathcal{L}_+$ restricted to $\mathcal{L}_-$) is given by $P_{\mathcal{L}}|_{\mathcal{L}_-} = T_+\Gamma_{\mathfrak{W}}T_-^{-1}$. Thus, the angle operator $P_{\mathcal{L}}|_{\mathcal{L}_-}$ between the subspaces $\mathcal{L}_-$ and $\mathcal{L}_+$ in $\mathcal{D}(\mathfrak{W})$ is a unitary image of $\Gamma_{\mathfrak{W}}$.

In Section 6 we develop the passive s/s systems theory further and introduce the input map $\mathfrak{B}_\Sigma$ and the output map $\mathfrak{C}_\Sigma$ of a passive s/s system Σ. Here $\mathfrak{B}_\Sigma$ is a contraction from $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$ to $\mathcal{X}$, and it is the unique extension to $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$ of the map from the equivalence class in $K^2_-(\mathcal{W})/\mathfrak{W}_-^{[\perp]}$ containing the signal part w of an externally generated stable trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ on $\mathbb{R}^-$ to $x(0)$. The operator $\mathfrak{C}_\Sigma$ is a contraction from $\mathcal{X}$ to $\mathcal{H}(\mathfrak{W}_+)$, and it is equal to the map from the initial state $x(0)$ of a stable trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ on $\mathbb{Z}^+$ to its signal part w factored over the future behavior $\mathfrak{W}_+$. As shown in Section 7, $\Gamma_{\mathfrak{W}} = \mathfrak{C}_\Sigma\mathfrak{B}_\Sigma$ whenever Σ is a passive s/s system with full behavior $\mathfrak{W}$.

Finally, in Sections 8–10 we present our canonical observable co-energy preserving s/s shift model, controllable energy preserving s/s shift model, and simple conservative s/s shift model, respectively, whose future, past, and full behaviors coincide with the given triple of passive behaviors $\mathfrak{W}_+$, $\mathfrak{W}_-$, and $\mathfrak{W}$, respectively. These models are canonical in the sense that they are uniquely determined by the given data $\mathfrak{W}_+$, $\mathfrak{W}_-$, or $\mathfrak{W}$ (any one of these three behaviors determines the other two uniquely). The state spaces in the co-energy preserving canonical model, the energy preserving canonical model, and the simple conservative canonical model are $\mathcal{H}(\mathfrak{W}_+)$, $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$, and $\mathcal{D}(\mathfrak{W})$, respectively. In all cases the dynamics of the models are described by means of a generalized compression of a shift acting in the state space.

This article may be regarded as an blend of [AS09a], [AS09b], and [AS10] on one hand and of [KS09] and [Kur10] on the other hand. In the first three of these canonical models of passive s/s systems were obtained in a discrete time setting, and in the last two a s/s theory is developed for the continuous time setting, including the passive case. Some preliminary steps towards the development of a s/s theory in continuous time were taken already in [BS06] by J. Ball and O. J. Staffans. See, in particular, [BS06] for a discussion of the connection with the theory of passive and conservative behaviors presented in the papers [Wil72a, Wil72b, WT98, WT02] and the monograph [PW98]. As explained in [AS05], part of the motivation comes from classical passive time-invariant circuit theory found in, e.g., [Bel68] and [Woh69].

List of Notations.

$\mathbb{R}, \mathbb{R}^+, \mathbb{R}^-$	$\mathbb{R} := (-\infty, \infty)$, $\mathbb{R}^+ := [0, \infty)$, $\mathbb{R}^- = (-\infty, 0]$.
$\mathbb{Z}, \mathbb{Z}^+, \mathbb{Z}^-$	$\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$, $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$, $\mathbb{Z}^- = \{-1, -2, \ldots\}$.
$\overline{\Omega}$	The closure of Ω.
$\mathcal{B}(\mathcal{U}; \mathcal{Y})$	The space of bounded linear operators from $\mathcal{U}$ to $\mathcal{Y}$.
$\operatorname{dom}(A), \operatorname{im}(A), \ker(A)$:	The domain, range, and kernel of the operator A.
$A\|_{\mathcal{Z}}$	The restriction of the operator A to $\mathcal{Z}$.
$1_{\mathcal{X}}$	The identity operator on $\mathcal{X}$.
$(\cdot, \cdot)_{\mathcal{X}}$	The inner product in the Hilbert space $\mathcal{X}$.
$[\cdot, \cdot]_{\mathcal{W}}$	The inner product in the Kreĭn space $\mathcal{W}$.
$-\mathcal{K}$	The anti-space of the Kreĭn space $\mathcal{K}$. This is the same topological vector space as $\mathcal{K}$, but it has a different inner product $[\cdot, \cdot]_{-\mathcal{K}} := -[\cdot, \cdot]_{\mathcal{K}}$.
τ^t	$(\tau^t w)(s) = w(s+t)$, $s, t \in \mathbb{R}$ (this is a left shift if $t > 0$).
τ^t_+	$(\tau^t_+ w)(s) = w(s+t)$, $s, t \in \mathbb{R}^+$ (this is a left shift if $t > 0$).
τ^t_-	$(\tau^t_- w)(s) = w(s+t)$ if $s + t \leq 0$, $(\tau^t_- w)(s) = 0$ if $s + t > 0$. Here $s \in \mathbb{R}^-$, $t \in \mathbb{R}^+$.
τ^{*t}	$(\tau^{*t} w)(s) = (\tau^{-1} w)(s) = w(s-t)$, $s, t \in \mathbb{R}$ (this is a right shift if $t > 0$).
τ^{*t}_+	$(\tau^{*t}_+ w)(s) = w(s-t)$ if $s-t \geq 0$ and $(\tau^{*t}_+ w)(s) = 0$ if $s-t < 0$. Here $s, t \in \mathbb{R}^+$.
τ^{*t}_-	$(\tau^{*t}_- w)(s) = w(s-t)$ for all $s \in \mathbb{R}^-$, $t \in \mathbb{R}^+$.
π_I, π_+, π_-	$(\pi_I w)(s) = w(s)$ if $s \in I$, $(\pi_I w)(s) = 0$ if $s \notin I$. We abbreviate $\pi_- = \pi_{\mathbb{R}^-}$ and $\pi_+ = \pi_{\mathbb{R}^+}$.

$C(I;\mathcal{X}), BUC(I;\mathcal{X}), C^1(I;\mathcal{X})$: The spaces of continuous, bounded uniformly continuous, or continuously differentiable functions, respectively, on I with values in $\mathcal{X}$, with the standard norms.

$L^2_{\rm loc}(I;\mathcal{W})$ — The space of functions from I to $\mathcal{W}$ which belong locally to L^2.

$K^2(I;\mathcal{X}), K^2(\mathcal{W}), K^2_+(\mathcal{W}), K^2_-(\mathcal{W})$: See (1.10) and (1.11).

$\mathcal{H}(\mathcal{Z}), \mathcal{H}^0(\mathcal{Z})$ — See Section 2.2.

$\mathfrak{W}, \mathfrak{W}_+, \mathfrak{W}_-$ — A passive full, future, or past behavior, respectively, on the Kreĭn signal space $\mathcal{W}$.

$\mathcal{H}_+, \mathcal{H}(\mathfrak{W}_+)$ — $\mathcal{H}_+ := \mathcal{H}(\mathfrak{W}_+)$ is defined in Theorem 5.1.

Q_+ — $Q_+ : w \mapsto w+\mathfrak{W}_+$ is the quotient map $K^2_+(\mathcal{W}) \mapsto K^2_+(\mathcal{W})/\mathfrak{W}_+$.

$\mathcal{H}^0_+, \mathcal{H}^0(\mathfrak{W}_+)$ — $\mathcal{H}^0_+ := \mathcal{H}^0(\mathfrak{W}_+) := Q_+\mathfrak{W}_+^{[\perp]}$.

$\mathcal{K}(\mathfrak{W}_+)$ — $\mathcal{K}(\mathfrak{W}_+) := Q_+^{-1}\mathcal{H}(\mathfrak{W}_+)$.

$\mathcal{H}_-, \mathcal{H}(\mathfrak{W}_-^{[\perp]})$ — $\mathcal{H}_- := \mathcal{H}(\mathfrak{W}_-^{[\perp]})$ is defined in Theorem 5.4.

Q_- — $Q_- : w \mapsto w+\mathfrak{W}_-^{[\perp]}$ is the quotient map $K^2_-(\mathcal{W}) \mapsto K^2_-(\mathcal{W})/\mathfrak{W}_-^{[\perp]}$.

$\mathcal{H}^0_-, \mathcal{H}^0(\mathfrak{W}_-)$ — $\mathcal{H}^0_- := \mathcal{H}^0(\mathfrak{W}_-) := Q_-\mathfrak{W}_-$.

$\mathcal{K}(\mathfrak{W}_-)$ — $\mathcal{K}(\mathfrak{W}_-) := Q_-^{-1}\mathcal{H}(\mathfrak{W}_-^{[\perp]})$.

$\Gamma_{\mathfrak{W}}$ — The past/future map of $\mathfrak{W}$. See Definition 5.8 and Section 7.

$\mathcal{D}(\mathfrak{W})$ — $\mathcal{D}(\mathfrak{W})$ is defined before Lemma 5.9.

Q — The quotient map $K^2(\mathcal{W}) \mapsto K^2(\mathcal{W})/(\mathfrak{W}_+ + \mathfrak{W}_-^{[\perp]})$.

$\mathcal{D}^0(\mathfrak{W})$ — $\mathcal{D}^0(\mathfrak{W}) = Q(\mathfrak{W} + \mathfrak{W}^{[\perp]})$.

$\mathcal{L}(\mathfrak{W})$ — $\mathcal{L}(\mathfrak{W}) := Q^{-1}\mathcal{D}(\mathfrak{W})$.

P_+ — The projection of $K^2(\mathcal{W})/(\mathfrak{W}_+ \dot{+} \mathfrak{W}_-^{[\perp]})$ onto $K^2_+(\mathcal{W})/\mathfrak{W}_+$

P_- — The projection of $K^2(\mathcal{W})/(\mathfrak{W}_+ \dot{+} \mathfrak{W}_-^{[\perp]})$ onto $K^2_-(\mathcal{W})/\mathfrak{W}_-^{[\perp]}$.

$\mathcal{L}_\pm$ — See Lemma 5.9.

$\Pi_\pm$ — The orthogonal projection of $\mathcal{H}_- \oplus \mathcal{H}_+$ onto $\mathcal{H}_\pm$.

$\mathfrak{B}_\Sigma, \mathfrak{C}_\Sigma$ — The input and output maps are defined in Section 6.

$\mathfrak{S}^\Sigma_+$ — See (6.3).

An (inner) direct sum decomposition of a Hilbert or Kreĭn space $\mathcal{W}$ into two closed subspaces $\mathcal{Y}$ and $\mathcal{U}$ will be denoted by $\mathcal{W} = \mathcal{Y} \dot{+} \mathcal{U}$, and the corresponding complementary projections onto $\mathcal{Y}$ and $\mathcal{U}$ will be denoted by $P_\mathcal{Y}^\mathcal{U}$ and $P_\mathcal{U}^\mathcal{Y}$. If, in addition, $\mathcal{Y}$ and $\mathcal{U}$ are orthogonal to each other, then we write $\mathcal{W} = \mathcal{Y} \oplus \mathcal{U}$ in the case of a Hilbert space and $\mathcal{W} = \mathcal{Y} \boxplus \mathcal{U}$ in the case of a Kreĭn space. In the orthogonal case the subspaces $\mathcal{Y}$ and $\mathcal{U}$ become Hilbert or Kreĭn spaces when we let them inherit the inner product from $\mathcal{W}$, and we denote the (orthogonal) projections of $\mathcal{W}$ onto $\mathcal{Y}$ and $\mathcal{U}$ by $P_{\mathcal{Y}}$ and $P_{\mathcal{U}}$, respectively.

We denote the (external) direct sum of two Hilbert or Kreĭn spaces $\mathcal{Y}$ and $\mathcal{U}$ by $\left[\begin{smallmatrix}\mathcal{Y}\\ \mathcal{U}\end{smallmatrix}\right]$. By this we mean the Cartesian product of $\mathcal{Y}$ and $\mathcal{U}$ equipped with the standard algebraic operations and standard product topology. We sometimes equip $\left[\begin{smallmatrix}\mathcal{Y}\\ \mathcal{U}\end{smallmatrix}\right]$ with the induced Kreĭn space inner product (in the Kreĭn space notation)

$$\left[\begin{bmatrix}y_1\\ u_1\end{bmatrix}, \begin{bmatrix}y_2\\ u_2\end{bmatrix}\right]_{\mathcal{Y}\boxplus\mathcal{U}} = [y_1, y_2]_{\mathcal{Y}} + [u_1, u_2]_{\mathcal{U}}. \tag{1.13}$$

After identifying $\left[\begin{smallmatrix}\mathcal{Y}\\ 0\end{smallmatrix}\right]$ with $\mathcal{Y}$ and $\left[\begin{smallmatrix}0\\ \mathcal{U}\end{smallmatrix}\right]$ with $\mathcal{U}$ we can in this case identify $\left[\begin{smallmatrix}\mathcal{Y}\\ \mathcal{U}\end{smallmatrix}\right]$ with $\mathcal{Y}\boxplus\mathcal{U}$. However, we shall often instead use a different Kreĭn space inner product in $\left[\begin{smallmatrix}\mathcal{Y}\\ \mathcal{U}\end{smallmatrix}\right]$ of the type

$$\left[\begin{bmatrix}y_1\\ u_1\end{bmatrix}, \begin{bmatrix}y_2\\ u_2\end{bmatrix}\right]_{\left[\begin{smallmatrix}\mathcal{Y}\\ \mathcal{U}\end{smallmatrix}\right]} = \left(\begin{bmatrix}y\\ u\end{bmatrix}, J\begin{bmatrix}y\\ u\end{bmatrix}\right)_{\mathcal{Y}\boxplus\mathcal{U}},$$

where J is a given signature operator in $\mathcal{Y}\boxplus\mathcal{U}$. With respect to this inner product $\mathcal{Y}$ and $\mathcal{U}$ may or may not be orthogonal. Analogous notations are used for direct sums with three or more components.

2 Kreĭn Spaces

2.1 Some Kreĭn space results

Throughout this work both the signal space $\mathcal{W}$ and the node space $\mathfrak{K}$ will be Kreĭn spaces. We therefore begin with a review of some Kreĭn space notions and results that will be needed here.

A Kreĭn space $\mathcal{W}$ is a vector space with an inner product $[\cdot,\cdot]_{\mathcal{W}}$ that satisfies all the standard properties required by an inner product, except for the condition $[w, w]_{\mathcal{W}} > 0$ for nonzero w, with the additional property that $\mathcal{W}$ can be decomposed into a direct sum $\mathcal{W} = -\mathcal{Y} \dotplus \mathcal{U}$ in such a way that the following conditions are satisfied:

(i) $\mathcal{U}$ and $-\mathcal{Y}$ are orthogonal to each other with respect to the inner product $[\cdot,\cdot]_{\mathcal{W}}$, i.e., $[y, u]_{\mathcal{W}} = 0$ for all $u \in \mathcal{U}$ and all $y \in -\mathcal{Y}$.

(ii) $\mathcal{U}$ is a Hilbert space with the inner product $(u, u')_{\mathcal{U}} := [u, u']_{\mathcal{W}}$, u, $u' \in \mathcal{U}$, inherited from $\mathcal{W}$.

(iii) $-\mathcal{Y}$ is an anti-Hilbert space with the inner product $[y, y']_{-\mathcal{Y}} := [y, y']_{\mathcal{W}}$, y, $y' \in -\mathcal{Y}$, inherited from $\mathcal{W}$.

Here and later we shall use the notation $-\mathcal{Y}$ for the *anti-space* of a vector space $\mathcal{Y}$ equipped with a (possibly indefinite) inner product. This is the same topological vector space as $\mathcal{Y}$, but the inner product $[\cdot, \cdot]_{\mathcal{Y}}$ in $\mathcal{Y}$ has been replaced by the inner product$[y, y']_{-\mathcal{Y}} := -[y, y']_{\mathcal{Y}}$, y, $y' \in -\mathcal{Y}$. The condition that $-\mathcal{Y}$ is an *anti-Hilbert space* with the inner product inherited from $\mathcal{W}$ is equivalent to saying that $\mathcal{Y}$ is a Hilbert space with the inner product $(y, y')_{\mathcal{Y}} := -[y, y']_{\mathcal{W}}$, y, $y' \in -\mathcal{Y}$, inherited from $-\mathcal{W}$. Since $\mathcal{Y}$ and $\mathcal{U}$ are orthogonal to each other we shall denote the direct sum by $\mathcal{W} = -\mathcal{Y} \boxplus \mathcal{U}$.

Any decomposition $\mathcal{W} = -\mathcal{Y} \boxplus \mathcal{U}$ with the properties listed above is called a *fundamental decomposition* of $\mathcal{W}$. If the space $\mathcal{W}$ itself is neither a Hilbert space nor an anti-Hilbert space, then it has infinite many fundamental decompositions. If $\mathcal{W} = -\mathcal{Y} \boxplus \mathcal{U}$ is a fundamental decomposition of $\mathcal{W}$, then

$$[w, w]_{\mathcal{W}} = -\|y\|_{\mathcal{Y}}^2 + \|u\|_{\mathcal{U}}^2, \ w = u + y, \ u \in \mathcal{U}, \ y \in \mathcal{Y}. \tag{2.1}$$

The dimensions of the positive space $\mathcal{U}$ and the negative space $-\mathcal{Y}$ do not depend on the particular fundamental decomposition. These dimensions are called the positive and negative indices of $\mathcal{W}$, respectively, and they are denoted by $\operatorname{ind}_+\mathcal{W}$ and $\operatorname{ind}_-\mathcal{W}$.

An arbitrary choice of fundamental decomposition $\mathcal{W} = -\mathcal{Y} \boxplus \mathcal{U}$ determines a Hilbert space norm on $\mathcal{W}$ by

$$\|w\|_{\mathcal{Y} \oplus \mathcal{U}}^2 = \|y\|_{\mathcal{Y}}^2 + \|u\|_{\mathcal{U}}^2, \ w = u + y, \ u \in \mathcal{U}, \ y \in \mathcal{Y}. \tag{2.2}$$

While the norm $\|\cdot\|_{\mathcal{Y} \oplus \mathcal{U}}$ itself depends on the choice of fundamental decomposition $\mathcal{W} = -\mathcal{Y} \boxplus \mathcal{U}$, all these norms are equivalent and the resulting strong and weak topologies are each independent of the choice of the fundamental decomposition. Thus, we can define topological notions, such as convergence, or closedness, with respect to any one of these norms. Any norm on $\mathcal{W}$ arising in this way from some choice of fundamental decomposition $\mathcal{W} = -\mathcal{Y} \boxplus \mathcal{U}$ for $\mathcal{W}$ we shall call an *admissible norm* on $\mathcal{W}$, and we shall refer to the corresponding positive inner product on $\mathcal{Y} \oplus \mathcal{U}$ as an *admissible Hilbert space inner product* on $\mathcal{W}$.

A subspace $\mathcal{Z}$ of $\mathcal{W}$ is *nonnegative* if every vector $w \in \mathcal{Z}$ is nonnegative ($[w, w]_{\mathcal{W}} \geq 0$), it is *neutral* if every vector $w \in \mathcal{Z}$ is neutral ($[w, w]_{\mathcal{W}} = 0$), and *nonpositive* if every vector $w \in \mathcal{Z}$ is nonpositive ($[w, w]_{\mathcal{W}} \leq 0$). A nonnegative subspace which is not strictly contained in any other nonnegative subspace is called *maximal nonnegative*, and the notion of a *maximal nonpositive subspace* is defined in an analogous way. Every nonnegative subspace is contained in some maximal nonnegative subspace, and every nonpositive subspace is contained in some maximal nonpositive subspace. Maximal nonnegative or nonpositive subspaces are always closed.

The *orthogonal companion* $\mathcal{Z}^{[\perp]}$ of an arbitrary subset $\mathcal{Z} \subset \mathcal{W}$ with respect to the Kreĭn space inner product $[\cdot,\cdot]_{\mathcal{W}}$ consists of all vectors in $\mathcal{W}$ that are orthogonal to all vectors in $\mathcal{Z}$, i.e.,

$$\mathcal{Z}^{[\perp]} = \{w' \in \mathcal{W} \mid [w', w]_{\mathcal{W}} = 0 \text{ for all } w \in \mathcal{Z}\}.$$

This is always a closed subspace of $\mathcal{W}$, and $\mathcal{Z} = (\mathcal{Z}^{[\perp]})^{[\perp]}$ if and only if $\mathcal{Z}$ is a closed subspace. If $\mathcal{W}$ is a Hilbert space, then we write $\mathcal{Z}^{\perp}$ instead of $\mathcal{Z}^{[\perp]}$. A subspace $\mathcal{Z}$ is neutral if and only if $\mathcal{Z} \subset \mathcal{Z}^{[\perp]}$. If instead $\mathcal{Z}^{[\perp]} \subset \mathcal{Z}$ (i.e., $\mathcal{Z}^{[\perp]}$ is neutral), then we call $\mathcal{Z}$ *co-neutral*. A subspace $\mathcal{Z} \subset \mathcal{W}$ is called *Lagrangian* if $\mathcal{Z} = \mathcal{Z}^{[\perp]}$.

A direct sum decomposition $\mathcal{W} = \mathcal{F} \dotplus \mathcal{E}$ of $\mathcal{W}$ where both $\mathcal{F}$ and $\mathcal{E}$ are neutral is called a *Lagrangian decomposition* of $\mathcal{W}$. The subspaces $\mathcal{F}$ and $\mathcal{E}$ are automatically Lagrangian in this case. Such a decomposition exists if and only if $\operatorname{ind}_{+}\mathcal{W} = \operatorname{ind}_{-}\mathcal{W}$ (this index may be finite or infinite).

If we fix a fundamental decomposition $\mathcal{W} = -\mathcal{Y} \boxplus \mathcal{U}$, then we may view elements of $\mathcal{W}$ as consisting of column vectors

$$w = \begin{bmatrix} y \\ u \end{bmatrix} \in \begin{bmatrix} -\mathcal{Y} \\ \mathcal{U} \end{bmatrix},$$

where we view $\mathcal{Y}$ and $\mathcal{U}$ as Hilbert spaces, and the Kreĭn space inner product on $\mathcal{W}$ is given by

$$\begin{aligned} \left[\begin{bmatrix} y \\ u \end{bmatrix}, \begin{bmatrix} y' \\ u' \end{bmatrix}\right]_{\mathcal{W}} &= \left(\begin{bmatrix} y \\ u \end{bmatrix}, \begin{bmatrix} -1_{\mathcal{Y}} & 0 \\ 0 & 1_{\mathcal{U}} \end{bmatrix} \begin{bmatrix} y' \\ u' \end{bmatrix}\right)_{\mathcal{Y}\oplus\mathcal{U}} \\ &= -(y, y')_{\mathcal{Y}} + (u, u')_{\mathcal{U}}. \end{aligned} \tag{2.3}$$

In this representation, nonnegative, neutral, nonpositive, and Lagrangian subspaces are characterized as follows.

Proposition 2.1. *Let $\mathcal{W}$ be a Kreĭn space represented in the form $\mathcal{W} = \left[\begin{smallmatrix} -\mathcal{Y} \\ \mathcal{U} \end{smallmatrix}\right]$ with Kreĭn space inner product equal to the quadratic form $[\cdot,\cdot]_J$ induced by the operator $J = \left[\begin{smallmatrix} -1_{\mathcal{Y}} & 0 \\ 0 & 1_{\mathcal{U}} \end{smallmatrix}\right]$ in the Hilbert space inner product of $\left[\begin{smallmatrix} \mathcal{Y} \\ \mathcal{U} \end{smallmatrix}\right]$ as explained above, and let $\mathcal{Z}$ be a subspace of $\mathcal{W}$. Then the following claims are true:*

(i) *$\mathcal{Z}$ is nonnegative if and only if there is a linear Hilbert space contraction $A_+ \colon \mathcal{Z}_+ \mapsto \mathcal{Y}$ from some domain $\mathcal{Z}_+ \subset \mathcal{U}$ into $\mathcal{Y}$ such that*

$$\mathcal{Z} = \begin{bmatrix} A_+ \\ 1_{\mathcal{U}} \end{bmatrix} \mathcal{Z}_+ = \left\{ \begin{bmatrix} A_+ z_+ \\ z_+ \end{bmatrix} \,\middle|\, z_+ \in \mathcal{Z}_+ \right\}. \tag{2.4}$$

$\mathcal{Z}$ is maximal nonnegative if and only if, in addition, $\mathcal{Z}_+ = \mathcal{U}$.

(ii) *$\mathcal{Z}$ is nonpositive if and only if there is a linear contraction $A_-\colon \mathcal{Z}_- \mapsto \mathcal{U}$ from some domain $\mathcal{Z}_- \subset \mathcal{Y}$ into $\mathcal{U}$ such that*

$$\mathcal{Z} = \begin{bmatrix} 1_{\mathcal{Y}} \\ A_- \end{bmatrix} \mathcal{Z}_- = \left\{ \begin{bmatrix} z_- \\ A_- z_- \end{bmatrix} \,\middle|\, z_- \in \mathcal{Z}_- \right\}. \tag{2.5}$$

$\mathcal{Z}$ is maximal nonpositive if and only if, in addition, $\mathcal{Z}_- = \mathcal{Y}$.

(iii) *$\mathcal{Z}$ is neutral if and only if there is an isometry A_+ mapping a subspace $\mathcal{Z}_+$ of $\mathcal{U}$ onto a subspace $\mathcal{Z}_-$ of $\mathcal{Y}$, or equivalently, an isometry A_- mapping $\mathcal{Z}_- \subset \mathcal{Y}$ isometrically onto $\mathcal{Z}_+ \subset \mathcal{U}$, such that*

$$\mathcal{Z} = \begin{bmatrix} A_+ \\ 1_{\mathcal{U}} \end{bmatrix} \mathcal{Z}_+ = \begin{bmatrix} 1_{\mathcal{Y}} \\ A_- \end{bmatrix} \mathcal{Z}_-. \tag{2.6}$$

$\mathcal{Z}$ is Lagrangian if and only if, in addition, $\mathcal{Z}_+ = \mathcal{U}$ and $\mathcal{Z}_- = \mathcal{Y}$.

(iv) *$\mathcal{Z}$ is maximal nonnegative if and only if $\mathcal{Z}$ is closed and $\mathcal{Z}^{[\perp]}$ is maximal nonpositive. More precisely, if $\mathcal{Z}$ has the representation* (2.4) *with $\mathcal{Z}_+ = \mathcal{U}$, then $\mathcal{Z}^{[\perp]}$ has the representation*

$$\mathcal{Z}^{[\perp]} = \begin{bmatrix} 1_{\mathcal{Y}} \\ A_+^* \end{bmatrix} \mathcal{Y}, \tag{2.7}$$

where A_+^* is computed with respect to the Hilbert space inner product in $\mathcal{Y}$ (instead of the anti-Hilbert space inner product in $-\mathcal{Y}$ inherited from $\mathcal{W}$).

(v) *$\mathcal{Z}$ is maximal nonnegative if and only if $\mathcal{Z}$ is closed and nonnegative and $\mathcal{Z}^{[\perp]}$ is nonpositive. In particular, $\mathcal{Z}$ is Lagrangian if and only if $\mathcal{Z}$ is both maximal nonnegative and maximal nonpositive.*

Proof. See [AI89, Section 1.8, pp. 48–64] or the following theorems in [Bog74]: Theorem 11.7 on p. 54, Theorems 4.2 and 4.4 on pp. 105–106, and Lemma 4.5 on p. 106. □

The fundamental decompositions that we have considered above are a special case of *orthogonal decompositions* $\mathcal{W} = -\mathcal{Y} \boxplus \mathcal{U}$ of $\mathcal{W}$, where $\mathcal{Y}$ and $\mathcal{U}$ are orthogonal with respect to $[\cdot,\cdot]_{\mathcal{W}}$, and both $\mathcal{Y}$ and $\mathcal{U}$ are Kreĭn spaces with the inner products inherited from $-\mathcal{W}$ and $\mathcal{W}$, respectively. Thus, if $w = y + u$ with $y \in \mathcal{Y}$ and $u \in \mathcal{U}$, then

$$[w,w]_{\mathcal{W}} = [y,y]_{\mathcal{W}} + [u,u]_{\mathcal{W}} = -[y,y]_{\mathcal{Y}} + [u,u]_{\mathcal{U}}. \tag{2.8}$$

This orthogonal decomposition is fundamental if and only if $\mathcal{Y}$ and $\mathcal{U}$ are Hilbert spaces, i.e., if they are both nonnegative Kreĭn spaces.

The next lemma, proved in [AS10], will be used later to find out if certain subspaces of a Kreĭn space with a special orthogonal decomposition are maximal nonnegative, or maximal nonpositive, or Lagrangian.

Lemma 2.2 ([AS10, Lemma 2.2]). *Let $\mathcal{X}$ and $\mathcal{Y}$ be two Hilbert spaces and $\mathcal{K}$ a Kreĭn space, and let $\mathfrak{K}$ be the Kreĭn space $\mathfrak{K} = \left[\begin{smallmatrix} -\mathcal{Y} \\ \mathcal{X} \\ \mathcal{K} \end{smallmatrix}\right] = -\mathcal{Y} \boxplus \mathcal{X} \boxplus \mathcal{K}$.*

(i) *A nonnegative subspace $\mathcal{Z}$ of $\mathfrak{K}$ is maximal nonnegative if and only if conditions (a) and (b) below hold:*

  (a) For each $x \in \mathcal{X}$ there exists some $y \in \mathcal{Y}$ and $w \in \mathcal{K}$ such that $\left[\begin{smallmatrix} y \\ x \\ w \end{smallmatrix}\right] \in \mathcal{Z}$;

  (b) The set of all $w \in \mathcal{K}$ for which there exists some $y \in \mathcal{Y}$ such that $\left[\begin{smallmatrix} y \\ 0 \\ w \end{smallmatrix}\right] \in \mathcal{Z}$ is maximal nonnegative in $\mathcal{K}$.

(ii) *A nonpositive subspace $\mathcal{Z}$ of $\mathfrak{K}$ is maximal nonpositive if and only if conditions (c) and (d) below hold:*

  (c) For each $y \in \mathcal{Y}$ there exists some $x \in \mathcal{X}$ and $w \in \mathcal{K}$ such that $\left[\begin{smallmatrix} y \\ x \\ w \end{smallmatrix}\right] \in \mathcal{Z}$;

  (d) The set of all $w \in \mathcal{K}$ for which there exists some $x \in \mathcal{X}$ such that $\left[\begin{smallmatrix} 0 \\ x \\ w \end{smallmatrix}\right] \in \mathcal{Z}$ is maximal nonpositive in $\mathcal{K}$.

(iii) *A neutral subspace $\mathcal{Z}$ of $\mathfrak{K}$ is Lagrangian if and only if conditions (a)–(d) above hold.*

Lemma 2.3. *Let $\mathfrak{K}$ be a Kreĭn space with the orthogonal decomposition $\mathfrak{K} = \mathfrak{K}_1 \boxplus \mathfrak{K}_2$, and let $\mathcal{Z}$ be a subspace of $\mathfrak{K}$. Then*

$$(P_{\mathfrak{K}_2}\mathcal{Z})^{[\perp]} = \mathcal{Z}^{[\perp]} \cap \mathfrak{K}_2 \text{ and } (\mathcal{Z}^{[\perp]} \cap \mathfrak{K}_2)^{[\perp]} = \overline{P_{\mathfrak{K}_2}\mathcal{Z}}, \tag{2.9}$$

where the orthogonal companions on the left-hand sides are computed with respect to $\mathfrak{K}_2$. In particular, if $P_{\mathfrak{K}_2}\mathcal{Z}$ is closed, then $P_{\mathfrak{K}_2}\mathcal{Z} = (\mathcal{Z}^{[\perp]} \cap \mathfrak{K}_2)^{[\perp]}$.

Proof. That $(P_{\mathfrak{K}_2}\mathcal{Z})^{[\perp]} = \mathcal{Z}^{[\perp]} \cap \mathfrak{K}_2$ follows from the following chain of equivalences:

$$\begin{aligned} & z^\dagger \in (P_{\mathfrak{K}_2}\mathcal{Z})^{[\perp]} \\ \Leftrightarrow & z^\dagger \in \mathfrak{K}_2 \text{ and } [z^\dagger, P_{\mathfrak{K}_2}z]_{\mathfrak{K}} = 0 \text{ for all } z \in \mathcal{Z} \\ \Leftrightarrow & z^\dagger \in \mathfrak{K}_2 \text{ and } [z^\dagger, z]_{\mathfrak{K}} = 0 \text{ for all } z \in \mathcal{Z} \\ \Leftrightarrow & z^\dagger \in Z^{[\perp]} \cap \mathfrak{K}_2. \end{aligned}$$

This implies that $\overline{P_{\mathfrak{K}_2}\mathcal{Z}} = ((P_{\mathfrak{K}_2}\mathcal{Z})^{[\perp]})^{[\perp]} = (Z^{[\perp]} \cap \mathfrak{K}_2)^{[\perp]}$. $\square$

Lemma 2.4. *Let $\mathfrak{K}$ be a Kreĭn space with the orthogonal decomposition $\mathfrak{K} = \mathfrak{K}_1 \boxplus \mathfrak{K}_2$, and let $\mathcal{Z}$ be a maximal nonnegative subspace of $\mathfrak{K}$. Then the following conditions are equivalent:*

(i) *$P_{\mathfrak{K}_2}\mathcal{Z}$ is a nonnegative subspace of $\mathfrak{K}_2$;*

(ii) *$P_{\mathfrak{K}_2}\mathcal{Z}$ is a maximal nonnegative subspace of $\mathfrak{K}_2$;*

(iii) *$\mathcal{Z} \cap \mathfrak{K}_1$ is a maximal nonnegative subspace of $\mathfrak{K}_1$;*

(iv) *$P_{\mathfrak{K}_1}|_{\mathcal{Z}}$ is a contraction $\mathcal{Z} \to \mathfrak{K}_1$, i.e.,*

$$[z, z]_{\mathfrak{K}} \leq [P_{\mathfrak{K}_1} z, P_{\mathfrak{K}_1} z]_{\mathfrak{K}_1} \text{ for all } z \in \mathcal{Z}.$$

(v) *$P_{\mathfrak{K}_1}\mathcal{Z}^{[\perp]}$ is a nonpositive subspace of $\mathfrak{K}_1$;*

(vi) *$P_{\mathfrak{K}_1}\mathcal{Z}^{[\perp]}$ is a maximal nonpositive subspace of $\mathfrak{K}_1$;*

(vii) *$\mathcal{Z}^{[\perp]} \cap \mathfrak{K}_2$ is a maximal nonpositive subspace of $\mathfrak{K}_2$;*

(viii) *$P_{\mathfrak{K}_2}|_{\mathcal{Z}^{[\perp]}}$ is an expansion $\mathcal{Z}^{[\perp]} \to \mathfrak{K}_2$, i.e.,*

$$[z^\dagger, z^\dagger]_{\mathfrak{K}} \geq [P_{\mathfrak{K}_2} z^\dagger, P_{\mathfrak{K}_2} z^\dagger]_{\mathfrak{K}_2} \text{ for all } z^\dagger \in \mathcal{Z}^{[\perp]}.$$

When these equivalent conditions holds we have $(\mathcal{Z} \cap \mathfrak{K}_1)^{[\perp]} = P_{\mathfrak{K}_1}\mathcal{Z}^{[\perp]}$ and $P_{\mathfrak{K}_2}\mathcal{Z} = (\mathcal{Z}^{[\perp]} \cap \mathfrak{K}_2)^{[\perp]}$, where the orthogonal companions on the right-hand sides are computed in $\mathfrak{K}_1$ and $\mathfrak{K}_2$, respectively.

Proof. We first show that (i), (ii), (iv), and (vii) are equivalent to each other, and that (analogously) (v), (vi), (viii), and (iii) are equivalent to each other, and then complete the proof of the equivalence of the conditions (i)–(viii) by showing that (iii) $\Rightarrow$ (i) and (vii) $\Rightarrow$ (v). The final claim follows from Lemma 2.3.

(i) $\Leftrightarrow$ (ii): Trivially (ii) $\Rightarrow$ (i). If $P_{\mathfrak{K}_2}\mathcal{Z}$ is not maximal nonnegative in $\mathfrak{K}_2$, then $P_{\mathfrak{K}_2}\mathcal{Z}$ is properly contained in some nonnegative subspace $\mathcal{Z}_2$ of $\mathfrak{K}_2$. This implies that $\mathcal{Z}$ is properly contained in the nonnegative subspace $\mathcal{Z} \vee \begin{bmatrix} \{0\} \\ \mathcal{Z}_2 \end{bmatrix}$ of $\mathfrak{K}$, and hence $\mathcal{Z}$ cannot be maximal. Thus (i) $\Rightarrow$ (ii).

(i) $\Leftrightarrow$ (iv): Since $\mathfrak{K} = \mathfrak{K}_1 \boxplus \mathfrak{K}_2$ we have

$$[z, z]_{\mathfrak{K}} = [P_{\mathfrak{K}_1} z, P_{\mathfrak{K}_1} z]_{\mathfrak{K}_1} + [P_{\mathfrak{K}_2} z, P_{\mathfrak{K}_2} z]_{\mathfrak{K}_2} \text{ for all } z \in \mathcal{Z}.$$

Thus, (i) $\Leftrightarrow$ (iv).

(ii) $\Leftrightarrow$ (vii): This follows from Proposition 2.1(iv) and Lemma 2.3.

(v) $\Leftrightarrow$ (vi): This follows from the equivalence (i) $\Leftrightarrow$ (ii) if we replace $\mathfrak{K}$ by its anti-space $-\mathfrak{K}$, interchange $\mathfrak{K}_1$ and $\mathfrak{K}_2$, and also interchange $\mathcal{Z}$ and $\mathcal{Z}^{[\perp]}$.

(v) $\Leftrightarrow$ (viii): This follows from the equivalence (i) $\Leftrightarrow$ (iv) if we replace $\mathfrak{K}$ by its anti-space $-\mathfrak{K}$, interchange $\mathfrak{K}_1$ and $\mathfrak{K}_2$, and also interchange $\mathcal{Z}$ and $\mathcal{Z}^{[\perp]}$.

(vi) $\Leftrightarrow$ (iii): This follows from Proposition 2.1(iv) and Lemma 2.3.

(iii) $\Rightarrow$ (i): Suppose that (i) does not hold. Then there exists a vector $z_0 \in \mathcal{Z}$ such that $[P_{\mathfrak{K}_2} z_0, P_{\mathfrak{K}_2} z_0]_{\mathfrak{K}_2} < 0$. In particular, since $\mathcal{Z}$ is nonnegative, this implies that $P_{\mathfrak{K}_2} z_0 \notin \mathcal{Z}$, and consequently, $P_{\mathfrak{K}_1} z_0 = z_0 - P_{\mathfrak{K}_2} z_0 \notin \mathcal{Z}$. Thus, $\mathcal{Z} \cap \mathfrak{K}_1$ is a proper subset of $P_{\mathfrak{K}_1} z_0 \vee (\mathcal{Z} \cap \mathfrak{K}_1)$. We claim that this subspace is nonnegative. This is true because for all $z \in \mathcal{Z}$ and all $\lambda \in \mathbb{C}$, we have $\lambda z_0 + z \in \mathcal{Z}$, and hence

$$[\lambda P_{\mathfrak{K}_1} z_0 + z, \lambda P_{\mathfrak{K}_1} z_0 + z]_{\mathfrak{K}} = [\lambda z_0 + z, \lambda z_0 + z]_{\mathfrak{K}} - |\lambda|^2 [P_{\mathfrak{K}_2} z_0, P_{\mathfrak{K}_2} z_0]_{\mathfrak{K}_2} \geq 0.$$

Thus, if (i) is false, then so is (iii).

(vii) $\Rightarrow$ (v): This follows from the implication (iii) $\Rightarrow$ (i) if we replace $\mathfrak{K}$ by its anti-space $-\mathfrak{K}$, interchange $\mathfrak{K}_1$ and $\mathfrak{K}_2$, and also interchange $\mathcal{Z}$ and $\mathcal{Z}^{[\perp]}$. $\square$

2.2 The Hilbert space $\mathcal{H}(\mathcal{Z})$

In [AS09a] a Hilbert space $\mathcal{H}(\mathcal{Z})$ was constructed, starting from an arbitrary maximal nonnegative subspace $\mathcal{Z}$ of a Kreĭn space. Below we give a short review of this construction. It will be used later in the construction of the state spaces of our canonical s/s signal realizations.

Let $\mathcal{Z}$ be a maximal nonnegative subspace of the Kreĭn space $\mathcal{K}$, and let $\mathcal{K}/\mathcal{Z}$ be the quotient of $\mathcal{K}$ modulo $\mathcal{Z}$. We define $\mathcal{H}(\mathcal{Z})$ by

$$\mathcal{H}(\mathcal{Z}) = \{h \in \mathcal{K}/\mathcal{Z} \mid \sup\{-[x,x]_{\mathcal{K}} \mid x \in h\} < \infty\}. \tag{2.10}$$

It turns out that $\sup\{-[x,x]_{\mathcal{K}} \mid x \in h\} \geq 0$ for all $h \in \mathcal{H}(\mathcal{Z})$, that $\mathcal{H}(\mathcal{Z})$ is a subspace of $\mathcal{K}/\mathcal{Z}$, that $\mathcal{H}(\mathcal{Z})$ is a Hilbert space with the norm

$$\|h\|_{\mathcal{H}(\mathcal{Z})} = \left(\sup\{-[x,x]_{\mathcal{K}} \mid x \in h\}\right)^{1/2}, \qquad h \in \mathcal{H}(\mathcal{Z}), \tag{2.11}$$

and that $\mathcal{H}(\mathcal{Z})$ is continuously contained in $\mathcal{K}/\mathcal{Z}$ (where we use the standard quotient topology in $\mathcal{K}/\mathcal{Z}$, induced by some arbitrarily chosen admissible Hilbert space norm in $\mathcal{K}$). We denote the equivalence class $h \in \mathcal{K}/\mathcal{Z}$ that contains a particular vector $x \in \mathcal{K}$ by $h = x + \mathcal{Z}$. Thus, with this notation, (2.10) and (2.11) can be rewritten in the form

$$\mathcal{H}(\mathcal{Z}) = \{x + \mathcal{Z} \in \mathcal{K}/\mathcal{Z} \mid \|x + \mathcal{Z}\|^2_{\mathcal{H}(\mathcal{Z})} < \infty\}, \tag{2.12}$$

$$\|x + \mathcal{Z}\|^2_{\mathcal{H}(\mathcal{Z})} = \sup\{-[x + z, x + z]_{\mathcal{K}} \mid z \in \mathcal{Z}\}, \qquad x \in \mathcal{K}. \tag{2.13}$$

A very important (and easily proved fact) is that if we define

$$\mathcal{H}^0(\mathcal{Z}) := \{z^\dagger + \mathcal{Z} \mid z^\dagger \in \mathcal{Z}^{[\perp]}\}, \tag{2.14}$$

then $\mathcal{H}^0(\mathcal{Z})$ is a subspace of $\mathcal{H}(\mathcal{Z})$. However, even more is true: $\mathcal{H}^0(\mathcal{Z})$ is a *dense subspace* of $\mathcal{H}(\mathcal{Z})$, and

$$[x + \mathcal{Z}, z^\dagger + \mathcal{Z}]_{\mathcal{H}(\mathcal{Z})} = -[x, z^\dagger]_{\mathcal{K}}, \quad x + \mathcal{Z} \in \mathcal{H}(\mathcal{Z}), \quad z^\dagger \in \mathcal{Z}^{[\perp]}, \tag{2.15}$$

$$\|z^\dagger + \mathcal{Z}\|^2_{\mathcal{H}(\mathcal{Z})} = -[z^\dagger, z^\dagger]_{\mathcal{K}}, \quad z^\dagger \in \mathcal{Z}^{[\perp]}. \tag{2.16}$$

Thus, $\mathcal{H}(\mathcal{Z})$ may be interpreted as a completion of $\mathcal{H}^0(\mathcal{Z})$. See [AS09a] for more details.

3 Passive and Conservative State/Signal Systems

3.1 Basic properties of trajectories of passive s/s systems

We already gave a short introduction to passive s/s (state/signal) systems, and now describe this notion in more detail.

In the following definition and throughout the remainder of this paper, the interval I is assumed to be *closed* and *nontrivial*, i.e., it should have a nonempty interior. Thus, it is either a finite interval $I = [t_0, t_1]$, or a semi-finite interval $I = (-\infty, t_1]$ or $I = [t_0, \infty)$, or the full real line $I = \mathbb{R} = (-\infty, \infty)$.

Definition 3.1. Let $\mathcal{X}$ be a Hilbert space and $\mathcal{W}$ a Kreĭn space.

(i) By a *passive s/s node* in continuous time we mean a triple $\Sigma = (V; \mathcal{X}, \mathcal{W})$ where V is a maximal nonnegative subspace of the Kreĭn *node space* $\mathfrak{K} := \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix}$ equipped with the inner product (1.7) satisfying (1.1).

(ii) A *classical trajectory* generated by a subspace V of $\mathfrak{K}$ on an interval I is a pair of functions $\begin{bmatrix} x \\ w \end{bmatrix} \in \begin{bmatrix} C^1(I;\mathcal{X}) \\ C(I;\mathcal{W}) \end{bmatrix}$ satisfying (1.2).

(iii) A *(generalized) trajectory* generated by a subspace V of $\mathfrak{K}$ on an interval I is a pair of functions $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \left[\begin{smallmatrix} C(I;\mathcal{X}) \\ L^2_{\mathrm{loc}}(I;\mathcal{W}) \end{smallmatrix}\right]$ which can be approximated by a sequence of classical trajectories $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right]$ in such a way that $x_n \to x$ in $\mathcal{X}$ locally uniformly on I, and $w_n \to w$ in $L^2_{\mathrm{loc}}(I;\mathcal{W})$.

(iv) The passive s/s node Σ together with its families of classical and generalized trajectories is called a *passive s/s system*, and it is denoted by the same symbols as the node.

(v) By a *past*, *full*, or *future* trajectory of Σ we mean a trajectory of Σ on $\mathbb{R}^-$, $\mathbb{R}$, or $\mathbb{R}^+$, respectively.

(vi) A (generalized) trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ of a passive s/s system $\Sigma = (V;\mathcal{X},\mathcal{W})$ on an interval I is *externally generated* if the following condition holds: If I has a finite left end-point t_0, then we require that $x(t_0) = 0$, and if the left end-point of I is $-\infty$, then we require that $\lim_{t\to-\infty} x(t) = 0$ and that $w \in L^2((-\infty,T];\mathcal{W})$ for every finite $T \in I$.

(vii) A (generalized) trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ of a passive s/s system $\Sigma = (V;\mathcal{X},\mathcal{W})$ is *stable* if x is bounded on I and $w \in L^2(I;\mathcal{W})$.

As the following lemma shows, the boundedness condition on x in Definition 3.1(vii) is often redundant.

Lemma 3.2. *Let $\Sigma = (V;\mathcal{X},\mathcal{W})$ be a passive s/s system, let I be an interval, and let $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ be a (generalized) trajectory of Σ on the closed interval I. Assume further that at least one of the conditions (i) or (ii) below holds:*

(i) *The interval I is bounded to the left.*

(ii) *$\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is externally generated.*

Then $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is stable if and only if $w \in L^2(I;\mathcal{W})$, or equivalently, if and only if $P_{\mathcal{U}}w \in L^2(I;\mathcal{U})$ for some fundamental decomposition $\mathcal{W} = -\mathcal{Y} \boxplus \mathcal{U}$ of $\mathcal{W}$. In the case where $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is externally generated we have, in addition,

$$\begin{aligned} \|x(t)\|^2_{\mathcal{X}} &\le \int_{s\in I;\ s\le t} [w(s),w(s)]_{\mathcal{W}}\,\mathrm{d}s \\ &= \int_{s\in I;\ s\le t} \left(\|P_{\mathcal{U}}w(s)\|^2_{\mathcal{U}} - \|P_{\mathcal{Y}}w(s)\|^2_{\mathcal{Y}}\right)\mathrm{d}s, \qquad t\in I. \end{aligned} \tag{3.1}$$

In particular, if $I = \mathbb{R}^-$ and $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is an externally generated stable past trajectory, then

$$\|x(0)\|^2_{\mathcal{X}} \le [w,w]_{K^2_-(\mathcal{W})} = \|P_{\mathcal{U}}w\|^2_{L^2_-(\mathcal{U})} - \|P_{\mathcal{Y}}w\|^2_{L^2_-(\mathcal{Y})}. \tag{3.2}$$

Proof. In Section 1 we outlined a proof of (1.9), and the results mentioned above follow from (1.9). □

In the following lemma we list some elementary properties of the set of all trajectories of a passive s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$. The notations τ^t and τ_+^t were explained at the end of Section 1.

Lemma 3.3. *The stable trajectories of a passive s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ have the following properties:*

(i) *If $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is a classical or generalized stable trajectory on some interval I and $t \in \mathbb{R}$, then $\left[\begin{smallmatrix} \tau^t x \\ \tau^t w \end{smallmatrix}\right]$ is a classical or generalized stable trajectory, respectively, of Σ on the interval $I - t := \{s \in \mathbb{R} \mid s + t \in I\}$, and $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is externally generated on I if and only if $\left[\begin{smallmatrix} \tau^t x \\ \tau^t w \end{smallmatrix}\right])$ is externally generated on $I - t$.*

(ii) *The restriction of a classical or generalized stable trajectory on some interval I' to a subinterval $I \subset I'$ is a classical or generalized stable trajectory of Σ on I, respectively, and if I and I' have the same left end-point, then the restricted trajectory is externally generated if and only if the original trajectory is externally generated.*

(iii) *If $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is a classical or generalized stable trajectory on $\mathbb{R}^+$, then $\left[\begin{smallmatrix} \tau_+^t x \\ \tau_+^t w \end{smallmatrix}\right]$ is a classical or generalized stable trajectory on $\mathbb{R}^+$ for all $t \in \mathbb{R}^+$.*

(iv) *The set of all stable (generalized) trajectories and the set of all externally generated stable (generalized) trajectories of Σ on some interval I (finite or infinite) are closed subspaces of $\left[\begin{smallmatrix} BUC(I;\mathcal{X}) \\ L^2(I;\mathcal{W}) \end{smallmatrix}\right]$.*

Proof. Claims (i)–(iii) follow immediately from Definition 3.1.

In order to prove (iv) we let $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right]$ be a sequence of stable trajectories of Σ on I converging to $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ in $\left[\begin{smallmatrix} BUC(I;\mathcal{X}) \\ L^2(I;\mathcal{W}) \end{smallmatrix}\right]$. Both $C(I; \mathcal{X})$ and $L^2_{\rm loc}(I; \mathcal{W})$ are Fréchet spaces, and by the definition of a generalized trajectory of Σ on I, we can find a sequence of classical trajectories $\left[\begin{smallmatrix} \tilde{x}_n \\ \tilde{w}_n \end{smallmatrix}\right]$ of Σ on I such that the distance from $\left[\begin{smallmatrix} \tilde{x}_n \\ \tilde{w}_n \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right]$ in $\left[\begin{smallmatrix} C(I;\mathcal{X}) \\ L^2_{\rm loc}(I;\mathcal{W}) \end{smallmatrix}\right]$ tends to zero as $n \to \infty$. Then $\left[\begin{smallmatrix} \tilde{x}_n \\ \tilde{w}_n \end{smallmatrix}\right]$ also tends to $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ in $\left[\begin{smallmatrix} C(I;\mathcal{X}) \\ L^2_{\rm loc}(I;\mathcal{W}) \end{smallmatrix}\right]$ as $n \to \infty$. Thus, $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is a trajectory of Σ on I. By assumption, $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \left[\begin{smallmatrix} BUC(I;\mathcal{X}) \\ L^2(I;\mathcal{W}) \end{smallmatrix}\right]$, and hence $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is stable. If all x_n tend to zero at the left end-point of I, then so does x (because of the uniform convergence), and hence $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is externally generated if all the trajectories $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right]$ are externally generated. □

In the proof of Lemma 3.3 we made only marginal use of the passivity of Σ (i.e., we did not use any other properties of V than the closedness). However, in the proof of the following lemma we shall make significant use of the passivity of Σ (or more precisely, of the fact that every passive s/s system is well-posed in the sense of [KS09]).

Lemma 3.4. *The set of (generalized) trajectories of a passive s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ has the following properties:*

(i) *Let $\mathcal{W} = -\mathcal{Y} \boxplus \mathcal{U}$ be a fundamental decomposition of $\mathcal{W}$. Then, for each $x_0 \in \mathcal{X}$, each closed interval I with a finite left end-point t_0, and each $u \in L^2(I;\mathcal{U})$, there exists a unique stable trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ of Σ on I satisfying $x(t_0) = x_0$ and $P_{\mathcal{U}} w = u$.*

(ii) *Let $\left[\begin{smallmatrix} x_1 \\ w_1 \end{smallmatrix}\right]$ be a stable trajectory of Σ on the finite interval $I_1 = [t_0, t_1]$, and let $\left[\begin{smallmatrix} x_2 \\ w_2 \end{smallmatrix}\right]$ be a stable trajectory of Σ on a closed interval I_2 with left end-point t_1. Then the concatenation $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ defined by*

$$\begin{bmatrix} x(t) \\ w(t) \end{bmatrix} := \begin{cases} \left[\begin{smallmatrix} x_1(t) \\ w_1(t) \end{smallmatrix}\right], & t \in I_1, \\ \left[\begin{smallmatrix} x_2(t) \\ w_2(t) \end{smallmatrix}\right], & t \in I_2 \setminus \{t_1\}, \end{cases} \tag{3.3}$$

is a stable trajectory of Σ on $I := I_1 \cup I_2$ if and only if $x_1(t_1) = x_2(t_1)$.

(iii) *Every stable trajectory on some finite interval $I = [t_0, t_1]$ can be extended to a stable trajectory of Σ on $[t_0, \infty)$. This extension can be chosen so that $\pi_{[t_1,\infty)} P_{\mathcal{U}} w = u$ for an arbitrary $u \in L^2([t_1, \infty);\mathcal{U})$, and it is uniquely determined by u.*

(iv) *A pair of functions $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ on an interval $[t_0, \infty)$ is a stable trajectory of Σ on $[t_0, \infty)$ if and only if the restriction of $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ to every finite subinterval $[t_0, t_1]$ of $[t_0, \infty)$ is a trajectory of Σ on $[t_0, t_1]$ and, in addition, $P_{\mathcal{U}} w \in L^2([t_0, \infty), \mathcal{U})$ for some fundamental decomposition $\mathcal{W} = -\mathcal{Y} \boxplus \mathcal{U}$ of $\mathcal{W}$.*

Proof. (i) By Lemma 3.3(i), it suffices to prove the case where $t_0 = 0$. By [Kur10, Prop. 5.8], the i/o pair $(\mathcal{U}, \mathcal{Y})$ is L^2-well-posed for Σ. Theorem 6.6 of [KS09] then yields that for every $x_0 \in \mathcal{X}$ and $u \in L^2_{\mathrm{loc}}(\mathbb{R}^+;\mathcal{U})$ the system Σ has a unique future trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$, such that $x(0) = x_0$ and $P_{\mathcal{U}} w = u$. According to Lemma 3.2, a sufficient condition for the stability of $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is that $w \in K^2(\mathbb{R}^+; \mathcal{W})$. Indeed, this condition is satisfied due to the fact that, by (1.9),

$$\|P_{\mathcal{Y}} w\|^2_{L^2(\mathbb{R}^+;\mathcal{Y})} \le \|x(0)\|^2_{\mathcal{X}} + \|u\|^2_{L^2(\mathbb{R}^+;\mathcal{U})}.$$

(ii) Claim (ii) is proved in [KS09, Prop. 3.8]

(iii) Claim (iii) follows from (i) and (ii).

(iv) According to [KS09, Proposition 3.9], $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is a trajectory of Σ on $[t_0, \infty)$ if and only if the restriction of $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ to every finite subinterval $[t_0, t_1]$ is a trajectory of Σ on $[t_0, t_1]$. By Lemma 3.2, this trajectory is stable if and only if $P_{\mathcal{U}} \in L^2([t_0, \infty); \mathcal{U})$. □

The following theorem plays a key role in our extension of many of the discrete time results developed in [AS09b] and [AS10] to a continuous time setting (see Remark 3.17 below). In this theorem we need the family $\mathfrak{K}_{0,t} := \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{X} \\ K^2([0,t];\mathcal{W}) \end{smallmatrix}\right]$, $t \in \mathbb{R}^+$, of Kreĭn spaces with the indefinite inner products (1.12) as well as the Kreĭn space $\mathfrak{L}_{0,\infty} := \left[\begin{smallmatrix} \mathcal{X} \\ K^2(\mathbb{R}^+;\mathcal{W}) \end{smallmatrix}\right]$ with the natural inner product (1.13). It follows immediately that $\mathfrak{K}_{0,t}$ and $\mathfrak{L}_{0,\infty}$ are Kreĭn spaces with fundamental decompositions $\mathfrak{K}_{0,t} = -\mathfrak{K}_{0,t,-} \boxplus \mathfrak{K}_{0,t,+}$ and $\mathfrak{L}_{0,\infty} = -\mathfrak{L}_{0,\infty,-} \boxplus \mathfrak{L}_{0,\infty,+}$, where

$$\begin{aligned} \mathfrak{K}_{0,t,+} &:= \begin{bmatrix} \{0\} \\ \mathcal{X} \\ L^2([0,t];\mathcal{U}) \end{bmatrix}, \qquad \mathfrak{K}_{0,t,-} := \begin{bmatrix} \mathcal{X} \\ \{0\} \\ L^2([0,t];\mathcal{Y}) \end{bmatrix}, \quad t \geq 0, \\ \mathfrak{L}_{0,\infty,+} &:= \begin{bmatrix} \mathcal{X} \\ L^2(\mathbb{R}^+;\mathcal{U}) \end{bmatrix}, \qquad \mathfrak{L}_{0,\infty,-} := \begin{bmatrix} \{0\} \\ L^2(\mathbb{R}^+;\mathcal{Y}) \end{bmatrix}. \end{aligned} \tag{3.4}$$

and $\mathcal{W} = -\mathcal{Y} \boxplus \mathcal{U}$ is an arbitrary fundamental decomposition of $\mathcal{W}$.

Theorem 3.5. *Let $\mathcal{X}$ be a Hilbert space, let $\mathcal{W}$ be a Kreĭn space, and let $\mathcal{T}_+$ be a subspace of $\left[\begin{smallmatrix} BUC(\mathbb{R}^+;\mathcal{X}) \\ K^2(\mathbb{R}^+;\mathcal{W}) \end{smallmatrix}\right]$. Define*

$$\begin{aligned} \mathcal{T}_{0,t} &:= \left\{ \left[\begin{smallmatrix} x(t) \\ x(0) \\ \pi_{[0,t]} w \end{smallmatrix}\right] \,\middle|\, \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathcal{T}_+ \right\}, \quad t \in \mathbb{R}^+, \\ \mathcal{S}_{0,\infty} &:= \left\{ \left[\begin{smallmatrix} x(0) \\ w \end{smallmatrix}\right] \,\middle|\, \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathcal{T}_+ \right\}. \end{aligned} \tag{3.5}$$

Then the subspace $\mathcal{T}_+$ is the family of all stable future trajectories of some passive s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ if and only if the following three conditions hold:

(i) *$\mathcal{T}_+$ is left-shift invariant, i.e.,*

$$\tau_+^t \mathcal{T}_+ := \{\tau_+^t \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \mid \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathcal{T}_+\} \subset \mathcal{T}_+, \qquad t \in \mathbb{R}^+; \tag{3.6}$$

(ii) *For all $t \in \mathbb{R}^+$, $\mathcal{T}_{0,t}$ is a maximal nonnegative subspace of $\mathfrak{K}_{0,t}$.*

(iii) *$\mathcal{S}_{0,\infty}$ is a maximal nonnegative subspace of $\mathfrak{L}_{0,\infty}$.*

Proof. This proof is based on [KS09, Lemma 4.7], and we refer the reader to [KS09] for the precise definitions of some of the s/s and i/s/o notions that we use in this proof. The monograph [Sta05] can be used as an alternative source for basic results on well-posed i/s/o systems.

Throughout this proof we let $\mathcal{W} = -\mathcal{U} \boxplus \mathcal{Y}$ be a fundamental decomposition of $\mathcal{W}$, and let $\mathfrak{K}_{0,t} = -\mathfrak{K}_{0,t,-} \boxplus \mathfrak{K}_{0,t,+}$ and $\mathfrak{L}_{0,\infty} = -\mathfrak{L}_{0,\infty,-} \boxplus \mathfrak{L}_{0,\infty,+}$ be the corresponding fundamental decompositions defined in (3.4).

Step 1: Necessity of properties (i) –(iii). Let $\mathcal{T}_+$ be the space of stable future trajectories of the passive s/s system Σ. Then the left-shift invariance of $\mathcal{T}_+$ follows from Lemma 3.3(iii). The nonnegativity of $\mathcal{T}_{0,t}$ in $\mathfrak{K}_{0,t}$ and the nonnegativity of $\mathcal{S}_{0,\infty}$ in $\mathfrak{L}_{0,\infty}$ follow from (1.9) with $t_1 = 0$ and $t_2 = t$. To see that $\mathcal{T}_{0,t}$ is *maximal* nonnegative we argue as follows. It follows from Lemma 3.4(i) that the projection of $\mathcal{T}_{0,t}$ onto $\mathfrak{K}_{0,t,+}$ is all of $\mathfrak{K}_{0,t,+}$ and the projection of $\mathcal{S}_{0,\infty}$ onto $\mathfrak{L}_{0,\infty,+}$ is all of $\mathfrak{L}_{0,\infty,+}$, and consequently, by Proposition 2.1(i), $\mathcal{T}_{0,t}$ and $\mathcal{S}_{0,\infty}$ are maximal nonnegative.

Step 2: Characterization of the closure of $\mathcal{T}_+$ in $\left[\begin{smallmatrix} C(\mathbb{R}^+;\mathcal{X}) \\ L^2_{\mathrm{loc}}(\mathbb{R}^+;\mathcal{W}) \end{smallmatrix}\right]$. For the proof of the sufficiency of (i)–(iii) we let $\mathcal{T}_+$ be a subspace of $\left[\begin{smallmatrix} BUC(\mathbb{R}^+;\mathcal{X}) \\ K^2(\mathbb{R}^+;\mathcal{W}) \end{smallmatrix}\right]$ with properties (i)–(iii). Since this part of our proof is based on Lemma [KS09, Lemma 4.7] we must show that the closure $\overline{\mathcal{T}_+}$ of $\mathcal{T}_+$ in $\left[\begin{smallmatrix} BUC(\mathbb{R}^+;\mathcal{X}) \\ L^2_{\mathrm{loc}}(\mathbb{R}^+;\mathcal{W}) \end{smallmatrix}\right]$ has the following three properties:

(i') $\overline{\mathcal{T}_+}$ is left-shift invariant;

(ii') For all $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \overline{\mathcal{T}_+}$ and all $t \in \mathbb{R}^+$ we have

$$\|x(t)\|^2_{\mathcal{X}} + \int_0^t \|P_{\mathcal{Y}} w(s)\|^2_{\mathcal{Y}}\, \mathrm{d}s \le \|x(0)\|^2_{\mathcal{X}} + \int_0^t \|P_{\mathcal{U}} w(s)\|^2_{\mathcal{U}}\, \mathrm{d}s; \tag{3.7}$$

(iii') For all $x_0 \in \mathcal{X}$ and $u \in L^2_{\mathrm{loc}}(\mathbb{R}^+;\mathcal{U})$ there exit a unique $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \overline{\mathcal{T}_+}$ satisfying $x(0) = x_0$ and $P_{\mathcal{U}} w = u$.

Clearly (i') follows from the left-shift invariance of $\mathcal{T}_+$. Moreover, the uniqueness in (iii') follows from (ii'), so it suffices to prove existence in (iii') in addition to (ii').

Fix $x_0 \in \mathcal{X}$ and $u \in L^2_{\mathrm{loc}}(\mathbb{R}^+;\mathcal{W})$, and let $n \in \mathbb{Z}^+$. By the maximal nonnegativity of $\mathcal{T}_{[0,n]}$ in $\mathfrak{K}_{[0,n]}$, the definition of $\mathcal{T}_{[0,n]}$, and Proposition 2.1(i), there exists some $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right] \in \mathcal{T}_+$ such that $x(0) = x_0$ and $\pi_{[0,n]} P_{\mathcal{U}} w_n = \pi_{[0,n]} u$. By the nonnegativity of $\mathcal{T}_{0,t}$ for all $t \in \mathbb{R}^+$, (3.7) holds with $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ replaced by $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right]$. We claim that $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right]$ tends to a limit in $\left[\begin{smallmatrix} C(\mathbb{R}^+;\mathcal{X}) \\ L^2_{\mathrm{loc}}(\mathbb{R}^+;\mathcal{W}) \end{smallmatrix}\right]$ as $n \to \infty$. Indeed, for all $T > 0$ and all $m, n \ge T$, we get from (3.7) applied to $\left[\begin{smallmatrix} x_n - x_m \\ w_n - w_m \end{smallmatrix}\right]$, combined with the conditions $x_n(0) = x_m(0) = x_0$ and $\pi_{[0,T]} P_{\mathcal{U}} w_n = \pi_{[0,T]} u = \pi_{[0,T]} P_{\mathcal{U}} w_m$, that

$$\sup_{0 \le t \le T} \|x_n(t) - x_m(t)\|^2_{\mathcal{X}} \le \|x_n(0) - x_m(0)\|^2_{\mathcal{X}} + \int_0^T \|P_{\mathcal{U}}(w_n(s) - w_m(s))\|^2_{\mathcal{W}} \, \mathrm{d}s = 0,$$

$$\int_0^T \|P_{\mathcal{Y}}(w_n(s) - w_m(s))\|^2_{\mathcal{W}} \, \mathrm{d}s \le \int_0^T \|P_{\mathcal{U}}(w_n(s) - w_m(s))\|^2_{\mathcal{W}} \, \mathrm{d}s = 0.$$

Thus, if we define $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ by $\left[\begin{smallmatrix} x(s) \\ w(s) \end{smallmatrix}\right] = \left[\begin{smallmatrix} x_n(s) \\ w_n(s) \end{smallmatrix}\right]$ for $s \in [n-1, n)$, $n \in \mathbb{Z}^+$, then $\pi_{[0,n)} \left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right] = \pi_{[0,n)} \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$, $n \in \mathbb{Z}^+$, and hence $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right] \to \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ in $\left[\begin{smallmatrix} C(\mathbb{R}^+;\mathcal{W}) \\ L^2_{\mathrm{loc}}(\mathbb{R}^+;\mathcal{W}) \end{smallmatrix}\right]$ as $n \to \infty$. Since each $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right] \in \mathcal{T}_+$, the limit $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ belongs to $\overline{\mathcal{T}_+}$, and (3.7) holds since we know that (3.7) holds with $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ replaced by $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right]$ for all n. This proves (ii') and (iii').

Step 3: Existence of L^2-well-posed scattering passive i/s/o representation. By Step 2 and [KS09, Lemma 4.7], there exists a L^2-well-posed i/s/o system $\left[\begin{smallmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{smallmatrix}\right]$, such that

$$\begin{aligned} \overline{\mathcal{T}_+} = \Bigg\{ \begin{bmatrix} x \\ w \end{bmatrix} \in \begin{bmatrix} C(\mathbb{R}^+;\mathcal{X}) \\ K^2_{\mathrm{loc}}(\mathbb{R}^+;\mathcal{W}) \end{bmatrix} \,\Bigg|\, \forall t \ge 0 : \\ \begin{bmatrix} x(t) \\ P_{\mathcal{Y}} w \end{bmatrix} = \begin{bmatrix} \mathfrak{A}^t & \mathfrak{B}\tau^t \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix} \begin{bmatrix} x(0) \\ P_{\mathcal{U}} w \end{bmatrix} \Bigg\}. \end{aligned} \tag{3.8}$$

That this system is scattering passive follows from (3.7) and [Sta05, Lemma 11.1.4].

Step 4: Existence of passive s/s system. By Step 2 and [KS09, Theorem 6.6], there exists a L^2-well-posed s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ such that $\overline{\mathcal{T}_+}$ is the set of all future (generalized) trajectories of Σ. The same theorem says that the decomposition $-\mathcal{Y} \boxplus \mathcal{U}$ is admissible, and it follows from (3.8) that the system $\left[\begin{smallmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{smallmatrix}\right]$ is an i/s/o representation of Σ. Every L^2-well-posed s/s system has a unique (maximal) generating subspace V in the sense of [KS09, Theorem 6.4], and by that theorem, this subspace V is given by

$$V = \begin{bmatrix} A\&B \\ [1\ 0] \\ C\&D + [0\ 1] \end{bmatrix} \operatorname{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]\right), \tag{3.9}$$

where $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right] : \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \supset \operatorname{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]\right) \to \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$ is the system node of $\left[\begin{smallmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{smallmatrix}\right]$ (see, e.g., [Sta05, Section 4.6] or [KS09, Sect. 5] for the definition of the system node of a well-posed linear i/s/o system). By [Sta05, Theorem 11.1.5] and [Kur10, Proposition 5.6], V is maximal nonnegative in the node space $\mathfrak{K}$ (note that condition (iii) in [Sta05, Theorem 11.1.5] is identical to condition (ii) in [Kur10, Proposition 5.6]). Consequently, Σ is passive.

Step 5: $\mathcal{T}_+$ is the space of stable future trajectories of Σ. By construction $\mathcal{T}_+ \subset \overline{\mathcal{T}_+} \cap \begin{bmatrix} BUC(\mathbb{R}^+;\mathcal{X}) \\ K^2(\mathbb{R}^+;\mathcal{W}) \end{bmatrix}$, and every $\left[\begin{smallmatrix} w \\ w \end{smallmatrix}\right] \in \overline{\mathcal{T}_+}$ is a future trajectory of Σ. Consequently, every $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathcal{T}_+$ is a stable future trajectory of Σ. Conversely, let $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ be a stable future trajectory of Σ. Since $\mathcal{S}_{0,\infty}$ is maximal nonnegative in $\mathfrak{L}_{0,\infty}$ and $\mathfrak{L}_{0,\infty} = -\mathfrak{L}_{0,\infty,-} \boxplus \mathfrak{L}_{0,\infty,+}$ is a fundamental decomposition of $\mathfrak{K}_+$, there exists some $\left[\begin{smallmatrix} \tilde{x} \\ \tilde{w} \end{smallmatrix}\right] \in \mathcal{T}_+$ with $\tilde{x}(0) = x(0)$ and $P_{\mathcal{U}}\tilde{w} = P_{\mathcal{U}}w$. Since $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] - \left[\begin{smallmatrix} \tilde{x} \\ \tilde{w} \end{smallmatrix}\right] \in \overline{\mathcal{T}_+}$, we can apply (3.7) to $\left[\begin{smallmatrix} x-\tilde{x} \\ w-\tilde{w} \end{smallmatrix}\right]$, and conclude that $x = \tilde{x}$ and $P_{\mathcal{U}}w = P_{\mathcal{U}}\tilde{w}$. Thus, $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] = \left[\begin{smallmatrix} \tilde{x} \\ \tilde{w} \end{smallmatrix}\right] \in \mathcal{T}_+$, and we have proved that every stable future trajectory of Σ belongs to $\mathcal{T}_+$. □

3.2 Classical trajectories and the generating subspace

We originally defined the notion of a trajectory of a passive s/s system $\Sigma = (V;\mathcal{X},\mathcal{W})$ by means of the generating subspace V. Below we shall study the converse problem: how to recreate the generating subspace from the family of all classical trajectories. (We have already encountered one result of this type in the proof of Theorem 3.5.) For simplicity we primarily restrict ourselves to future trajectories, i.e., trajectories defined on $\mathbb{R}^+$.

We begin with a preliminary lemma which gives a universal method to construct a sequence of classical approximations of an arbitrary future trajectory.

Lemma 3.6. *Let $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ be a future trajectory of the passive s/s system $\Sigma = (V;\mathcal{X},\mathcal{W})$. For each $n \in \mathbb{Z}^+$, define $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right]$ by*

$$\begin{bmatrix} x_n(t) \\ w_n(t) \end{bmatrix} := n \int_t^{t+1/n} \begin{bmatrix} x(s) \\ w(s) \end{bmatrix} \mathrm{d}s, \qquad t \in \mathbb{R}^+. \tag{3.10}$$

Then $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right]$ is a classical future trajectory of Σ, and $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right] \to \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ in $\begin{bmatrix} C(\mathbb{R}^+;\mathcal{X}) \\ L^2_{\mathrm{loc}}(\mathbb{R}^+;\mathcal{W}) \end{bmatrix}$ as $n \to \infty$. If $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is stable, then so is $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right]$, and $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right] \to \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ in $\begin{bmatrix} BUC(\mathbb{R}^+;\mathcal{X}) \\ L^2(\mathbb{R}^+;\mathcal{W}) \end{bmatrix}$ as $n \to \infty$.

This result is essentially contained in [KS11, Corollary 2.4]. For the convenience of the reader we have included a proof.

Proof of Lemma 3.6. Clearly, each $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right] \in \begin{bmatrix} C^1(\mathbb{R}^+;\mathcal{X}) \\ C(\mathbb{R}^+;\mathcal{W}) \end{bmatrix}$. Let $\left[\begin{smallmatrix} x^k \\ w^k \end{smallmatrix}\right]$ be a sequence of classical future trajectories of Σ converging to $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \begin{bmatrix} C(\mathbb{R}^+;\mathcal{X}) \\ L^2_{\mathrm{loc}}(\mathbb{R}^+;\mathcal{W}) \end{bmatrix}$.

Since the trajectories $\left[\begin{smallmatrix} x^k \\ w^k \end{smallmatrix}\right]$ are classical, they satisfy $\left[\begin{smallmatrix} \dot{x}^k(s) \\ x^k(s) \\ w^k(s) \end{smallmatrix}\right] \in V$ for all $s \in \mathbb{R}^+$, and since V is closed, also

$$n \int_t^{t+1/n} \begin{bmatrix} \dot{x}^k(s) \\ x^k(s) \\ w^k(s) \end{bmatrix} \, \mathrm{d}s \in V.$$

As $k \to \infty$,

$$\begin{bmatrix} x_n^k(t) \\ w_n^k(t) \end{bmatrix} = n \int_t^{t+1/n} \begin{bmatrix} x^k(s) \\ w^k(s) \end{bmatrix} \, \mathrm{d}s \to n \int_t^{t+1/n} \begin{bmatrix} x(s) \\ w(s) \end{bmatrix} \, \mathrm{d}s = \begin{bmatrix} x_n(t) \\ w_n(t) \end{bmatrix}$$

and

$$n \int_t^{t+1/n} \dot{x}^k(s) \, \mathrm{d}s = n[x^k(t+1/n) - x^k(t)] \to n[x(t+1/n) - x(t)] = \dot{x}_n(t).$$

Consequently, $n \int_t^{t+1/n} \left[\begin{smallmatrix} \dot{x}^k(s) \\ x^k(s) \\ w^k(s) \end{smallmatrix}\right] \mathrm{d}s \to \left[\begin{smallmatrix} \dot{x}_n(t) \\ x_n(t) \\ w_n(t) \end{smallmatrix}\right]$ as $k \to \infty$, and since V is closed, we find that $\left[\begin{smallmatrix} \dot{x}_n(t) \\ x_n(t) \\ w_n(t) \end{smallmatrix}\right] \in V$ for all $t \in \mathbb{R}^+$. Thus, $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right]$ is a classical future trajectory of Σ.

All the additional claims in Lemma 3.6 follow from standard properties of approximate identities (the scalar versions of these results are found in many places, such as [GLS90, p. 67], and the vector-valued versions can be proved in the same way). □

Proposition 3.7. *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive s/s system. Then the following claims are true:*

(i) *If $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is a (generalized) trajectory of Σ on some interval $[t_0, t_0 + h]$, and if both $z_0 := \lim_{t \to t_0+} \frac{1}{t}(x(t) - x(t_0)$ and $w_0 := \lim_{t \to t_0+} \frac{1}{t} \int_{t_0}^t w(s) \, \mathrm{d}s$ exist, then $\left[\begin{smallmatrix} z_0 \\ x(t_0) \\ w_0 \end{smallmatrix}\right] \in V$.*

(ii) *For each $\left[\begin{smallmatrix} z_0 \\ x_0 \\ w_0 \end{smallmatrix}\right] \in V$ there exists a stable future classical trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ satisfying $\left[\begin{smallmatrix} \dot{x}(0) \\ x(0) \\ w(0) \end{smallmatrix}\right] = \left[\begin{smallmatrix} z_0 \\ x_0 \\ w_0 \end{smallmatrix}\right]$ with the additional property that w is locally absolutely continuous and $\left[\begin{smallmatrix} \dot{x} \\ \dot{w} \end{smallmatrix}\right]$ is a stable future trajectory of Σ. In particular,*

$$V = \left\{ \begin{bmatrix} \dot{x}(0) \\ x(0) \\ w(0) \end{bmatrix} \,\middle|\, \begin{bmatrix} x \\ w \end{bmatrix} \text{ is a future classical trajectory of } \Sigma \right\}. \tag{3.11}$$

(iii) *A (generalized) trajectory* $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ *of* Σ *on some interval* I *is classical if and only if* $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \left[\begin{smallmatrix} C^1(I;\mathcal{X}) \\ C(I;\mathcal{W}) \end{smallmatrix}\right]$.

(iv) *There is a one-to-one correspondence between the passive s/s system* $\Sigma = (V;\mathcal{X},\mathcal{W})$ *and the set of all classical future trajectories of* Σ*, and also between* Σ *and the set of all generalized future trajectories of* Σ*.*

Parts of this proposition are also found in [KS11, Theorem 3.1 and Corollary 3.2].

Proof of Proposition 3.7. (i) By the shift-invariance expressed in Lemma 3.3(i) it suffices to treat the case $t_0 = 0$. If $h < \infty$, then we first extend $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ to a trajectory defined on all of $\mathbb{R}^+$ in an arbitrary way; cf. Lemma 3.4(iii). Let $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right]$ be the family of classical approximations of $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ defined in Lemma 3.6. Then, for all $n \in \mathbb{Z}^+$,

$$\begin{bmatrix} \dot{x}_n(0) \\ x_n(0) \\ w_n(0) \end{bmatrix} = \frac{1}{n} \begin{bmatrix} x(1/n) - x(0) \\ \int_0^{1/n} x(s)\,\mathrm{d}s \\ \int_0^{1/n} w(s)\,\mathrm{d}s \end{bmatrix} \in V.$$

This tends to $\left[\begin{smallmatrix} z_0 \\ x(0) \\ w_0 \end{smallmatrix}\right]$ as $n \to \infty$, and since V is closed, it follows that $\left[\begin{smallmatrix} z_0 \\ x(0) \\ w_0 \end{smallmatrix}\right] \in V$.

(ii) Let $\mathcal{W} = -\mathcal{Y} \boxplus \mathcal{U}$ be a fundamental decomposition of $\mathcal{W}$, and let $\left[\begin{smallmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{smallmatrix}\right]$ be the i/s/o representation of Σ constructed in the proof of Theorem 3.5. Since $\left[\begin{smallmatrix} z_0 \\ x_0 \\ w_0 \end{smallmatrix}\right] \in V$, it follows from (3.9) that $\left[\begin{smallmatrix} x_0 \\ w_0 \end{smallmatrix}\right] \in \operatorname{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]\right)$, and that

$$\begin{bmatrix} z_0 \\ w_0 \end{bmatrix} = \begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} + \begin{bmatrix} 0 \\ u_0 \end{bmatrix}.$$

Let u be an arbitrary function in $C^\infty(\mathbb{R}^+;\mathcal{U})$ with compact support and with $u(0) = u_0$, define

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} := \begin{bmatrix} \mathfrak{A}^t & \mathfrak{B}\tau^t \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix} \begin{bmatrix} x(0) \\ u \end{bmatrix}, \quad t \in \mathbb{R}^+,$$

and take $w = u+y$. By Theorem 3.5 and its proof, $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is a future trajectory of Σ, and it follows from (3.7) that this trajectory is stable. Moreover, $\left[\begin{smallmatrix} x(0) \\ w(0) \end{smallmatrix}\right] = \left[\begin{smallmatrix} x_0 \\ w_0 \end{smallmatrix}\right]$. By [Sta05, Theorem 4.6.11], $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \left[\begin{smallmatrix} C^1(\mathbb{R}^+;\mathcal{X}) \\ C(\mathbb{R}^+;\mathcal{W}) \end{smallmatrix}\right]$, $\dot{x}(0) = z_0$, and y is locally absolutely continuous with a distribution derivative $\dot{y} \in L^2_{\text{loc}}(\mathbb{R}^+;\mathcal{Y})$. In particular, by part (i), $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is a classical trajectory of Σ, and w is locally absolutely continuous with $\dot{w} \in L^2_{\text{loc}}(\mathbb{R}^+;\mathcal{W})$.

For each $n \in \mathbb{Z}^+$, we define the sequence $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right]$ by

$$\begin{bmatrix} x_n(t) \\ w_n(t) \end{bmatrix} = n \int_t^{t+1/n} \begin{bmatrix} \dot{x}(s) \\ \dot{w}(s) \end{bmatrix} \mathrm{d}s = \begin{bmatrix} n(x(t+1/n) - x(t)) \\ n(w(t+1/n) - w(t)) \end{bmatrix}, \qquad n \in \mathbb{Z}^+.$$

The set of all classical future trajectory of Σ is a left-shift invariant subspace, and consequently each $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right]$ is a classical future trajectory of Σ. In the same way as in the proof of Lemma 3.6 (with $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ replaced by $\left[\begin{smallmatrix} \dot{x} \\ \dot{w} \end{smallmatrix}\right]$) we find that $x_n \to \dot{x}$ in $C(\mathbb{R}^+; \mathcal{X})$ and $w_n \to \dot{w}$ in $L^2_{\text{loc}}(\mathbb{R}^+; \mathcal{W})$ as $n \to \infty$. It then follows from Lemma 3.3(iv) that the restriction of $\left[\begin{smallmatrix} \dot{x} \\ \dot{w} \end{smallmatrix}\right]$ to each finite interval $[0, t_2]$ is trajectory of Σ on $[0, t_2]$, and from Lemma 3.4(iv) that $\left[\begin{smallmatrix} \dot{x} \\ \dot{w} \end{smallmatrix}\right]$ is a stable future trajectory of Σ.

(iii) In the case where the right end-point of I is $+\infty$ claim (iii) follows from (i) (combined with the obvious fact that every classical trajectory is also a generalized trajectory). If I has a finite right end-point t_1, then $\left[\begin{smallmatrix} \dot{x}(t_1-1/n) \\ x(t_1-1/n) \\ w(t_1-1/n) \end{smallmatrix}\right] \in V$ for all sufficiently large n and $\left[\begin{smallmatrix} \dot{x}(t_1-1/n) \\ x(t_1-1/n) \\ w(t_1-1/n) \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \dot{x}(t_1) \\ x(t_1) \\ w(t_1) \end{smallmatrix}\right]$ as $n \to \infty$. Since V is closed, this implies that $\left[\begin{smallmatrix} \dot{x}(t_1) \\ x(t_1) \\ w(t_1) \end{smallmatrix}\right] \in V$. Thus, also in this case $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is a classical trajectory on the full interval I.

(iv) Clearly, the generating subspace V of Σ determines the sets of all future smooth and generalized trajectories of Σ uniquely. Conversely, formula (3.11) defines V uniquely in terms of the set of all classical future trajectories of Σ, and (iii) defines the set of all classical future trajectories of Σ uniquely in terms of the set of all generalized future trajectories of Σ. □

Proposition 3.8. *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive s/s system, and let $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ be a future trajectory of Σ for which w is locally absolutely continuous and $\dot{w} \in L^2_{\text{loc}}(\mathbb{R}^+; \mathcal{W})$. Then $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is a classical trajectory if and only if $\dot{x}(0) := \lim_{t \to 0+} \frac{1}{t}(x(t) - x(0))$ exists.*

Proof. The existence of $\dot{x}(0)$ is necessary for $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ to be a classical solution. Conversely, if $\dot{x}(0)$ exists, then it follows from Proposition 3.7(i) that $\left[\begin{smallmatrix} \dot{x}(0) \\ x(0) \\ w(0) \end{smallmatrix}\right] \in V$. That $x \in C^1(\mathbb{R}^+; \mathcal{X})$ then follows from [Sta05, Theorem 4.6.11] in the same way as in the preceding proof, and by Proposition 3.7, this implies that $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is a classical solution. □

3.3 More on externally generated stable trajectories

Here we continue our study of externally generated trajectories begun in Section 3.1. In particular, we now allow the left end-point of the interval I on which the trajectories are defined to be $-\infty$.

Lemma 3.9. *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive s/s system.*

(i) *Let $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ be an externally generated stable trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ of Σ on $[t_0, \infty)$. Then there exists a sequence of stable classical trajectories $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right]$ of Σ on $[t_0, \infty)$ which satisfies $\left[\begin{smallmatrix} x_n(t_0) \\ w_n(t_0) \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$ for all n and tends to $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ in $\left[\begin{smallmatrix} BUC([t_0,\infty);\mathcal{X}) \\ L^2([t_0,\infty);\mathcal{W}) \end{smallmatrix}\right]$ as $n \to \infty$.*

(ii) *If $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is an externally generated stable trajectory of Σ on the interval $[t_0, \infty)$, and if we define $x(t) = 0$ and $w(t) = 0$ for $t < t_0$, then this extended pair of functions is an externally generated stable full trajectory of Σ.*

(iii) *Let $\mathcal{W} = -\mathcal{Y} \boxplus \mathcal{U}$ be a fundamental decomposition of $\mathcal{W}$, and let I be a nontrivial closed interval. Then, for each $u \in L^2(I; \mathcal{U})$ there exists a unique externally generated stable full trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ of Σ on I satisfying $P_{\mathcal{U}} w = u$.*

(iv) *Let $\left[\begin{smallmatrix} x_1 \\ w_1 \end{smallmatrix}\right]$ be a stable externally generated trajectory of Σ on the interval $I_1 = (-\infty, t_1]$, and let $\left[\begin{smallmatrix} x_2 \\ w_2 \end{smallmatrix}\right]$ be a stable trajectory of Σ on an interval I_2 with left end-point t_1. Then the concatenation $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ defined by (3.3) is a stable trajectory of Σ on $I := I_1 \cup I_2$ if and only if $x_1(t_1) = x_2(t_1)$.*

(v) *Every stable trajectory on the interval interval $I = (-\infty, t_1]$ can be extended to a stable full trajectory of Σ. This extension can be chosen so that $\pi_{[t_1,\infty)} P_{\mathcal{U}} w = u$ for an arbitrary $u \in L^2([t_1, \infty); \mathcal{U})$, and it is uniquely determined by u.*

Proof. (i) By Lemma 3.3(i), it suffices to prove the case where $t_0 = 0$. Let $\mathcal{W} = -\mathcal{Y} \boxplus \mathcal{U}$ be a fundamental decomposition of $\mathcal{W}$, and let $\{u_n\}$ be a sequence of $\mathcal{U}$-valued C^∞ functions with compact support such that $u_n(0) = 0$ for each n and $u_n \to P_{\mathcal{U}} w$ in $L^2(\mathbb{R}^+, \mathcal{U})$ as $n \to \infty$. Let $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right]$ be the stable future trajectory of Σ with $x_n(0) = 0$ given by Lemma 3.4(i). By [Sta05, Theorem 4.6.11], $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \left[\begin{smallmatrix} C^1(\mathbb{R}^+;\mathcal{X}) \\ C(\mathbb{R}^+;\mathcal{W}) \end{smallmatrix}\right]$ and $w_n(0) = 0$. By Proposition 3.7(iii), each $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right]$ is a classical future trajectory of Σ. It follows from Lemma 3.2 that $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right]$ is stable, and that $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right] \to \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ in $\left[\begin{smallmatrix} BUC(\mathbb{R}^+;\mathcal{X}) \\ L^2(\mathbb{R}^+;\mathcal{W}) \end{smallmatrix}\right]$ as $n \to \infty$.

(ii) Let $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right]$ be a sequence of classical stable future trajectories of Σ with the properties listed in (i). If we define $\left[\begin{smallmatrix} x_n(t) \\ w_n(t) \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$ for $t < 0$, then each $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right]$ is a classical stable full trajectory of Σ (note that $\dot{x}_n(0) = 0$ since $\left[\begin{smallmatrix} x_n(t) \\ w_n(t) \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} \dot{x}(0) \\ x(0) \\ w(0) \end{smallmatrix}\right] \in V$.) This extended sequence converges to the extended version of $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ in $\left[\begin{smallmatrix} BUC(\mathbb{R};\mathcal{X}) \\ L^2(\mathbb{R};\mathcal{W}) \end{smallmatrix}\right]$ as $n \to \infty$, and by Lemma 3.3(iv), the limit is a stable externally generated full trajectory of Σ.

(iii) In the case where I has a finite left end-point the claim (iii) follows from Lemma 3.4(i). Thus, we may here assume that the left end-point of I is $-\infty$. If the right end-point of I is finite, then we start by extending u to all of $\mathbb{R}$ by defining $u(t) = 0$ for $t \notin I$.

For each $n \in I$, define $u_n = P_{[-n,\infty)}u$. Then $u_n \to u$ in $L^2(\mathbb{R};\mathcal{U})$ as $n \to \infty$. Let $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right]$ be the externally generated trajectory of Σ on $[-n, \infty)$ satisfying $P_{\mathcal{U}}w_n = u_n$ given by Lemma 3.4(i), and use (ii) to extend this trajectory to a full externally generated trajectory, which we still denote by $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right]$. It follows from Lemma 3.2 that $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right]$ is a Cauchy sequence in $\left[\begin{smallmatrix} BUC(\mathbb{R};\mathcal{X}) \\ L^2(\mathbb{R};\mathcal{W}) \end{smallmatrix}\right]$ as $n \to \infty$, and hence it converges to a limit $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ in this space. By Lemma 3.3, this limit is an externally generated full trajectory of Σ. Clearly $P_{\mathcal{U}}w = u$. The uniqueness of this trajectory follows from Lemma 3.2.

(iv) The necessity of the condition $x_1(t_1) = x_2(t)$ for $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ to be a trajectory is obvious, since x is required to be continuous at t_1.

Let $\mathcal{W} = -\mathcal{Y} \boxplus \mathcal{U}$ be a fundamental decomposition of $\mathcal{W}$, and let $\left[\begin{smallmatrix} x' \\ w' \end{smallmatrix}\right]$ be the unique externally generated full trajectory of Σ given by (iii) which satisfies $P_{\mathcal{U}}w' = P_{\mathcal{U}}w$. Then the restriction of $\left[\begin{smallmatrix} x' \\ w' \end{smallmatrix}\right]$ to I_1 is an externally generated trajectory of Σ on I_1, and by (iii), this restriction is equal to $\left[\begin{smallmatrix} x_1 \\ w_1 \end{smallmatrix}\right]$. On the other hand, the restriction of $\left[\begin{smallmatrix} x' \\ w' \end{smallmatrix}\right]$ to I_2 is a trajectory of Σ on I_2, and by Lemma 3.4(i), this restriction is equal to $\left[\begin{smallmatrix} x_2 \\ w_2 \end{smallmatrix}\right]$. Thus, $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] = \left[\begin{smallmatrix} x' \\ w' \end{smallmatrix}\right]$, and so $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is an externally generated trajectory of Σ on I.

(v) That (v) is true follows from (iv) and Lemma 3.4(i). □

3.4 Passive past, full, and future behaviors

We recall the following definition from Section 1.

Definition 3.10. By the (stable) *behavior* $\mathfrak{W}^\Sigma(I)$ of the passive s/s system Σ on the closed and nontrivial interval I we mean the set of all the signal parts w of all externally generated stable trajectories $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ of Σ on I.

In the special cases $I = \mathbb{R}^-$, $I = \mathbb{R}$, and $I = \mathbb{R}^+$ we denote these behaviors by $\mathfrak{W}^\Sigma_-$, $\mathfrak{W}^\Sigma$, and $\mathfrak{W}^\Sigma_+$, and refer to them as the *past, full, and future behaviors* of Σ, respectively.

Lemma 3.11. *To each $w \in \mathfrak{W}^\Sigma_+$ there exists a unique $x \in C(\mathbb{R}^+;\mathcal{X})$ such that $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is an externally generated stable trajectory of Σ on $\mathbb{R}^+$, and this function satisfies $x \in BUC(\mathbb{R}^+;\mathcal{X})$. The same statement remains true if we replace $\mathfrak{W}^\Sigma_+$ by $\mathfrak{W}^\Sigma$ or by $\mathfrak{W}^\Sigma_-$ and at the same time replace $\mathbb{R}^+$ by $\mathbb{R}$ or $\mathbb{R}^-$, respectively.*

Proof. This follows from the definitions of $\mathfrak{W}_+^\Sigma$, $\mathfrak{W}^\Sigma$, and $\mathfrak{W}_-^\Sigma$ and Lemmas 3.4(i) and 3.9(iii). □

The shift (semi)groups τ^t, τ_-^t, and τ_+^t used in the following lemma were defined at the end of Section 1.

Lemma 3.12. *The past, full, and future behaviors $\mathfrak{W}_-^\Sigma$, $\mathfrak{W}^\Sigma$, and $\mathfrak{W}_+^\Sigma$ of a passive s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ have the following properties:*

(i) *$\mathfrak{W}_\pm$ are right-shift invariant and $\mathfrak{W}^\Sigma$ is bilaterally shift-invariant, i.e.,*

$$\begin{aligned} \tau_\pm^{*t}\mathfrak{W}_\pm^\Sigma &\subset \mathfrak{W}_\pm^\Sigma, \qquad t \in \mathbb{R}^+, \\ \tau^t\mathfrak{W}^\Sigma &= \mathfrak{W}^\Sigma, \qquad t \in \mathbb{R}. \end{aligned} \tag{3.12}$$

(ii) *$\mathfrak{W}_\pm^\Sigma$ can be recovered from $\mathfrak{W}^\Sigma$ by the formulas*

$$\begin{aligned} \mathfrak{W}_-^\Sigma &= \pi_-\mathfrak{W}^\Sigma := \{w_- \in K_-^2(\mathcal{W}) \mid w_- = \pi_- w \textit{ for some } w \in \mathfrak{W}^\Sigma\}, \\ \mathfrak{W}_+^\Sigma &= \mathfrak{W}^\Sigma \cap K_+^2(\mathcal{W}) := \{w \in \mathfrak{W}^\Sigma \mid w(t) = 0 \textit{ for } t < 0\}. \end{aligned} \tag{3.13}$$

(iii) *$\mathfrak{W}_\pm^\Sigma$ is a maximal nonnegative subspace of $K_\pm^2(\mathcal{W})$ and $\mathfrak{W}^\Sigma$ is a maximal nonnegative subspace of $K^2(\mathcal{W})$.*

Proof. (i) By Lemma 3.3(i), $\tau^t\mathfrak{W}^\Sigma = \mathfrak{W}^\Sigma$ for all $t \in \mathbb{R}$. By Lemma 3.3(i)–(ii), $\tau_-^{*t}\mathfrak{W}_-^\Sigma \subset \mathfrak{W}_-^\Sigma$ (and actually even $\tau_-^{*t}\mathfrak{W}_-^\Sigma = \mathfrak{W}_-^\Sigma$) for all $t \in \mathbb{R}^+$. That $\tau_+^{*t}\mathfrak{W}_+^\Sigma \subset \mathfrak{W}_+^\Sigma$ for all $t \in \mathbb{R}^+$ follows from Lemmas 3.3(i)–(ii) and 3.9(ii).

(ii) If $w \in \mathfrak{W}^\Sigma$, then by Lemma 3.3(ii), $w_- := \pi_- w \in \mathfrak{W}_-^\Sigma$. Conversely, according to Lemma 3.9, every $w_- \in \mathfrak{W}_-^\Sigma$ can be extended to a function $w \in \mathfrak{W}^\Sigma$. Analogously, by Lemma 3.3(ii), if $w \in \mathfrak{W}^\Sigma \cap K_+^2(\mathcal{W})$ then $w \in \mathfrak{W}_+^\Sigma$, and if $w \in \mathfrak{W}_+^\Sigma$ and we extend w to $K^2(\mathcal{W})$ by defining $w(t) = 0$ for $t < 0$, then by Lemma 3.9(ii), the extended function belongs to $\mathfrak{W}^\Sigma$.

(iii) That $\mathfrak{W}_-^\Sigma$, $\mathfrak{W}^\Sigma$, and $\mathfrak{W}_+^\Sigma$ are nonnegative follows from Lemma 3.2. To see that they are *maximal* nonnegative it suffices to take an arbitrary fundamental decomposition $\mathcal{W} = -\mathcal{Y} \boxplus \mathcal{U}$ of $\mathcal{W}$ and use Lemmas 3.4(i) and 3.9(iii) and Proposition 2.1(i). □

At this point we must warn the reader that if $\mathfrak{W}$ is an arbitrary maximal nonnegative bilaterally shift-invariant subspace of $K^2(\mathcal{W})$ and if we define $\mathfrak{W}_- = \pi_-\mathfrak{W}$ and $\mathfrak{W}_+ = \mathfrak{W} \cap K_+^2(\mathcal{W})$ (as in (3.12)) then it need *not be true* that $\mathfrak{W}_-$ is maximal nonnegative in $K_-^2(\mathcal{W})$ or that $\mathfrak{W}_+$ is maximal nonnegative in $K_+^2(\mathcal{W})$. A discrete time counter example is given in [AS09b, Examples 2.7 and 2.14], and the same example can easily be modified to become a continuous time counter example. The following lemma clarifies the situation.

Lemma 3.13. *Let $\mathfrak{W}$ be a maximal nonnegative subspace $\mathfrak{W}$ of $K^2(\mathcal{W})$, and define $\mathfrak{W}_-$ and $\mathfrak{W}_+$ by*

$$\mathfrak{W}_- := \pi_-\mathfrak{W}, \qquad \mathfrak{W}_+ := \mathfrak{W} \cap K^2_+(\mathcal{W}), \tag{3.14}$$

Then the following conditions are equivalent:

(i) *$\mathfrak{W}_-$ is a maximal nonnegative subspace of $K^2_-(\mathcal{W})$.*

(ii) *$\mathfrak{W}_+$ is a maximal nonnegative subspace of $K^2_+(\mathcal{W})$.*

(iii) *For* some *fundamental decomposition $\mathcal{W} = -\mathcal{Y} \boxplus \mathcal{U}$ the following implication is valid: If $w \in \mathfrak{W}$ and $\pi_-\pi_{\mathcal{U}}w = 0$, then $\pi_-\pi_{\mathcal{Y}}w = 0$.*

(iv) *For* every *fundamental decomposition $\mathcal{W} = -\mathcal{Y} \boxplus \mathcal{U}$ the following implication is valid: If $w \in \mathfrak{W}$ and $\pi_-\pi_{\mathcal{U}}w = 0$, then $\pi_-\pi_{\mathcal{Y}}w = 0$.*

Proof. This follows from Lemma 2.4 with the substitutions $\mathfrak{K} \to K^2(\mathcal{W})$, $\mathfrak{K}_1 \to K^2_+(\mathcal{W})$, $\mathfrak{K}_2 \to K^2_-(\mathcal{W})$, $\mathcal{Z} \to \mathfrak{W}$, $\mathcal{Y}_2 \to L^2_-(\mathcal{Y})$, and $\mathcal{U}_2 \to L^2_-(\mathcal{U})$. Note that conditions (i) and (ii) in Lemma 2.4 do not depend on the particular fundamental decomposition used in part (iii) of that lemma, so if (iii) holds for one fundamental decomposition, then it holds for every fundamental decomposition. □

Motivated by Lemmas 3.12 and 3.13 we make the following definition:

Definition 3.14. Let $\mathcal{W}$ be a Kreĭn space.

(i) A maximal nonnegative right-shift invariant subspace of $K^2_-(\mathcal{W})$ is called a *passive past behavior* on the (signal) space $\mathcal{W}$.

(ii) A maximal nonnegative right-shift invariant subspace $\mathfrak{W}_+$ of $K^2_+(\mathcal{W})$ is called a *passive future behavior* on the (signal) space $\mathcal{W}$.

(iii) A maximal nonnegative bilaterally shift invariant subspace $\mathfrak{W}$ of $K^2(\mathcal{W})$ which satisfies the equivalent conditions (i)–(iv) listed in Lemma 3.13 is called a *passive full behavior* on the Kreĭn (signal) space $\mathcal{W}$.

Proposition 3.15. *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive s/s system. Then the past, full, and future behaviors of Σ are passive past, full, and future behaviors, respectively, on $\mathcal{W}$ in the sense of Definition 3.14.*

Proof. This follows from Lemma 3.12 and Definition 3.14 □

As we shall see in Lemma 3.18 below, each one of these behaviors determine the two others uniquely.

Passive past and future behaviors actually have slightly stronger shift-invariance properties than what is explicitly required in Definition 3.14.

Lemma 3.16. *Let $\mathcal{W}$ be a Kreĭn space.*

(i) *Every passive past behavior $\mathfrak{W}_-$ on $\mathcal{W}$ satisfies $\tau_-^{*t}\mathfrak{W}_- = \mathfrak{W}_-$ for all $t \in \mathbb{R}^+$.*

(ii) *Every passive future behavior $\mathfrak{W}_+$ on $\mathcal{W}$ satisfies*

$$\tau_+^{*t}\mathfrak{W}_+ = \{w \in \mathfrak{W}_+ \mid w(s) = 0 \text{ for almost all } s \in [0,t]\}$$

for all $t \in \mathbb{R}^+$.

Before proving this lemma we make the following remark.

Remark 3.17. Many of our subsequent results (as well as Lemmas 3.13 and 3.18 above) can be regarded as continuous time versions of the corresponding discrete time results given in [AS09b] and [AS10]. In many cases the proofs given in [AS09b] and [AS10] can be adapted to the present setting by performing some simple substitutions. As a general rule, all those notions defined [AS09b] and [AS10] have a natural counterpart presented here, they should be replaced by that counterpart. The discrete time right shifts S_-, S, and S_+ are replaced by τ_-^{*t}, τ^{-t}, and τ_+^{*t}, $t \in \mathbb{R}^+$, and the discrete time left shifts S_-^*, S^{-1}, and S_+^* are replaced by τ_-^t, τ^t, and τ_+^t, $t \in \mathbb{R}^+$. The discrete time trajectories in [AS09b] and [AS10] are throughout replaced by *generalized continuous time trajectories* (i.e., no *classical* trajectories enter in these translations). The main difference between the discrete time and the continuous time cases is that in the proofs one should *not* replace the discrete time generating subspace V by the continuous time generating subspace V. Instead, in computations involving *future trajectories* one should through *replace V by the subspaces $\mathcal{T}_{0,t}$ defined in* (3.5), and the discrete time node space $\mathfrak{K}$ *should be replaced by the Kreĭn space* $\mathfrak{K}_{0,t}$ with the inner product (1.12), and we throughout *use Theorem 3.5 to characterize the passivity of a continuous time s/s system*, and not the original Definition 3.1. In connection with *past* trajectories we replace the discrete time generating subspace V by a left-shifted versions $\mathcal{T}_{[-t,0]}$ and $\mathfrak{K}_{[-t,0]}$ of $\mathcal{T}_{0,t}$ and $\mathfrak{K}_{0,t}$. This has the consequence that whenever a discrete time formula contains the term $[w(0), w(0)]_{\mathcal{W}}$ it should be replaced by $\int_0^t [w(s), w(s)]_{\mathcal{W}}\,\mathrm{d}s$, and analogously $[w(-1), w(-1)]_{\mathcal{W}}$ should be replaced by $\int_{-t}^0 [w(s), w(s)]_{\mathcal{W}}\,\mathrm{d}s$. As a consequence of these changes, the continuous time proofs are often slightly shorter than the discrete time proofs, since there is no need to build trajectories on a finite time interval $[0,T]$ from scratch, as sometimes happens in the discrete time setting.

If the proof of some particular result given below can be obtained from the corresponding result in [AS09b] or [AS10] by performing the substitutions listed above, then we sometimes omit the proof, and refer the reader to [AS09b] or [AS10]. We do this, in particular, if the conversion is straightforward and the proof is of significant length. If the proof is short, or if the conversion is less straightforward, or if the proof is important for the general understanding of the theory we write it out in full detail.

As an example on how to convert discrete time results to continuous time results, let us look at the graph representation of passive behaviors used in the proof of [AS09b, Theorem 2.11]. (These graph representations are needed, among others, for the proof of Lemma 3.16.) Let $\mathcal{W} = -\mathcal{Y} \boxplus \mathcal{U}$ be a fundamental decomposition of $\mathcal{W}$. Then $K^2(\mathcal{W}) = -L^2(\mathcal{Y}) \boxplus L^2(\mathcal{U})$ and $K^2_\pm(\mathcal{W}) = -L^2_\pm(\mathcal{Y}) \boxplus L^2_\pm(\mathcal{U})$ are fundamental decompositions of of the Kreĭn spaces $K^2(\mathcal{W})$ and $K^2_\pm(\mathcal{W})$, respectively. By assertion (i) and (iv) of Proposition 2.1, every passive past, full, and future behavior $\mathfrak{W}_-$, $\mathfrak{W}$, and $\mathfrak{W}_+$ on $\mathcal{W}$ and their orthogonal companions have graph representations with respect to the above fundamental decompositions of the type

$$\begin{aligned} \mathfrak{W}_\pm &= \left\{ \left[\begin{smallmatrix} \mathfrak{D}_\pm u \\ u \end{smallmatrix}\right] \middle| \, u \in L^2_\pm(\mathcal{U}) \right\}, & \mathfrak{W} &= \left\{ \left[\begin{smallmatrix} \mathfrak{D} u \\ u \end{smallmatrix}\right] \middle| \, u \in L^2 \right\}, \\ \mathfrak{W}_\pm^{[\perp]} &= \left\{ \left[\begin{smallmatrix} y \\ \mathfrak{D}_\pm^* y \end{smallmatrix}\right] \middle| \, y \in L^2_\pm(\mathcal{Y}) \right\}, & \mathfrak{W}^{[\perp]} &= \left\{ \left[\begin{smallmatrix} y \\ \mathfrak{D}^* y \end{smallmatrix}\right] \middle| \, y \in L^2(\mathcal{Y}) \right\}, \end{aligned} \tag{3.15}$$

where $\mathfrak{D}_\pm$ and $\mathfrak{D}$ are linear contractions between the respective L^2-spaces. It follows from Lemma 3.13 and Definition 3.14 hat $\pi_-\mathfrak{D}|_{L^2_+(\mathcal{U})} = 0$ and $\pi_+\mathfrak{D}^*|_{L^2_-(\mathcal{U})} = 0$, i.e., $\mathfrak{D}$ and $\mathfrak{D}^*$ are *causal* and *anti-causal*, respectively. Since $\tau_\pm^{*t}\mathfrak{W}_\pm \subset \mathfrak{W}_\pm$ for all $t \in \mathbb{R}^+$ and $\tau^t\mathfrak{W} = \mathfrak{W}$ for all $t \in \mathbb{R}$, it follows from (3.15) that $\mathfrak{D}_\pm$ are right-shift invariant and $\mathfrak{D}$ is bilaterally shift invariant, i.e.,

$$\tau_\pm^{*t}\mathfrak{D}_\pm = \mathfrak{D}_\pm\tau_\pm^{*t} \text{ for all } t \in \mathbb{R}^+ \text{ and } \tau^t\mathfrak{D} = \mathfrak{D}\tau^t \text{ for all } t \in \mathbb{R}. \tag{3.16}$$

Furthermore, if the three behaviors $\mathfrak{W}_\pm$ and $\mathfrak{W}$ are related to each other by the relations (3.14)–(3.19), then $\mathfrak{D}_\pm$ and $\mathfrak{D}$ are related to each other by

$$\begin{aligned} \mathfrak{D}_+ &= \mathfrak{D}|_{L^2_+(\mathcal{U})}, & \mathfrak{D}_- &= \pi_-\mathfrak{D}|_{L^2_-(\mathcal{U})}, \\ \mathfrak{D}_+^* &= \pi_+\mathfrak{D}^*|_{L^2_+(\mathcal{U})}, & \mathfrak{D}_-^* &= \mathfrak{D}^*|_{L^2_-(\mathcal{U})}. \end{aligned} \tag{3.17}$$

Proof of Lemma 3.16. Let $\mathcal{W} = -\mathcal{Y} \boxplus \mathcal{U}$ be a fundamental decomposition of $\mathcal{W}$.

(i) By (3.15), the right-shift invariance of $\mathfrak{D}_-$, and the fact that $\tau_-^{*t}L^2_-(\mathcal{U}) = L^2_-(\mathcal{U})$, we have

$$\begin{aligned}\tau_-^{*t}\mathfrak{W}_- &= \left\{\left[\begin{smallmatrix}\tau_-^{*t}\mathfrak{D}_-u\\ \tau_-^{*t}u\end{smallmatrix}\right]\middle|\, u\in L^2_-(\mathcal{U})\right\} = \left\{\left[\begin{smallmatrix}\mathfrak{D}_-\tau_-^{*t}u\\ \tau_-^{*t}u\end{smallmatrix}\right]\middle|\, u\in L^2_-(\mathcal{U})\right\}\\ &= \left\{\left[\begin{smallmatrix}\mathfrak{D}_-u\\ u\end{smallmatrix}\right]\middle|\, u\in L^2_-(\mathcal{U})\right\} = \mathfrak{W}_-.\end{aligned}$$

(ii) If $u \in \tau_+^{*t}\mathfrak{W}_+$, then by the shift-invariance of $\mathfrak{W}_+$, $\tau_+^{*t}w \in \mathfrak{W}_+$, and of course, w vanishes on $[0,t]$. Conversely, let $w \in \mathfrak{W}_+$ vanish on $[0,t]$. Define $u_1 = \tau_+^t P_{\mathcal{U}}w$, and $w_1 = \left[\begin{smallmatrix}\mathfrak{D}_+u_1\\ u_1\end{smallmatrix}\right]$. Then

$$\tau_+^{*t}w_1 = \left[\begin{smallmatrix}\tau_+^{*t}\mathfrak{D}_+u_1\\ \tau_+^{*t}u_1\end{smallmatrix}\right] = \left[\begin{smallmatrix}\mathfrak{D}_+\tau_+^{*t}u_1\\ \tau_+^{*t}u_1\end{smallmatrix}\right] = \left[\begin{smallmatrix}\mathfrak{D}_+P_{\mathcal{U}}w\\ P_{\mathcal{U}}w\end{smallmatrix}\right] = w.$$

Consequently, $w \in \tau_+^{*t}\mathfrak{W}_+$. □

The following lemma complements Lemmas 3.12 and 3.13.

Lemma 3.18. *Let $\mathcal{W}$ be a Kreĭn space.*

(i) *If $\mathfrak{W}_-$ is a passive past behavior on $\mathcal{W}$, and if we define $\mathfrak{W}$ by*

$$\mathfrak{W} = \bigcap_{t\in\mathbb{R}^+}\left\{w\in K^2(\mathcal{W})\,\middle|\,\pi_-\tau^t w\in\mathfrak{W}_-\right\}, \tag{3.18}$$

then $\mathfrak{W}$ is a passive full behavior on $\mathcal{W}$ and $\mathfrak{W}_- = \pi_-\mathfrak{W}$.

(ii) *If $\mathfrak{W}_+$ is a passive future behavior on $\mathcal{W}$, and if we define $\mathfrak{W}$ by*

$$\mathfrak{W} = \bigvee_{t\in\mathbb{R}^+}\tau^t\mathfrak{W}_+, \tag{3.19}$$

then $\mathfrak{W}$ is a passive full behavior on $\mathcal{W}$, and $\mathfrak{W}_+ = \mathfrak{W}\cap K^2_+(\mathcal{W})$.

(iii) *Let $\mathfrak{W}$ be a passive full behavior on the Kreĭn signal space $\mathcal{W}$, and define $\mathfrak{W}_-$ and $\mathfrak{W}_+$ by* (3.14)*. Then $\mathfrak{W}_-$ is a passive past behavior on $\mathcal{W}$, $\mathfrak{W}_+$ is a passive future behavior on $\mathcal{W}$, and $\mathfrak{W}$ can be recovered from $\mathfrak{W}_+$ and from $\mathfrak{W}_-$ by means of formulas* (3.18) *and* (3.19)*.*

Proof. (i) Let $\mathfrak{W}_-$ be a passive past behavior on $\mathcal{W}$, and define $\mathfrak{W}$ by (3.19). Denote

$$\mathfrak{W}_-^t = \left\{w\in K^2(\mathcal{W})\,\middle|\,\pi_-\tau^t w\in\mathfrak{W}_-\right\},\qquad t\in\mathbb{R},$$

so that $\mathfrak{W} = \bigcap_{t\in\mathbb{R}^+}\mathfrak{W}_-^t$. Then $\mathfrak{W}_-^t = \tau^{-t}\mathfrak{W}_-^0$ since

$$w\in\mathfrak{W}_-^t \Leftrightarrow \pi_-\tau^t w\in\mathfrak{W}_- \Leftrightarrow \tau^t w\in\mathfrak{W}_-^0 \Leftrightarrow w\in\tau^{-t}\mathfrak{W}_-^0.$$

Thus $\tau^s\mathfrak{W}^t_- = \tau^{s-t}\mathfrak{W}^0_- = \mathfrak{W}^{t-s}_-$ for all s, $t \in \mathbb{R}$. If $w \in \mathfrak{W}^t_-$, or equivalently, $\pi_-\tau^t w \in \mathfrak{W}_-$, then the right-shift invariance of $\mathfrak{W}_-$ implies that for all $s \in \mathbb{R}^+$,

$$\pi_-\tau^{t-s}w = \tau^{*s}_-\pi_-\tau^t w \in \mathfrak{W}_-.$$

and consequently $\mathfrak{W}^t_- \subset \mathfrak{W}^{t-s}_-$ for all $t \in \mathbb{R}$ and $s \in \mathbb{R}^+$. In particular, this implies that for all $t \in \mathbb{R}$,

$$\tau^t\mathfrak{W} = \tau^t \bigcap_{s\in\mathbb{R}^+} \mathfrak{W}^s_- = \bigcap_{s\in\mathbb{R}^+} \tau^t\mathfrak{W}^s_- = \bigcap_{s\in\mathbb{R}^+} \mathfrak{W}^{s-t}_- = \bigcap_{r\geq -t} \mathfrak{W}^r_- = \bigcap_{r\in\mathbb{R}^+} \mathfrak{W}^r_- = \mathfrak{W}.$$

Thus, $\mathfrak{W}$ is bilaterally shift-invariant.

We next show that $\mathfrak{W}$ is nonnegative in $K^2(\mathcal{W})$. If $w \in \mathfrak{W}^t_-$ for some $t \in \mathbb{R}$, then it follows from the definition of $\mathfrak{W}^t_-$ and the maximal nonnegativity of $\mathfrak{W}_-$ that

$$0 \leq \int_{\mathbb{R}^-} [(\tau^t w)(s), (\tau^t w)(s)]_{\mathcal{W}}\,\mathrm{d}s = \int_{-\infty}^t [w(s), w(s)]_{\mathcal{W}}\,\mathrm{d}s.$$

If $w \in \mathfrak{W} = \bigcap_{t\in\mathbb{R}^+} \mathfrak{W}^t_-$, then we can let $t \to \infty$ to get $[w, w]_{K^2(\mathcal{W})} \geq 0$. Thus, $\mathfrak{W}$ is a nonnegative subspace of $K^2(\mathcal{W})$.

To prove that $\mathfrak{W}$ is maximal nonnegative in $K^2(\mathcal{W})$ we let $K^2(\mathcal{W}) = -L^2(\mathcal{Y}) \boxplus L^2(\mathcal{U})$ be a fundamental decomposition of $K^2(\mathcal{W})$, where $\mathcal{W} = -\mathcal{Y} \boxplus \mathcal{U}$ is a fundamental decomposition of $\mathcal{W}$. Let u be an arbitrary function in $L^2(\mathcal{U})$. By the definition of $\mathfrak{W}^t_-$, the maximal nonnegativity of $\mathfrak{W}_-$, and Proposition 2.1(i), for each $n \in \mathbb{Z}^+$ there exists some $w_n \in \mathfrak{W}^n_-$ such that $\pi_- P_{\mathcal{U}}\tau^n w_n = \pi_- u$, or equivalently, $\pi_{(-\infty,n]}P_{\mathcal{U}}w_n = \pi_{(-\infty,n]}u$. Moreover, $\pi_{(-\infty,n]}P_{\mathcal{Y}}w_n$ is uniquely determined by $\pi_{(-\infty,n]}u$. Since $\mathfrak{W}^m_- \subset \mathfrak{W}^n_-$ for all m, $n \in \mathbb{Z}^+$, $n \geq m$, this implies that $\pi_{(-\infty,m]}w_n = \pi_{(-\infty,m]}w_m$ for all $n \geq m$. If we use the Hilbert space norm in $\mathcal{W}$ induced by the decomposition $\mathcal{W} = -\mathcal{Y}\boxplus\mathcal{U}$, then

$$\|\pi_{(-\infty,n]}w_n\|_{L^2(-\infty,n];\mathcal{W})} \leq 2\|\pi_{(-\infty,n]}u\|_{L^2(-\infty,n];\mathcal{U})} \leq 2\|u\|_{L^2(\mathcal{U})}.$$

Define $w(t) = w_0(t)$ for $t \leq 0$, and $w(t) = w_n(t)$ for $t \in (n-1, n]$, $n \geq 1$. Then $\pi_{(-\infty,n]}\tau^n w = \pi_{(-\infty,n]}\tau^n w_n \in \mathfrak{W}_-$, and consequently $w \in \bigcap_{n\in\mathbb{Z}^+} \mathfrak{W}^n_- = \mathfrak{W}$. By Proposition 2.1(i), $\mathfrak{W}$ is maximal nonnegative.

Trivially, $\pi_-\mathfrak{W} \subset \pi_-\mathfrak{W}^0_- = \mathfrak{W}_-$. Conversely, take some arbitrary $w_- \in \mathfrak{W}_-$. Let $\mathcal{W} = -\mathcal{Y} \boxplus \mathcal{U}$ be a fundamental decomposition of $\mathcal{W}$, and define $u(t) = P_{\mathcal{U}}w_-(t)$ for $t \in \mathbb{R}^-$ and $u(t) = 0$ for $t > 0$. Let w be the corresponding function in $\mathcal{W}$ constructed in the preceding paragraph. Then $\pi_- w \in \mathfrak{W}_-$ and $P_{\mathcal{U}}\pi_- w = P_{\mathcal{U}}w_-$. Consequently, since every function in $\mathfrak{W}_-$ is uniquely determined by its $\mathcal{U}$-component, we have $\pi_- w = w_-$. Thus, $\mathfrak{W}_- \subset \pi_-\mathfrak{W}$. Together with the inclusion $\pi_-\mathfrak{W} \subset \mathfrak{W}_-$ this gives $\pi_-\mathfrak{W} = \mathfrak{W}_-$. By Definition 3.14, $\mathfrak{W}$ is a passive full behavior.

(ii) Let $\mathfrak{W}_+$ be a passive future behavior on $\mathcal{W}$, and define $\mathfrak{W}$ by (3.19). Denote

$$\mathfrak{W}_+^t = \tau^t \mathfrak{W}_+, \qquad t \in \mathbb{R},$$

so that $\mathfrak{W} = \bigvee_{t\in\mathbb{R}^+} \mathfrak{W}_+^t$. Trivially $\tau^s \mathfrak{W}_+^t = \tau^{s+t}\mathfrak{W}_+ = \mathfrak{W}_+^{s+t}$ for all s, $t \in \mathbb{R}$. The right-shift invariance of $\mathfrak{W}_+$ implies that $\mathfrak{W}_+^s \subset \mathfrak{W}_+^t$ for all $s \le t$. In particular, for all $t \in \mathbb{R}$,

$$\tau^t\mathfrak{W} = \tau^t \bigvee_{s\in\mathbb{R}^+} \mathfrak{W}_+^s = \bigvee_{s\in\mathbb{R}^+} \tau^t\mathfrak{W}_+^s = \bigvee_{s\in\mathbb{R}^+} \mathfrak{W}_+^{s+t} = \bigvee_{r\ge t} \mathfrak{W}_+^r = \bigvee_{r\in\mathbb{R}^+} \mathfrak{W}_+^r = \mathfrak{W}.$$

Thus, $\mathfrak{W}$ is bilaterally shift-invariant.

We next show that $\mathfrak{W}$ is nonnegative in $K^2(\mathcal{W})$. If $w \in \mathfrak{W}_+^t$ for some $t \in \mathbb{R}$, then it follows from the definition of $\mathfrak{W}_+^t$ and the maximal nonnegativity of $\mathfrak{W}_+$ that

$$0 \le \int_{\mathbb{R}^+} [(\tau^{-t}w)(s), (\tau^{-t}w)(s)]_{\mathcal{W}} \, \mathrm{d}s = \int_{\mathbb{R}} [w(s), w(s)]_{\mathcal{W}} \, \mathrm{d}s.$$

Thus, each of the subspaces $\mathfrak{W}_+^t$ is nonnegative, and hence so is the closed linear hull $\mathfrak{W} = \bigvee_{t\in\mathbb{R}^+} \mathfrak{W}_+^t$.

To prove that $\mathfrak{W}$ is maximal nonnegative in $K^2(\mathcal{W})$ we let $K^2(\mathcal{W}) = -L^2(\mathcal{Y}) \boxplus L^2(\mathcal{U})$ be a fundamental decomposition of $K^2(\mathcal{W})$, where $\mathcal{W} = -\mathcal{Y} \boxplus \mathcal{U}$ is a fundamental decomposition of $\mathcal{W}$. Let u be an arbitrary function in $L^2(\mathcal{U})$. By the definition of $\mathfrak{W}_+^t$, the maximal nonnegativity of $\mathfrak{W}_+$, and Proposition 2.1(i), for each $n \in \mathbb{Z}^+$ there exists some $w_n \in \mathfrak{W}_+^n$ such that $P_{\mathcal{U}} w_n = \pi_{[-n,\infty)} u$. If we use the Hilbert space norm in $\mathcal{W}$ induced by the decomposition $\mathcal{W} = -\mathcal{Y} \boxplus \mathcal{U}$, then for all m, $n \in \mathbb{Z}^+$, $m \ge n$,

$$\|w_m - w_n\|_{L^2(\mathbb{R};\mathcal{W})} \le 2\|u\|_{L^2([-m,-n];\mathcal{U})}.$$

Thus, w_n is a Cauchy sequence in $L^2(\mathcal{W})$ which converges to a limit w in $L^2(\mathcal{W})$. Since each $w_n \in \mathfrak{W}_+^n$, we have $w \in \bigvee_{t\in\mathbb{R}^+} \mathfrak{W}_+^t = \mathfrak{W}$. Thus, $P_{\mathcal{U}}\mathfrak{W} = L^2(\mathcal{U})$, and by Proposition 2.1(i), $\mathfrak{W}$ is maximal nonnegative.

By Lemma 3.16, for each $t \in \mathbb{R}^+$ we have $\mathfrak{W}_+ = \mathfrak{W}_+^t \cap K_+^2(\mathcal{W})$. Thus $\mathfrak{W}_+ \subset \big(\bigvee_{t\in\mathbb{R}^+} \mathfrak{W}_+^t\big) \cap K_+^2(\mathcal{W}) = \mathfrak{W} \cap K_+^2(\mathcal{W})$. On the other hand, $\mathfrak{W} \cap K_+^2(\mathcal{W})$ is a nonnegative subspace of $K_+^2(\mathcal{W})$ whereas $\mathfrak{W}_+$ is a *maximal* nonnegative subspace of $K_+^2(\mathcal{W})$ contained in $K_+^2(\mathcal{W}) \cap \mathfrak{W}$. Thus, $\mathfrak{W}_+ = K_+^2(\mathcal{W}) \cap \mathfrak{W}$. By Definition 3.14, $\mathfrak{W}$ is a passive full behavior.

(iii) Let $\mathfrak{W}$ be a passive full behavior on $\mathcal{W}$, and define $\mathfrak{W}_-$ and $\mathfrak{W}_+$ by (3.14). It follows from Definition 3.14 that $\mathfrak{W}_+$ is a maximal nonnegative subspace of $K^2_+(\mathcal{W})$ and that $\mathfrak{W}_-$ is a maximal nonnegative subspace of $K^2_-(\mathcal{W})$. The right-shift invariance of $\mathfrak{W}_\pm$ follows from the bilateral shift invariance of $\mathfrak{W}$ and (3.14). Thus, $\mathfrak{W}_+$ and $\mathfrak{W}_-$ are passive future and past behaviors, respectively. This proves the first two claims in (i).

We continue with the proof of (3.18). Denote the right-hand side of (3.18) by $\widetilde{\mathfrak{W}}$. By (i), $\widetilde{\mathfrak{W}}$ is a (maximal) nonnegative subspace of $K^2(\mathcal{W})$, and it follows from Definition 3.14 that $\mathfrak{W} \subset \widetilde{\mathfrak{W}}$. Since $\mathfrak{W}$ is maximal nonnegative, we must have $\mathfrak{W} = \widetilde{\mathfrak{W}}$, and consequently (3.18) holds.

We finally prove (3.19). Denote the right-hand side of (3.19) by $\widetilde{\mathfrak{W}}$. By (ii), $\widetilde{\mathfrak{W}}$ is a maximal nonnegative subspace of $K^2(\mathcal{W})$, and it follows from Definition 3.14 that $\widetilde{\mathfrak{W}} \subset \mathfrak{W}$. Since $\mathfrak{W}$ is nonnegative, we must have $\mathfrak{W} = \widetilde{\mathfrak{W}}$, and consequently (3.19) holds.

□

Lemma 3.19. *Let $\mathfrak{W}_-$ be a passive past behavior on a Kreĭn space $\mathcal{W}$. Then the set of all $w \in \mathfrak{W}_-$ with compact support is a dense subspace of $\mathfrak{W}_-$.*

Proof. By Lemma 3.18(iii),

$$\mathfrak{W}_- = \pi_-\mathfrak{W} = \pi_- \bigvee_{t\in\mathbb{R}^+} \tau^t\mathfrak{W}_+ = \bigvee_{t\in\mathbb{R}^+} \pi_-\tau^t\mathfrak{W}_+,$$

where each function in $\pi_-\tau^t\mathfrak{W}_+$ has compact support. □

Lemma 3.20. *Let $\mathfrak{W}_+$ be a passive future behavior on $\mathcal{W}$, and define the $[0,t]$-sections $\mathfrak{W}_{[0,t]}$ of $\mathfrak{W}_+$ by*

$$\mathfrak{W}_{[0,t]} := \pi_{[0,t]}\mathfrak{W}_+, \qquad t \in \mathbb{R}^+. \tag{3.20}$$

Then each $\mathfrak{W}_{[0,t]}$ is a maximal nonnegative subspace of $K^2([0,t];\mathcal{W})$.

Proof. By Lemma 3.16, $\mathfrak{W}_+ \cap K^2([t,\infty);\mathcal{W}) = \tau_+^{*t}\mathfrak{W}_+$, and therefore $\mathfrak{W}_+ \cap K^2([t,\infty);\mathcal{W})$ is maximal nonnegative in $K^2([t,\infty);\mathcal{W})$. This fact, combined with Lemma 2.4 with the substitutions $\mathfrak{K} \to K^2_+(\mathcal{W})$, $\mathfrak{K}_1 \to K^2([t,\infty);\mathcal{W})$, $\mathfrak{K}_2 \to K^2([0,t]\mathcal{W})$, and $\mathcal{Z} \to \mathfrak{W}_+$, implies that $\pi_{[0,t]}\mathfrak{W}_+$ is maximal nonnegative in $K^2([0,t];\mathcal{W})$. □

3.5 Intertwined systems

Definition 3.21. Let $\Sigma_1 = (V_1;\mathcal{X}_1;\mathcal{W})$ and $\Sigma_2 = (V_2;\mathcal{X}_2;\mathcal{W})$ be two passive s/s systems (with the same signal space $\mathcal{W}$).

(i) A bounded linear operator $E\colon \mathcal{X}_1 \to \mathcal{X}_2$ *intertwines* the two passive s/s systems Σ_1 and Σ_2 if the formula

$$(x_1, w) \mapsto (Ex_1, w) \tag{3.21}$$

defines a map from the set of all stable future trajectories $\left[\begin{smallmatrix} x_1 \\ w \end{smallmatrix}\right]$ of Σ_1 onto the set of all stable future trajectories $\left[\begin{smallmatrix} x_2 \\ w \end{smallmatrix}\right]$ of Σ_2 satisfying $x_2(0) \in \operatorname{im}(E)$.

(ii) Σ_1 and Σ_2 are *boundedly intertwined* if there exists an operator $E \in \mathcal{B}(\mathcal{X}_1; \mathcal{X}_2)$ which intertwines Σ_1 and Σ_2. The operator E is called an intertwining operator between Σ_1 and Σ_2.

(iii) Σ_1 and Σ_2 are *contractively intertwined* if there exists a contraction $E \in \mathcal{B}(\mathcal{X}_1; \mathcal{X}_2)$ which intertwines Σ_1 and Σ_2.

(iv) Σ_1 and Σ_2 are *similar* if there exists a boundedly invertible operator $E \in \mathcal{B}(\mathcal{X}_1; \mathcal{X}_2)$ which intertwines Σ_1 and Σ_2. The operator E is called a similarity operator between Σ_1 and Σ_2.

(v) Σ_1 and Σ_2 are *unitarily similar* if there exists a unitary operator $E \in \mathcal{B}(\mathcal{X}_1; \mathcal{X}_2)$ which intertwines Σ_1 and Σ_2.

Note, in particular, that if Σ_1 and Σ_2 are boundedly intertwined, then they have the same future behavior.

Definition 3.22. (i) The s/s system $\widetilde{\Sigma} = (\widetilde{V}; \widetilde{\mathcal{X}}, \mathcal{W})$ is called an *orthogonal outgoing dilation* of the s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ and Σ is called an *orthogonal outgoing compression onto* $\mathcal{X}$ of $\widetilde{\Sigma}$, if $\mathcal{X} \subset \widetilde{\mathcal{X}}$ and the orthogonal projection of $\widetilde{\mathcal{X}}$ onto $\mathcal{X}$ intertwines $\widetilde{\Sigma}$ and Σ.

(ii) The s/s system $\widetilde{\Sigma}$ is called an *incoming dilation* of Σ and Σ is called an *incoming compression* of $\widetilde{\Sigma}$ if $\mathcal{X} \subset \widetilde{\mathcal{X}}$ and the embedding operator $\mathcal{X} \hookrightarrow \widetilde{\mathcal{X}}$ intertwines Σ and $\widetilde{\Sigma}$.

4 The Anti-Passive Adjoint State/Signal Systems

4.1 Anti-passive state/signal systems

According to Definition 3.1, the generating subspace V of a passive s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is required to be maximal nonnegative. Consequently, by Proposition 2.1(iv), its orthogonal companion $V^{[\perp]}$ is maximal nonpositive.

Definition 4.1. Let $\mathcal{X}$ be a Hilbert space and $\mathcal{W}$ a Kreĭn space.

(i) By an *anti-passive s/s node* in continuous time we mean a triple $\Sigma^\dagger = (V^\dagger; \mathcal{X}, \mathcal{W})$ where $V^\dagger$ is a maximal nonpositive subspace of the Kreĭn *node space* $\mathfrak{K} := \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix}$ equipped with the inner product (1.7), with the additional property that if $\begin{bmatrix} z \\ 0 \\ 0 \end{bmatrix} \in V^\dagger$, then $z = 0$.

(ii) *Classical* and *generalized trajectories* of an anti-passive s/s system $\Sigma^\dagger$ are defined in the same way as in the case of a passive system (see Definition 3.1(i)–(ii)).

(iii) The anti-passive s/s node $\Sigma^\dagger$ together with its families of classical and generalized trajectories is called an *anti-passive s/s system*, and it denoted by the same symbols as the node.

(iv) By a *past*, *full*, or *future* trajectory of an anti-passive system $\Sigma^\dagger$ we mean a trajectory of Σ on $\mathbb{R}^-$, $\mathbb{R}$, or $\mathbb{R}^+$, respectively.

(v) A (generalized) trajectory $\begin{bmatrix} x^\dagger \\ w^\dagger \end{bmatrix}$ of an anti-passive s/s system $\Sigma^\dagger = (V^\dagger; \mathcal{X}, \mathcal{W})$ on an interval I is *backward externally generated* if the following condition holds: If I has a finite right end-point t_1, then we require that $x^\dagger(t_1) = 0$, and if the right end-point of I is ∞, then we require that $\lim_{t\to\infty} x^\dagger(t) = 0$ and that $w^\dagger \in L^2([T,\infty); \mathcal{W})$ for every finite $T \in I$.

(vi) A (generalized) trajectory $\begin{bmatrix} x^\dagger \\ w^\dagger \end{bmatrix}$ of an anti-passive s/s system $\Sigma^\dagger = (V^\dagger; \mathcal{X}, \mathcal{W})$ is *stable* if x is bounded on I and $w \in L^2(I; \mathcal{W})$.

To distinguish between trajectories of a passive s/s system and an anti-passive system we often denote the trajectories of an anti-passive system by $\begin{bmatrix} x^\dagger \\ w^\dagger \end{bmatrix}$.

Remark 4.2. Since the generating subspace $V^\dagger$ of an anti-passive system is maximal nonpositive in the node space $\mathfrak{K}$, it is maximal nonnegative in the anti-space $-\mathfrak{K}$. The inner product in $-\mathfrak{K}$ is given by

$$\left[\begin{bmatrix} z_1 \\ x_1 \\ w_1 \end{bmatrix}, \begin{bmatrix} z_2 \\ x_2 \\ w_2 \end{bmatrix}\right]_{-\mathfrak{K}} = -(-z_1, x_2) - (x_1, -z_2) - [w_1, w_2]_{\mathcal{W}}. \tag{4.1}$$

Recall that the z-component represents the time derivative $\dot{x}^\dagger(t)$ of a classical trajectory $\begin{bmatrix} x^\dagger \\ w^\dagger \end{bmatrix}$, the x-component represents the state $x^\dagger(t)$ itself, and the w-component represents the signal $w^\dagger(t)$. The change of sign in the z-component in (4.1) compared to (1.7) can be interpreted as a reflection of the time direction, since $\frac{\mathrm{d}}{\mathrm{d}t}x^\dagger(-t) = -\dot{x}^\dagger(-t)$, and the change of sign in the w-component amounts to the replacement of the signal space $\mathcal{W}$ by its anti-space. This means that the theory of anti-passive s/s systems is identical to the theory of passive s/s systems, apart from a reflection of the time axis, and a change of sign in the signal component. Because of the reflection, externally generated stable trajectories of a passive s/s system correspond to backward externally generated trajectories of the corresponding reflected system. All the results listed in Sections 3.1–3.3 have anti-passive counterparts, where the past and future have changed places. Note, in particular, that the basic inequalities (1.6), (1.8), and (1.9) are reversed. We shall not give a complete list here, but only formulate those results that we actually use. See also Remark 4.12 below.

Lemma 4.3. $\Sigma = (V; \mathcal{X}, \mathcal{W})$ *is a passive s/s system if and only if* $\Sigma^{[\perp]} = (V^{[\perp]}; \mathcal{X}, \mathcal{W})$ *is an anti-passive s/s system.*

Proof. Suppose that $\Sigma := (V; \mathcal{X}, \mathcal{W})$ is a passive s/s system. By Proposition 2.1(iv), $V^{[\perp]}$ is maximal nonpositive subspace of the node space $\mathfrak{K}$ since V is maximal nonnegative in $\mathfrak{K}$. That $V^{[\perp]}$ also satisfies the additional condition that if $\begin{bmatrix} z^\dagger \\ 0 \\ 0 \end{bmatrix} \in V^\dagger$, then $z^\dagger = 0$ follows from [Kur10, Corollary 4.8]. That also the converse claim is true follows from Remark 4.2. $\square$

Definition 4.4. The *anti-passive dual* of a passive s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is the anti-passive s/s system $\Sigma^{[\perp]} := (V^{[\perp]}; \mathcal{X}, \mathcal{W})$.

Above we have defined the anti-passive dual of a s/s system by means of its generating subspace. It can alternatively be characterized by means of the orthogonality between the trajectories of the original system and its dual, as described in the following theorem.

Theorem 4.5. *Let* $\Sigma = (V; \mathcal{X}, \mathcal{W})$ *be a passive s/s system, and let* $\Sigma^{[\perp]} = (V^{[\perp]}; \mathcal{X}, \mathcal{W})$ *be its anti-passive dual.*

(i) *For each interval* I *with finite right end-point* t_2*, the pair of functions* $\begin{bmatrix} x^\dagger \\ w^\dagger \end{bmatrix} \in \begin{bmatrix} C(I;\mathcal{X}) \\ L^2(I;\mathcal{W}) \end{bmatrix}$ *is a stable trajectory of* $\Sigma^{[\perp]}$ *in* I *if and only if*

$$(x^\dagger(t_2), x(t_2))_{\mathcal{X}} = (x^\dagger(t_1), x(t_1))_{\mathcal{X}} + \int_{t_1}^{t_2} [w^\dagger(s), w(s)]_{\mathcal{W}} \,\mathrm{d}s \qquad (4.2)$$

for all $t_1 \in I$ *and all stable trajectories* $\begin{bmatrix} x \\ w \end{bmatrix}$ *of* Σ *on* $[t_1, t_2]$.

(ii) *For each interval I with right end-point ∞, the pair of functions $\left[\begin{smallmatrix} x^\dagger \\ w^\dagger \end{smallmatrix}\right] \in \left[\begin{smallmatrix} C(I;\mathcal{X}) \\ L^2(I;\mathcal{W}) \end{smallmatrix}\right]$ is a backward externally generated stable trajectory of $\Sigma^{[\perp]}$ in I if and only if $x^\dagger(t) \to 0$ as $t \to \infty$ and*

$$0 = (x^\dagger(t_1), x(t_1))_{\mathcal{X}} + \int_{t_1}^{\infty} [w^\dagger(s), w(s)]_{\mathcal{W}} \,\mathrm{d}s \tag{4.3}$$

for all $t_1 \in I$ and all stable trajectories $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ of Σ on $[t_1, \infty)$.

Proof. The proofs of claims (i) and (ii) are almost identical to each other, so we only prove (i), and leave the proof of (ii) to the reader.

The necessity of (4.2) is quite obvious: if both of the trajectories are classical, then one gets (4.2) by integrating the equation

$$0 = \left[\left[\begin{bmatrix} \dot{x}^\dagger(s) \\ x^\dagger(s) \\ w^\dagger(s) \end{bmatrix}, \begin{bmatrix} \dot{x}(s) \\ x(s) \\ w(s) \end{bmatrix}\right]\right]_{\mathfrak{K}} = -\frac{\mathrm{d}}{\mathrm{d}s}(x^\dagger(s), x(s))_{\mathcal{X}} + [w^\dagger(s), w(s)]_{\mathcal{W}}$$

over the interval $[t_1, t_2]$. In the case of generalized trajectories we first approximate $\left[\begin{smallmatrix} x^\dagger \\ w^\dagger \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ by sequences of classical trajectories, and then pass to the limit to get (4.2) for generalized trajectories $\left[\begin{smallmatrix} x^\dagger \\ w^\dagger \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$.

Conversely, let $\left[\begin{smallmatrix} x^\dagger \\ w^\dagger \end{smallmatrix}\right] \in \left[\begin{smallmatrix} C(I;\mathcal{X}) \\ L^2(I;\mathcal{W}) \end{smallmatrix}\right]$ satisfy (4.2) for all $t_1 \le t_2$, $t_1 \in I$ and all trajectories $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ of Σ on $[t_1, t_2]$. Fix $t_0 \in I$. By Lemma 3.4(i) and Remark 4.2, there exists a stable trajectory $\left[\begin{smallmatrix} x_1^\dagger \\ w_1^\dagger \end{smallmatrix}\right]$ of $\Sigma^{[\perp]}$ on $[t_0, t_2]$ with $x_1^\dagger(t_2) = x^\dagger(t_2)$ and $P_{\mathcal{Y}}\pi_{[t_0,t_2]}w_1^\dagger = P_{\mathcal{Y}}\pi_{[t_0,t_2]}w^\dagger$. By the first part of the proof and by our assumption on $\left[\begin{smallmatrix} x^\dagger \\ w^\dagger \end{smallmatrix}\right]$, for all $t_1 \in [t_0, t_2]$,

$$\begin{aligned} 0 &= (x^\dagger(t_1) - x_1^\dagger(t_1), x(t_1))_{\mathcal{X}} + \int_{t_1}^{t_2} [w^\dagger(s) - w_1^\dagger(s), w(s)]_{\mathcal{W}} \,\mathrm{d}s \\ &= (x^\dagger(t_1) - x_1^\dagger(t_1), x(t_1))_{\mathcal{X}} + \int_{t_1}^{t_2} [P_{\mathcal{U}}(w^\dagger(s) - w_1^\dagger(s)), P_{\mathcal{U}}w(s)]_{\mathcal{W}} \,\mathrm{d}s. \end{aligned}$$

By Lemma 3.4(i), the pair $\left[\begin{smallmatrix} x(t_1) \\ P_{\mathcal{U}}P_{[t_1,t_2]}w \end{smallmatrix}\right]$ can be an arbitrary vector in $\mathcal{X} \times L^2([t_1, t_2]; \mathcal{U})$, and consequently $x^\dagger(t_1) = x_1^\dagger(t_1)$ and $\pi_{[t_1,t_2]}w^\dagger = \pi_{[t_1,t_2]}w_1^\dagger$. Thus, the restriction of $\left[\begin{smallmatrix} x^\dagger \\ w^\dagger \end{smallmatrix}\right]$ to any finite interval $[t_0, t_2]$ of I is a trajectory of $\Sigma^{[\perp]}$ on $[t_0, t_2]$, and by Lemma 3.4(iv) and Remark 4.2, $\left[\begin{smallmatrix} x^\dagger \\ w^\dagger \end{smallmatrix}\right]$ is a stable trajectory of $\Sigma^{[\perp]}$ on I. □

Corollary 4.6. *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive s/s system, and let $\Sigma^{[\perp]} = (V^{[\perp]}; \mathcal{X}, \mathcal{W})$ be the anti-passive dual of Σ.*

(i) *If* $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ *is a stable past trajectory of* Σ *and* $\left[\begin{smallmatrix} x^\dagger \\ w^\dagger \end{smallmatrix}\right]$ *a stable past trajectory of* $\Sigma^{[\perp]}$*, then* $\lim_{t\to-\infty}(x(t), x^\dagger(t))_{\mathcal{X}}$ *exists, and*

$$(x^\dagger(0), x(0))_{\mathcal{X}} = \lim_{t\to-\infty}(x^\dagger(t), x(t))_{\mathcal{X}} + [w^\dagger, w]_{K^2_-(\mathcal{W})}. \tag{4.4}$$

(ii) *If* $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ *is a stable future trajectory of* Σ *and* $\left[\begin{smallmatrix} x^\dagger \\ w^\dagger \end{smallmatrix}\right]$ *a stable future trajectory of* $\Sigma^{[\perp]}$*, then* $\lim_{t\to\infty}(x(t), x^\dagger(t))_{\mathcal{X}}$ *exists, and*

$$\lim_{t\to\infty}(x^\dagger(t), x(t))_{\mathcal{X}} = (x^\dagger(0), x(0))_{\mathcal{X}} + [w^\dagger, w]_{K^2(\mathbb{R}^+;\mathcal{W})}. \tag{4.5}$$

(iii) *If* $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ *is a stable full trajectory of* Σ *and* $\left[\begin{smallmatrix} x^\dagger \\ w^\dagger \end{smallmatrix}\right]$ *is a stable full trajectory of* $\Sigma^\dagger$*, then* $\lim_{t\to-\infty}(x(t), x^\dagger(t))_{\mathcal{X}}$ *and* $\lim_{t\to\infty}(x(t), x^\dagger(t))_{\mathcal{X}}$ *exist, and*

$$\lim_{t\to\infty}(x^\dagger(t), x(t))_{\mathcal{X}} = \lim_{t\to-\infty}(x^\dagger(t), x(t))_{\mathcal{X}} + [w^\dagger, w]_{K^2(\mathbb{R};\mathcal{W})}. \tag{4.6}$$

Proof. This follows immediately from Theorem 4.5. □

Definition 4.7. A passive s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is *energy preserving* if $V \subset V^{[\perp]}$, it is *co-energy preserving* if $V^{[\perp]} \subset V$, and it is *conservative* if $V = V^{[\perp]}$. Analogously, an anti-passive s/s system $\Sigma^\dagger = (V^\dagger; \mathcal{X}, \mathcal{W})$ is energy preserving, co-energy preserving, or conservative if $V^\dagger \subset (V^\dagger)^{[\perp]}$, $(V^\dagger)^{[\perp]} \subset V^\dagger$, or $V^\dagger = (V^\dagger)^\perp$, respectively.

Thus, in particular, *a conservative s/s system is at the same time both passive and anti-passive.*

Lemma 4.8. *Let* $\Sigma = (V; \mathcal{X}, \mathcal{W})$ *be a passive s/s system, and let* $\Sigma^{[\perp]} = (V^{[\perp]}; \mathcal{X}, \mathcal{W})$ *be its anti-passive dual.*

(i) Σ *is energy preserving if and only if every trajectory of* Σ *on every nontrivial interval* I *is also a trajectory of* $\Sigma^{[\perp]}$ *on* I*.*

(ii) Σ *is co-energy preserving if and only if every trajectory of* $\Sigma^{[\perp]}$ *on every nontrivial interval* I *is also a trajectory of* Σ *on* I*.*

(iii) Σ *is conservative if and only if* Σ *and* $\Sigma^{[\perp]}$ *have the same set of trajectories on every nontrivial interval* I*.*

The same claims remain true if we restrict I *to belong to the family of all nontrivial finite subintervals of* $\mathbb{R}^+$*.*

Proof. This follows from Definitions 3.1 and 4.7 combined with Lemma 3.3 and Proposition 3.7, which imply that the generating subspace is uniquely determined by the set of all trajectories on some arbitrarily small interval. □

Lemma 4.9. *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive s/s node with the space $\mathcal{T}_+$ of stable future trajectories, and define $\mathcal{T}_{0,t}$ and $\mathfrak{K}_{0,t}$ by* (3.5) *and* (1.12)*, respectively. Then Σ is energy preserving, co-energy preserving or conservative if and only if $\mathcal{T}_{0,t}$ is a neutral, co-neutral or Lagrangian subspace, respectively of $\mathfrak{K}_{0,t}$ for all $t \in \mathbb{R}^+$.*

Proof. If Σ is energy preserving, then the argument leading up to (1.9) shows that $\mathcal{T}_{0,t}$ is a neutral subspace of $\mathfrak{K}_{0,t}$ for all $t \in \mathbb{R}^+$. Conversely, suppose that $\mathcal{T}_{0,t}$ is a neutral subspace of $\mathfrak{K}_{0,t}$ for all $t \in \mathbb{R}^+$. Then it follows from Theorem 4.5 that every trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ of Σ on some interval $[0,t]$ is also a trajectory of $\Sigma^{[\perp]}$ on $[0,t]$, and Lemma 4.8 then shows that Σ is neutral.

Recall that Σ is co-energy preserving if and only if its anti-passive dual $\Sigma^{[\perp]}$ is energy preserving. Let $\mathcal{T}_-^\dagger$ be the family of all stable past trajectories of $\Sigma^{[\perp]}$, and denote

$$\mathcal{T}_{t,0}^\dagger := \left\{ \begin{bmatrix} x^\dagger(t) \\ x^\dagger(0) \\ \pi_{[t,0]} w^\dagger \end{bmatrix} \middle| \left[\begin{smallmatrix} x^\dagger \\ w^\dagger \end{smallmatrix}\right] \in \mathcal{T}_-^\dagger \right\}, \quad t \in \mathbb{R}^-. \tag{4.7}$$

By Remark 4.2 and the part of Lemma 4.9 which we have already established, $\Sigma^{[\perp]}$ is energy-preserving if and only if $\mathcal{T}_{t,0}^\dagger$ is a neutral subspace of $\mathfrak{K}_{t,0}$ for all $t \in \mathbb{R}^-$, or equivalently, if and only if $\tau^{-t}\mathcal{T}_{-t,0}^\dagger$ is a $\mathfrak{K}_{0,t}$ for all $t \in \mathbb{R}^+$.

We claim that $\tau^{-t}\mathcal{T}_{-t,0}^\dagger = \mathcal{T}_{0,t}^{[\perp]}$. It follows from Theorem 4.5 that $\tau^{-t}\mathcal{T}_{-t,0}^\dagger \subset \mathcal{T}_{0,t}^{[\perp]}$. On the other hand, by Theorem 3.5 and Remark 4.2, $\tau^{-t}\mathcal{T}_{-t,0}^\dagger$ is maximal nonpositive, whereas by Theorem 3.5 and Proposition 2.1(iv), $\mathcal{T}_{0,t}^{[\perp]}$ is (maximal) nonpositive. Thus, $\tau^{-t}\mathcal{T}_{-t,0}^\dagger = \mathcal{T}_{0,t}^{[\perp]}$, as claimed.

Since $\tau^{-t}\mathcal{T}_{-t,0}^\dagger = \mathcal{T}_{0,t}^{[\perp]}$, we find that Σ is energy-preserving if and only if $\mathcal{T}_{0,t}$ is a neutral subspace of $\mathfrak{K}_{0,t}$ for all $t \in \mathbb{R}^+$.

Finally, Σ is conservative if and only if Σ is at the same time both energy preserving and co-energy preserving, and by the above argument, this is equivalent to the condition that $\mathcal{T}_{0,t}$ is both a neutral and a co-neutral subspace of $\mathfrak{K}_{0,t}$. □

4.2 Anti-passive behaviors

Definition 4.10. Let $\mathcal{W}$ be a Kreĭn space.

(i) A maximal nonpositive left-shift invariant subspace of $K_-^2(\mathcal{W})$ is called an *anti-passive past behavior* on the (signal) space $\mathcal{W}$.

(ii) A maximal nonpositive left-shift invariant subspace of $K^2_+(\mathcal{W})$ is called an *anti-passive future behavior* on the Kreĭn (signal) space $\mathcal{W}$.

(iii) A maximal nonpositive bilaterally shift-invariant subspace $\mathfrak{W}^\dagger$ of $K^2(\mathcal{W})$ is called an *anti-passive full behavior* on the Kreĭn (signal) space $\mathcal{W}$ if $\mathfrak{W}^\dagger_+ := \pi_+\mathfrak{W}^\dagger$ is a maximal nonpositive subspace of $K^2_+(\mathcal{W})$, or equivalently, if $\mathfrak{W}^\dagger_- := \mathfrak{W}^\dagger \cap K^2_-(\mathcal{W})$ is a maximal nonpositive subspace of $K^2_-(\mathcal{W})$

Indeed, by Lemma 3.13 and Remark 4.2, the two conditions given in part (iii) of the above definition are equivalent.

Lemma 4.11. (i) *A closed subspace $\mathfrak{W}_+$ of $K^2_+(\mathcal{W})$ is a passive future behavior on $\mathcal{W}$ if and only if $\mathfrak{W}_+^{[\perp]}$ is an anti-passive future behavior on $\mathcal{W}$.*

(ii) *A closed subspace $\mathfrak{W}_-$ of $K^2_-(\mathcal{W})$ is a passive past behavior on $\mathcal{W}$ if and only if $\mathfrak{W}_-^{[\perp]}$ is an anti-passive past behavior on $\mathcal{W}$.*

(iii) *A closed subspace $\mathfrak{W}$ of $K^2(\mathcal{W})$ is a passive full behavior on $\mathcal{W}$ if and only if $\mathfrak{W}^{[\perp]}$ is an anti-passive full behavior on $\mathcal{W}$.*

Proof. (i) By definition, $\mathfrak{W}_+$ is a passive future behavior if and only if $\mathfrak{W}_+$ is maximal nonnegative in $K^2_+(\mathcal{W})$ and right-shift invariant. Since $\mathfrak{W}_+$ is assumed to be closed, according to Proposition 2.1(iv), $\mathfrak{W}_+$ is maximal nonnegative if and only if $\mathfrak{W}_+^{[\perp]}$ is maximal nonpositive. It is also easy to see that $\mathfrak{W}_+$ is right-shift invariant if and only if $\mathfrak{W}_+^{[\perp]}$ is left-shift invariant. Thus, $\mathfrak{W}_+$ is a passive future behavior if and only if $\mathfrak{W}_+^{[\perp]}$ is an anti-passive future behavior.

(ii) The proof of the claim about the past behaviors is analogous.

(iii) In the case of full behaviors, by arguing in the same way as above we find that $\mathfrak{W}$ is maximal nonnegative and bilaterally shift-invariant if and only if $\mathfrak{W}^{[\perp]}$ is maximal nonpositive and bilaterally shift-invariant. By the continuous time version of [AS09b, Lemma 3.5] (cf. Remark 3.17),

$$\mathfrak{W}^{[\perp]} \cap K^2_-(\mathfrak{W}) = (\pi_-\mathfrak{W})^{[\perp]}.$$

Thus, by Lemma 3.13, Remark 4.2, and Definitions 3.14 and 4.10, $\mathfrak{W}$ is a passive full behavior if and only if $\mathfrak{W}^{[\perp]}$ is an anti-passive full behavior. □

Remark 4.12. It is easy to see that $\mathfrak{W}_+$, $\mathfrak{W}$, and $\mathfrak{W}_-$ are passive future, full, or past behaviors on the signal space $\mathcal{W}$ if and only if the time-reflected versions of these behaviors are anti-passive past, full, or future behaviors, respectively, on the signal space $-\mathcal{W}$. This implies that all the results in Section 3.4 have anti-passive counterparts, where the past and the future have been interchanged with each other. We shall not give a complete list here, but only formulate those results that we actually use. See also Remark 4.2 above.

Lemma 4.13. *Let $\mathfrak{W}_+$ be a passive future behavior on a Kreĭn space $\mathcal{W}$. Then the set of all $w^\dagger \in \mathfrak{W}_+^{[\perp]}$ with compact support is a dense subspace of $\mathfrak{W}_+^{[\perp]}$.*

Proof. This follows from Lemmas 4.11 and 3.19 and Remark 4.12. □

Definition 4.14. By the (stable) *backward behavior* induced by the anti-passive s/s system $\Sigma^\dagger$ on the closed and nontrivial interval I we mean the set of all the signal parts $w^\dagger$ of all backward externally generated stable trajectories $\left[\begin{smallmatrix} x^\dagger \\ w^\dagger \end{smallmatrix}\right]$ of $\Sigma^\dagger$.

In the special cases $I = \mathbb{R}^-$, $I = \mathbb{R}$, and $I = \mathbb{R}^+$ we denote these behaviors by ${}^\dagger\mathfrak{W}_-^{\Sigma^\dagger}$, ${}^\dagger\mathfrak{W}^{\Sigma^\dagger}$, and ${}^\dagger\mathfrak{W}_+^{\Sigma^\dagger}$, and refer to them as the *past, full, and future backward behaviors* of $\Sigma^\dagger$, respectively.

Proposition 4.15. *The past, full, and future backward behaviors of an anti-passive s/s system are anti-passive past, full, and future behaviors, respectively, in the sense of Definition 4.10.*

Proof. This follows from Proposition 3.15 and Remark 4.2. □

Proposition 4.16. *The past, full, and future backward behaviors ${}^\dagger\mathfrak{W}_-^{\Sigma^{[\perp]}}$, ${}^\dagger\mathfrak{W}^{\Sigma^{[\perp]}}$, and ${}^\dagger\mathfrak{W}_+^{\Sigma^{[\perp]}}$ of the anti-causal dual $\Sigma^{[\perp]} = (V^{[\perp]}; \mathcal{X}, \mathcal{W})$ of the passive s/s systems $\Sigma = (V; \mathcal{X}, \mathcal{W})$ are given by*

$$
{}^\dagger\mathfrak{W}_-^{\Sigma^{[\perp]}} = (\mathfrak{W}_-^{\Sigma})^{[\perp]}, \quad {}^\dagger\mathfrak{W}^{\Sigma^{[\perp]}} = (\mathfrak{W}^{\Sigma})^{[\perp]}, \quad \text{and } {}^\dagger\mathfrak{W}_+^{\Sigma^{[\perp]}} = (\mathfrak{W}_+^{\Sigma})^{[\perp]}, \tag{4.8}
$$

where $\mathfrak{W}_-^{\Sigma}$, $\mathfrak{W}^{\Sigma}$, and $\mathfrak{W}_+^{\Sigma}$ are the past, full, and future behaviors of Σ.

Proof. These three identities are in principle proved in the same way, so we only prove one of them. If $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} x^\dagger \\ w^\dagger \end{smallmatrix}\right]$ are stable externally and backward externally generated trajectories of Σ and $\Sigma^{[\perp]}$, respectively, then by Corollary 4.6, $[w, w^\dagger]_{k^2(\mathcal{W})} = 0$. This implies that ${}^\dagger\mathfrak{W}^{\Sigma^{[\perp]}} \subset (\mathfrak{W}^{\Sigma})^{[\perp]}$. Since ${}^\dagger\mathfrak{W}^{\Sigma^{[\perp]}}$ is maximal nonpositive and $(\mathfrak{W}^{\Sigma})^{[\perp]}$ is nonpositive, this implies that ${}^\dagger\mathfrak{W}^{\Sigma^{[\perp]}} = (\mathfrak{W}^{\Sigma})^{[\perp]}$. □

5 The Hilbert Spaces $\mathcal{H}(\mathfrak{W}_+)$, $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$, and $\mathcal{D}(\mathfrak{W})$

In this subsection we shall present three special Hilbert spaces that play a central role throughout the rest of this article. These Hilbert spaces will be used as the state spaces of three of our canonical passive s/s realizations of a given passive behavior. These first two of them are special cases of the Hilbert space $\mathcal{H}(\mathcal{Z})$ constructed in [AS09a] and described in Section 2.2, where $\mathcal{Z}$ is a maximal nonnegative subspace of a Kreĭn space $\mathcal{K}$, and the third is constructed from the first two and an angle operator, called the past/future map.

We begin by adapting the spaces $\mathcal{H}(\mathcal{Z})$ from Section 2.2 to the case where $\mathcal{Z}$ is either a passive future or an anti-passive past behavior.

5.1 The Hilbert space $\mathcal{H}(\mathfrak{W}_+)$

Let $\mathfrak{W}_+$ be a given passive future behavior on a Kreĭn signal space $\mathcal{W}$, i.e., $\mathfrak{W}_+$ is a maximal nonnegative right-shift invariant subspace of $K^2_+(\mathcal{W})$. We take $\mathcal{K} = K^2_+(\mathcal{W})$ and $\mathcal{Z} = \mathfrak{W}_+$ in the discussion in Section 2.2. Adapting our earlier formulas to this case we get the following result.

Theorem 5.1. *Let $\mathfrak{W}_+$ be a passive future behavior on the Kreĭn space $K^2_+(\mathcal{W})$. Denote the quotient map $K^2_+(\mathcal{W}) \mapsto K^2_+(\mathcal{W})/\mathfrak{W}_+$ by Q_+, and define $\mathcal{H}(\mathfrak{W}_+)$ and $\|\cdot\|_{\mathcal{H}(\mathfrak{W}_+)}$ by*

$$\|h_+\|^2_{\mathcal{H}(\mathfrak{W}_+)} = \sup\bigl\{-[w_+, w_+]_{K^2_+(\mathcal{W})} \bigm| w_+ \in h_+\bigr\}, \quad h_+ \in K^2_+(\mathcal{W})/\mathfrak{W}_+, \tag{5.1}$$

$$\mathcal{H}(\mathfrak{W}_+) = \bigl\{h_+ \in K^2_+(\mathcal{W})/\mathfrak{W}_+ \bigm| \|h_+\|^2_{\mathcal{H}(\mathfrak{W}_+)} < \infty\bigr\}. \tag{5.2}$$

(i) *$\mathcal{H}(\mathfrak{W}_+)$ equipped with the norm $\|\cdot\|_{\mathcal{H}(\mathfrak{W}_+)}$ is a Hilbert space that is continuously contained in $K^2_+(\mathcal{W})/\mathfrak{W}_+$.*

(ii) *The image*

$$\mathcal{H}^0(\mathfrak{W}_+) := Q_+\mathfrak{W}_+^{[\perp]} \tag{5.3}$$

of $\mathfrak{W}_+^{[\perp]}$ under Q_+ is a dense subspace of $\mathcal{H}(\mathfrak{W}_+)$, and

$$\|Q_+w_+^\dagger\|^2_{\mathcal{H}(\mathfrak{W}_+)} = -[w_+^\dagger, w_+^\dagger]_{K^2_+(\mathcal{W})}, \qquad w_+^\dagger \in \mathfrak{W}_+^{[\perp]}. \tag{5.4}$$

(iii) *Denote the inverse image of $\mathcal{H}(\mathfrak{W}_+)$ under Q_+ by*

$$\mathcal{K}(\mathfrak{W}_+) := Q_+^{-1}\mathcal{H}(\mathfrak{W}_+). \tag{5.5}$$

Then

$$(Q_+w_+^\dagger, Q_+w_+)_{\mathcal{H}(\mathfrak{W}_+)} = -[w_+^\dagger, w_+]_{K_+^2(\mathcal{W})}, \qquad \text{if } w_+^\dagger \in \mathfrak{W}_+^{[\perp]} \text{ and } w_+ \in \mathcal{K}(\mathfrak{W}_+). \tag{5.6}$$

(iv) *The restriction $Q_+|_{\mathcal{K}(\mathfrak{W}_+)}$ is closed and surjective as an operator $K_+^2(\mathcal{W}) \to \mathcal{H}(\mathfrak{W}_+)$, and it has a bounded right-inverse.*

(v) *If $w_+^k \in \mathcal{K}(\mathfrak{W}_+)$ and $Q_+w_+^k \to Q_+w_+$ in $\mathcal{H}(\mathfrak{W}_+)$ for some $w_+ \in \mathcal{K}(\mathfrak{W}_+)$, then there exists a sequence $z_+^k \in \mathfrak{W}_+$ such that $w_+^k + z_+^k \to w_+$ in $K_+^2(\mathcal{W})$.*

Proof. Claims (i)–(iii) follow from the discussion in Section 2.2. Claims (iv) and (v) follow from the more detailed discussion of $\mathcal{H}(\mathcal{Z})$ given in [AS09a, p. 2597]. □

Lemma 5.2. *Let $\mathfrak{W}_+$ be a passive future behavior on the Kreĭn space $\mathcal{W}$. Then the set*

$$\mathcal{H}_0^0(\mathfrak{W}_+) := \{Q_+w_+^\dagger \mid w_+^\dagger \in \mathfrak{W}_+^{[\perp]} \text{ has compact support}\}$$

(which is contained in $\mathcal{H}^0(\mathfrak{W}_+)$) is a dense subspace of $\mathcal{H}(\mathfrak{W}_+)$.

Proof. Let $w_+^\dagger \in \mathfrak{W}_+^{[\perp]}$. Then by Lemma 4.13, there exists a sequence $w_+^k \in \mathfrak{W}_+^{[\perp]}$, where each w_+^k has compact support, such that $w_+^k \to w_+^\dagger$ in $K_+^2(\mathcal{W})$ as $k \to \infty$. This implies that $[w_+^k - w_+^\dagger, w_+^k - w_+^\dagger]_{K_+^2(\mathcal{W})} \to 0$ as $n \to \infty$, and according to (5.4), this means that $w_+^k + \mathfrak{W}_+ \to w_+^\dagger + \mathfrak{W}_+$ in $\mathcal{H}(\mathfrak{W}_+)$ as $k \to \infty$. Since $\mathcal{H}^0(\mathfrak{W}_+)$ is dense in $\mathcal{H}(\mathfrak{W}_+)$, this proves the lemma. □

Lemma 5.3. *If $w_+ \in \mathcal{K}(\mathfrak{W}_+)$, where $\mathfrak{W}_+$ is a passive future behavior on the Kreĭn space $\mathcal{W}$, then $\tau_+^t w_+ \in \mathcal{K}(\mathfrak{W}_+)$ for all $t \in \mathbb{R}^+$, and*

$$\|Q_+\tau_+^t w_+\|^2_{\mathcal{H}(\mathfrak{W}_+)} \le \|Q_+w_+\|^2_{\mathcal{H}(\mathfrak{W}_+)} + \int_0^t [w_+(s), w_+(s)]_{\mathcal{W}}\,\mathrm{d}s. \tag{5.7}$$

If $w_+ \in \mathfrak{W}_+^{[\perp]}$, then $w_+ \in \mathcal{K}(\mathfrak{W}_+)$ and (5.7) holds with equality.

Proof. We have for all $w_+ \in \mathcal{K}(\mathfrak{W}_+)$, all $z \in \mathfrak{W}_+$, and all $t \in \mathbb{R}^+$ (recall that $\tau_+^{*t}\mathfrak{W}_+ \subset \mathfrak{W}_+$)

$$\begin{aligned}
-[\tau_+^t w_+ + z, \tau_+^t w_+ + z]_{K_+^2(\mathcal{W})} &= -[\tau_+^t(w_+ + \tau_+^{*t}z), \tau_+^t(w_+ + \tau_+^{*t}z)]_{K_+^2(\mathcal{W})} \\
&= -[w_+ + \tau_+^{*t}z, w_+ + \tau_+^{*t}z]_{K_+^2(\mathcal{W})} + \int_0^t [w_+(s), w_+(s)]_{\mathcal{W}}\,\mathrm{d}s \\
&\le \|Q_+w_+\|^2_{\mathcal{H}(\mathfrak{W}_+)} + \int_0^t [w_+(s), w_+(s)]_{\mathcal{W}}\,\mathrm{d}s.
\end{aligned}$$

From here we get (5.7) by taking the supremum over all $z \in \mathfrak{W}_+$. If $w_+ \in \mathfrak{W}_+^{[\perp]}$, then $Q_+w_+ \in \mathcal{H}^0(\mathfrak{W}_+) \subset \mathcal{H}(\mathfrak{W}_+)$, and by (5.1),

$$\|Q_+\tau_+^t w_+\|^2_{\mathcal{H}(\mathfrak{W}_+)} - \|Q_+w_+\|^2_{\mathcal{H}(\mathfrak{W}_+)}$$
$$= -[\tau_+^t w_+, \tau_+^t w_+]_{K_+^2(\mathcal{W})} + [w_+, w_+]_{K_+^2(\mathcal{W})} = \int_0^t [w_+(s), w_+(s)]_{\mathcal{W}}\, \mathrm{d}s. \quad \square$$

5.2 The Hilbert space $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$

Let $\mathfrak{W}_-$ be a given passive past behavior on a Kreĭn signal space $\mathcal{W}$, i.e., $\mathfrak{W}_-$ is a maximal nonnegative right-shift invariant subspace of $K_-^2(\mathcal{W})$. Then $\mathfrak{W}_-^{[\perp]}$ is a maximal nonpositive left-shift invariant subspace of $K_-^2(\mathcal{W})$, and hence it can be interpreted as a maximal nonnegative left-shift invariant subspace of the anti-space $-K_-^2(\mathcal{W})$ of $K_-^2(\mathcal{W})$. This time we take $\mathcal{K} = -K_-^2(\mathcal{W})$ and $\mathcal{Z} = \mathfrak{W}_-^{[\perp]}$ in the definition of $\mathcal{H}(\mathcal{Z})$. Adapting our earlier formulas to this case we get the following result.

Theorem 5.4. *Let $\mathfrak{W}_-$ be a passive past behavior on the Kreĭn space $K_-^2(\mathcal{W})$, and interpret $\mathfrak{W}_-^{[\perp]}$ as a maximal nonnegative left-shift invariant subspace of the anti-space $-K_-^2(\mathcal{W})$. Denote the quotient map $-K_-^2(\mathcal{W}) \mapsto -K_-^2(\mathcal{W})/\mathfrak{W}_-^{[\perp]}$ by Q_-, and define $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$ and $\|\cdot\|_{\mathcal{H}(\mathfrak{W}_-^{[\perp]})}$ by*

$$\|h_-\|^2_{\mathcal{H}(\mathfrak{W}_-^{[\perp]})} = \sup\big\{[w_-, w_-]_{K_-^2(\mathcal{W})} \,\big|\, w_- \in h_-\big\}, \quad h_- \in -K_-^2(\mathcal{W})/\mathfrak{W}_-^{[\perp]}, \tag{5.8}$$

$$\mathcal{H}(\mathfrak{W}_-^{[\perp]}) = \big\{h_- \in -K_-^2(\mathcal{W})/\mathfrak{W}_-^{[\perp]} \,\big|\, \|h_-\|^2_{\mathcal{H}(\mathfrak{W}_-^{[\perp]})} < \infty\big\}. \tag{5.9}$$

(i) *$\mathcal{H}(\mathfrak{W}_-^{[\perp]})$ equipped with the norm $\|\cdot\|_{\mathcal{H}(\mathfrak{W}_-^{[\perp]})}$ is a Hilbert space that is continuously contained in $-K_-^2(\mathcal{W})/\mathfrak{W}_-^{[\perp]}$.*

(ii) *The image*

$$\mathcal{H}^0(\mathfrak{W}_-^{[\perp]}) = Q_-\mathfrak{W}_- \tag{5.10}$$

of $\mathfrak{W}_-$ under Q_- is a dense subspace of $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$, and

$$\|Q_-w_-\|^2_{\mathcal{H}(\mathfrak{W}_-^{[\perp]})} = [w_-, w_-]_{K_-^2(\mathcal{W})}, \qquad w_- \in \mathfrak{W}_-. \tag{5.11}$$

(iii) *Denote the inverse image of $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$ under Q_- by*

$$\mathcal{K}(\mathfrak{W}_-^{[\perp]}) := Q_-^{-1}\mathcal{H}(\mathfrak{W}_-^{[\perp]}). \tag{5.12}$$

Then

$$(Q_-w_-, Q_-v_-)_{\mathcal{H}(\mathfrak{W}_-^{[\perp]})} = [w_-, v_-]_{K_-^2(\mathcal{W})}, \qquad \text{if } w_- \in \mathfrak{W}_- \text{ and } v_- \in \mathcal{K}(\mathfrak{W}_-^{[\perp]}). \tag{5.13}$$

(iv) *The restriction* $Q_-|_{\mathcal{K}(\mathfrak{W}_-^{[\perp]})}$ *is closed and surjective as an operator* $K_-^2(\mathcal{W}) \to \mathcal{H}(\mathfrak{W}_-^{[\perp]})$*, and it has a bounded right-inverse.*

(v) *If* $w_-^k \in \mathcal{K}(\mathfrak{W}_-^{[\perp]})$ *and* $Q_-w_-^k \to Q_-w_-$ *in* $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$ *for some* $w_- \in \mathcal{K}(\mathfrak{W}_-^{[\perp]})$*, then there exists a sequence* $z_-^k \in \mathfrak{W}_-^{[\perp]}$ *such that* $w_-^k + z_-^k \to w_-$ *in* $K_-^2(\mathcal{W})$*.*

Proof. The proof is analogous to the proof of Theorem 5.1. □

Lemma 5.5. *Let* $\mathfrak{W}_-$ *be a passive past behavior on the Kreĭn space* $\mathcal{W}$*. Then the set*

$$\mathcal{H}_0^0(\mathfrak{W}_-^{[\perp]}) := \{Q_-w_- \mid w_- \in \mathfrak{W}_- \text{ has compact support}\}$$

(which is contained in $\mathcal{H}^0(\mathfrak{W}_-^{[\perp]})$*) is a dense subspace of* $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$ *.*

Proof. This follows from Lemma 5.2 and Remark 4.12. □

Lemma 5.6. *If* $w_- \in \mathcal{K}(\mathfrak{W}_-^{[\perp]})$*, then* $\tau_-^{*t}w_- \in \mathcal{K}(\mathfrak{W}_-^{[\perp]})$ *and*

$$\|Q_-\tau_-^{*t}w_-\|^2_{\mathcal{H}(\mathfrak{W}_-^{[\perp]})} \le \|Q_-w_-\|^2_{\mathcal{H}(\mathfrak{W}_-^{[\perp]})} - \int_{-t}^0 [w_-(s), w_-(s)]_{\mathcal{W}}\,\mathrm{d}s. \tag{5.14}$$

If $w_- \in \mathfrak{W}_-$*, then* $w_- \in \mathcal{K}(\mathfrak{W}_-^{[\perp]})$ *and* (5.14) *holds with equality.*

Proof. This follows from Lemma 5.3 and Remark 4.12. □

5.3 The past/future map $\Gamma_{\mathfrak{W}}$ and the Hilbert space $\mathcal{D}(\mathfrak{W})$

In Section 10 we shall also need the quotient space $K^2(\mathcal{W})/(\mathfrak{W}_+ \dotplus \mathfrak{W}_-^{[\perp]})$. Here $\mathfrak{W}_+ \dotplus \mathfrak{W}_-^{[\perp]}$ is a closed subspace of $K^2(\mathcal{W})$ since the sum $\mathfrak{W}_+ \dotplus \mathfrak{W}_-^{[\perp]}$ is direct in $K^2(\mathcal{W})$.

We denote the quotient map $K^2(\mathcal{W}) \mapsto K^2(\mathcal{W})/(\mathfrak{W}_+ \dotplus \mathfrak{W}_-^{[\perp]})$ by Q. Thus,

$$\begin{aligned} Q_-w_- &:= w_- + \mathfrak{W}_-^{[\perp]}, \quad w_- \in K_-^2(\mathcal{W}),\\ Q_+w_+ &:= w_+ + \mathfrak{W}_+, \quad w_+ \in K_+^2(\mathcal{W}),\\ Qw &:= w + (\mathfrak{W}_+ \dotplus \mathfrak{W}_-^{[\perp]}), \quad w \in K^2(\mathcal{W}). \end{aligned} \tag{5.15}$$

To shorten the notations we furthermore define

$$\begin{aligned}\mathcal{H}_- &:= \mathcal{H}(\mathfrak{W}_-^{[\perp]}), \quad \mathcal{H}_-^0 := \mathcal{H}^0(\mathfrak{W}_-^{[\perp]}),\\ \mathcal{H}_+ &:= \mathcal{H}(\mathfrak{W}_+), \quad \mathcal{H}_+^0 := \mathcal{H}^0(\mathfrak{W}_+).\end{aligned} \tag{5.16}$$

Each vector in the quotient space $K^2(\mathcal{W})/(\mathfrak{W}_+ \dotplus \mathfrak{W}_-^{[\perp]})$ is an equivalence class of the type $x := w + (\mathfrak{W}_+ \dotplus \mathfrak{W}_-^{[\perp]})$ for some $w \in K^2(\mathcal{W})$. Since $K^2(\mathcal{W}) = K_+^2(\mathcal{W}) \boxplus K_-^2(\mathcal{W})$, and since $\mathfrak{W}_+$ is a closed subspace of $K_+^2(\mathcal{W})$ and $\mathfrak{W}_-^{[\perp]}$ is a closed subspace of $K_-^2(\mathcal{W})$, it follows that we can identify $K^2(\mathcal{W})/(\mathfrak{W}_+ \dotplus \mathfrak{W}_-^{[\perp]})$ with the product space $\begin{bmatrix} K_+^2(\mathcal{W})/\mathfrak{W}_+ \\ K_-^2(\mathcal{W})/\mathfrak{W}_-^{[\perp]} \end{bmatrix}$.

We denote the projections of $K^2(\mathcal{W})/(\mathfrak{W}_+ \dotplus \mathfrak{W}_-^{[\perp]})$ onto $K_+^2(\mathcal{W})/\mathfrak{W}_+$ and $K_-^2(\mathcal{W})/\mathfrak{W}_-^{[\perp]}$ by P_+ and P_-, respectively. Thus, $P_\pm$ is the operator which for each $w \in K^2(\mathcal{W})$ maps $x = Qw$ into $Q_\pm \pi_\pm w$. Since $\mathcal{H}_+$ is continuously contained in $K_+^2(\mathcal{W})/\mathfrak{W}_+$ and $\mathcal{H}_-$ is continuously contained in $K_-^2(\mathcal{W})/\mathfrak{W}_-^{[\perp]}$, this means that $\begin{bmatrix} \mathcal{H}_+ \\ \mathcal{H}_- \end{bmatrix}$ can be interpreted as a continuously contained subspace of $K^2(\mathcal{W})/(\mathfrak{W}_+ \dotplus \mathfrak{W}_-^{[\perp]})$.

Lemma 5.7. *Let $\mathfrak{W}$ be a passive full behavior on $\mathcal{W}$ with the corresponding passive past behavior $\mathfrak{W}_- = \pi_- \mathfrak{W}$ and passive future behavior $\mathfrak{W}_+ = \mathfrak{W} \cap K_+^2(\mathcal{W})$. Then there exists a unique contraction $\Gamma_\mathfrak{W} \colon \mathcal{H}_- \to \mathcal{H}_+$ satisfying*

$$\Gamma_\mathfrak{W}\, Q_- \pi_- w = Q_+ \pi_+ w, \qquad w \in \mathfrak{W}, \tag{5.17}$$

where Q_- is the quotient map $K_-^2 \mapsto K_-^2/\mathfrak{W}_-^{[\perp]}$ and Q_+ is the quotient map $K_+^2 \mapsto K_+^2/\mathfrak{W}_+$.

Proof. The proof is essentially the same as the proof of [AS09b, Lemma 6.1] (see Remark 3.17). □

Definition 5.8. The contraction $\Gamma_\mathfrak{W} \colon \mathcal{H}(\mathfrak{W}_-^{[\perp]}) \to \mathcal{H}(\mathfrak{W}_+)$ in Lemma 5.7 is called the *past/future map* of the full behavior $\mathfrak{W}$.

Throughout the rest of this section we let $\mathfrak{W}$ be a passive full behavior on $\mathcal{W}$, and define the corresponding passive past and future behaviors $\mathfrak{W}_-$ and $\mathfrak{W}_+$ by (3.14).

Let

$$A_\mathfrak{W} := \begin{bmatrix} 1_{\mathcal{H}_+} & \Gamma_\mathfrak{W} \\ \Gamma_\mathfrak{W}^* & 1_{\mathcal{H}_-} \end{bmatrix}. \tag{5.18}$$

This is a bounded linear operator on $\mathcal{H}_+ \oplus \mathcal{H}_-$. It is nonnegative since $\Gamma_{\mathfrak{W}}$ is a contraction $\mathcal{H}_- \to \mathcal{H}_+$, and by the Schwarz inequality, for all $\left[\begin{smallmatrix} x_+ \\ x_- \end{smallmatrix}\right] \in \mathcal{H}_+ \oplus \mathcal{H}_-$,

$$\begin{aligned}\left(\begin{bmatrix} x_+ \\ x_- \end{bmatrix}, A_{\mathfrak{W}} \begin{bmatrix} x_+ \\ x_- \end{bmatrix} \right)_{\mathcal{H}_+ \oplus \mathcal{H}_-} &= \|x_+\|^2_{\mathcal{H}_+} + 2\Re(x_+, \Gamma_{\mathfrak{W}} x_-)_{\mathcal{H}_+} + \|x_-\|^2_{\mathcal{H}_-} \\ &\geq \|x_+\|^2_{\mathcal{H}_+} - 2\|x_+\|_{\mathcal{H}_+}\|x_-\|_{\mathcal{H}_-} + \|x_-\|^2_{\mathcal{H}_-} \geq 0. \end{aligned}$$

We define $\mathcal{D}(\mathfrak{W})$ to be the range of $A_{\mathfrak{W}}^{1/2}$, with the range norm, i.e.,

$$\left\| \begin{bmatrix} x_+ \\ x_- \end{bmatrix} \right\|_{\mathcal{D}(\mathfrak{W})} = \left\| (A_{\mathfrak{W}}^{1/2})^{[-1]} \begin{bmatrix} x_+ \\ x_- \end{bmatrix} \right\|_{\mathcal{H}_+ \oplus \mathcal{H}_-},$$

where $(A_{\mathfrak{W}}^{1/2})^{[-1]}$ is the pseudo-inverse of $A_{\mathfrak{W}}^{1/2}$, i.e., $\left[\begin{smallmatrix} x'_+ \\ x'_- \end{smallmatrix}\right] := (A_{\mathfrak{W}}^{1/2})^{[-1]} \left[\begin{smallmatrix} x_+ \\ x_- \end{smallmatrix}\right]$ is the unique vector in $\overline{\operatorname{im}(A_{\mathfrak{W}})} = \overline{\operatorname{im}\left(A_{\mathfrak{W}}^{1/2}\right)}$ which satisfies $\left[\begin{smallmatrix} x_+ \\ x_- \end{smallmatrix}\right] = A_{\mathfrak{W}}^{1/2} \left[\begin{smallmatrix} x'_+ \\ x'_- \end{smallmatrix}\right]$. With respect to this inner product in the range space the operator $A_{\mathfrak{W}}^{1/2}|_{\overline{\operatorname{im}(A_{\mathfrak{W}})}}$ is a unitary operator mapping $\overline{\operatorname{im}(A_{\mathfrak{W}})}$ onto $\mathcal{D}(\mathfrak{W})$. In particular, $\mathcal{D}(\mathfrak{W})$ is a Hilbert space.

Lemma 5.9. *Define $A_{\mathfrak{W}}$ by* (5.18)*.*

(i) $\operatorname{im}(A_{\mathfrak{W}})$ *is a dense subset of the Hilbert space $\mathcal{D}(\mathfrak{W})$, $\mathcal{D}(\mathfrak{W})$ is a dense subspace of* $\overline{\operatorname{im}(A_{\mathfrak{W}})}$*, and $\mathcal{D}(\mathfrak{W})$ is continuously contained in $\mathcal{H}_+ \oplus \mathcal{H}_-$.*

(ii) *$A_{\mathfrak{W}}$ is bounded as an operator $\mathcal{H}_+ \oplus \mathcal{H}_- \to \mathcal{D}(\mathfrak{W})$.*

(iii) *If $x \in \mathcal{D}(\mathfrak{W})$ and $y = A_{\mathfrak{W}} y'$, then $y \in \mathcal{D}(\mathfrak{W})$, and $(x, y)_{\mathcal{D}(\mathfrak{W})} = (x, y')_{\mathcal{H}_- \oplus \mathcal{H}_+}$.*

(iv) *$T_- := A_{\mathfrak{W}}|_{\mathcal{H}_-} = \left[\begin{smallmatrix} \Gamma_{\mathfrak{W}} \\ 1_{\mathcal{H}_-} \end{smallmatrix}\right]$ is an isometry $\mathcal{H}_- \to \mathcal{D}(\mathfrak{W})$.*

(v) *$T_+ := A_{\mathfrak{W}}|_{\mathcal{H}_+} = \left[\begin{smallmatrix} 1_{\mathcal{H}_+} \\ \Gamma_{\mathfrak{W}}^* \end{smallmatrix}\right]$ is an isometry $\mathcal{H}_+ \to \mathcal{D}(\mathfrak{W})$.*

Proof. The proof is essentially the same as the proof of [AS10, Lemma 3.1] (see Remark 3.17). □

Lemma 5.10. *Denote $\mathcal{L}_\pm := \operatorname{im}(T_\pm)$, where $T_\pm$ are the operators defined in Lemma 5.9. Then $\mathcal{L}_+ + \mathcal{L}_-$ is dense in $\mathcal{D}(\mathfrak{W})$, and*

$$P_{\mathcal{L}_+}|_{\mathcal{L}_-} = T_+ \Gamma_{\mathfrak{W}} T_-^*|_{\mathcal{L}_-}. \tag{5.19}$$

Proof. That $\mathcal{L}_+ + \mathcal{L}_- = \operatorname{im}(A_{\mathfrak{W}})$ is dense in $\mathcal{D}(\mathfrak{W})$ is contained in part (i) of Lemma 5.9.

Let $x \in \mathcal{D}(\mathfrak{W})$, $x_- \in \mathcal{L}_-$, and define $h_- := T_-^* x_-$. Then $x_- = T_- h_-$ (since $T_- \colon \mathcal{H}_- \to \mathcal{L}_-$ is unitary). Define $x_+ := P_{\mathcal{L}_+} x$ and $h_+ = T_+^* x_+$, so that $x_+ = T_+ h_+$. Then

$$\begin{aligned}
(x, P_{\mathcal{L}_+} x_-)_{\mathcal{D}(\mathfrak{W})} &= (P_{\mathcal{L}_+} x, x_-)_{\mathcal{D}(\mathfrak{W})} = \big(P_{\mathcal{L}_+} x, A_{\mathfrak{W}} \left[\begin{smallmatrix} 0 \\ h_- \end{smallmatrix}\right]\big)_{\mathcal{D}(\mathfrak{W})} \\
&= \big(P_{\mathcal{L}_+} x, \left[\begin{smallmatrix} 0 \\ h_- \end{smallmatrix}\right]\big)_{\mathcal{H}_+ \oplus \mathcal{H}_-} = \left(\left[\begin{smallmatrix} h_+ \\ \Gamma_{\mathfrak{W}}^* h_+ \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ h_- \end{smallmatrix}\right]\right)_{\mathcal{H}_+ \oplus \mathcal{H}_-} \\
&= (\Gamma_{\mathfrak{W}}^* h_+, h_-)_{\mathcal{H}_-} = (h_+, \Gamma_{\mathfrak{W}} h_-)_{\mathcal{H}_+} \\
&= \big(T_+^* P_{\mathcal{L}_+} x, \Gamma_{\mathfrak{W}} T_-^* x_-\big)_{\mathcal{H}_+} = \big(P_{\mathcal{L}_+} x, T_+ \Gamma_{\mathfrak{W}} T_-^* x_-\big)_{\mathcal{H}_+} \\
&= \big(x, T_+ \Gamma_{\mathfrak{W}} T_-^* x_-\big)_{\mathcal{H}_+}
\end{aligned}$$

This proves (5.19). □

Remark 5.11. Lemma 5.10 may be reformulated as follows. The space $\mathcal{D}(\mathfrak{W})$ is equal to the closed linear span of its subspaces $\mathcal{L}_+ \vee \mathcal{L}_-$, where $\mathcal{L}_\pm$ are the unitary images in $\mathcal{D}(\mathfrak{W})$ of $\mathcal{H}_\pm$ under $T_\pm$, and the angle operator $K := P_{\mathcal{L}_+}|_{\mathcal{L}_-}$ between $\mathcal{L}_-$ and $\mathcal{L}_+$ is given by (5.19) in terms of $T_\pm$ and the past/future map $\Gamma_{\mathfrak{W}}$. In particular,

$$\begin{aligned}
1_{\mathcal{L}_-} - K^* K &= 1_{\mathcal{L}_-} - T_- \Gamma_{\mathfrak{W}}^* \Gamma_{\mathfrak{W}} T_-^*|_{\mathcal{L}_-} = T_-(1_{\mathcal{L}_-} - \Gamma_{\mathfrak{W}}^* \Gamma_{\mathfrak{W}}) T_-^*|_{\mathcal{L}_-}, \\
1_{\mathcal{L}_+} - K K^* &= 1_{\mathcal{L}_+} - T_+ \Gamma_{\mathfrak{W}} \Gamma_{\mathfrak{W}}^* T_-^*|_{\mathcal{L}_+} = T_+(1_{\mathcal{L}_+} - \Gamma_{\mathfrak{W}} \Gamma_{\mathfrak{W}}^*) T_+^*|_{\mathcal{L}_+}.
\end{aligned}$$

This leads to the following conclusions:

(i) The following conditions are equivalent:

  (a) K is an isometry;

  (b) $\mathcal{D}(\mathfrak{W}) = \mathcal{L}_+$;

  (c) $\Gamma_{\mathfrak{W}}$ is an isometry $\mathcal{H}_- \to \mathcal{H}_+$.

(ii) The following conditions are equivalent:

  (a) K is a co-isometry;

  (b) $\mathcal{D}(\mathfrak{W}) = \mathcal{L}_-$;

  (c) $\Gamma_{\mathfrak{W}}$ is a co-isometry $\mathcal{H}_- \to \mathcal{H}_+$.

In the sequel we shall throughout *interpret* $A_{\mathfrak{W}}$ *as a bounded linear operator* $\mathcal{H}_+ \oplus \mathcal{H}_- \to \mathcal{D}(\mathfrak{W})$, instead of interpreting $A_{\mathfrak{W}}$ as a self-adjoint operator in $\mathcal{H}_+ \oplus \mathcal{H}_-$. In particular, in this setting *the operator* $A_{\mathfrak{W}}$ *is not self-adjoint unless* $\mathcal{D}(\mathfrak{W}) = \mathcal{H}_+ \oplus \mathcal{H}_-$, *i.e., unless* $\Gamma_{\mathfrak{W}} = 0$. When the duality in the range space is taken with respect to the inner product in $\mathcal{D}(\mathfrak{W})$ instead of the inner product in $\mathcal{H}_+ \oplus \mathcal{H}_-$ then *the operator* $A_{\mathfrak{W}}^*$ *becomes a bounded linear operator* $\mathcal{D}(\mathfrak{W}) \to \mathcal{H}_+ \oplus \mathcal{H}_-$.

Recall that we denoted the projections of $K^2(\mathcal{W})/(\mathfrak{W}_+ \dotplus \mathfrak{W}_-^{[\perp]})$ onto $K_+^2(\mathcal{W})/\mathfrak{W}_+$ and $K_-^2(\mathcal{W})/\mathfrak{W}_-^{[\perp]}$ by P_+ and P_-, respectively. We denote the restrictions of $P_\pm$ to $\begin{bmatrix} \mathcal{H}_- \\ \mathcal{H}_+ \end{bmatrix}$ by $\Pi_\pm$, so that $\Pi_\pm \left[\begin{smallmatrix} x_+ \\ x_- \end{smallmatrix}\right] = x_\pm$ for all $\left[\begin{smallmatrix} x_+ \\ x_- \end{smallmatrix}\right] \in \begin{bmatrix} \mathcal{H}_+ \\ \mathcal{H}_- \end{bmatrix}$.

Lemma 5.12. *Let* $A_{\mathfrak{W}}$ *be the operator defined in* (5.18)*, interpreted as bounded linear operator* $\mathcal{H}_+ \oplus \mathcal{H}_- \to \mathcal{D}(\mathfrak{W})$*, whose adjoint* $A_{\mathfrak{W}}^*$ *is a bounded linear operator* $\mathcal{D}(\mathfrak{W}) \to \mathcal{H}_+ \oplus \mathcal{H}_-$*.*

(i) $A_{\mathfrak{W}}^*$ *is equal to the embedding operator* $\mathcal{D}(\mathfrak{W}) \hookrightarrow \begin{bmatrix} \mathcal{H}_+ \\ \mathcal{H}_- \end{bmatrix}$.

(ii) $(A_{\mathfrak{W}}|_{\mathcal{H}_+})^* = \Pi_+|_{\mathcal{D}(\mathfrak{W})}$ *and* $(A_{\mathfrak{W}}|_{\mathcal{H}_-})^* = \Pi_-|_{\mathcal{D}(\mathfrak{W})}$*. (In the computation of these adjoints we interpret* $A_{\mathfrak{W}}|_{\mathcal{H}_\pm}$ *as operators* $\mathcal{H}_\pm \to \mathcal{D}(\mathfrak{W})$*.)*

Proof. By Lemma 5.9(iii), for all $x \in \mathcal{D}(\mathfrak{W})$ and all $y' \in \mathcal{H}_+ \oplus \mathcal{H}_-$,

$$(x, A_{\mathfrak{W}} y')_{\mathcal{D}(\mathfrak{W})} = (x, y')_{\mathcal{H}_- \oplus \mathcal{H}_+}.$$

This proves claim (i). If we in the same computation replace $y' \in \mathcal{H}_+ \oplus \mathcal{H}_-$ by either $y' \in \mathcal{H}_+$ or $y' \in \mathcal{H}_-$ we get claim (ii). □

We let $\mathcal{L}(\mathfrak{W})$ be the inverse image of $\mathcal{D}(\mathfrak{W})$ under Q, i.e.,

$$\mathcal{L}(\mathfrak{W}) := Q^{-1}\mathcal{D}(\mathfrak{W}) := \{w \in K^2(\mathcal{W}) \mid Qw \in \mathcal{D}(\mathfrak{W})\}, \tag{5.20}$$

and denote

$$\mathcal{D}^0(\mathfrak{W}) := Q(\mathfrak{W} + \mathfrak{W}^{[\perp]}) := \left\{Q(z + z^\dagger) \bigm| z \in \mathfrak{W},\ z^\dagger \in \mathfrak{W}^{[\perp]}\right\}. \tag{5.21}$$

As we shall see in the following lemma, $\mathcal{D}^0(\mathfrak{W})$ is a dense subspace of $\mathcal{D}(\mathfrak{W})$.

Lemma 5.13. (i) *If* $z \in \mathfrak{W}$ *and* $z^\dagger \in \mathfrak{W}^{[\perp]}$*, then* $Q(z+z^\dagger) = A_{\mathfrak{W}} \begin{bmatrix} Q_+\pi_+ z^\dagger \\ Q_-\pi_- z \end{bmatrix}$*. In particular,* $\mathcal{D}^0(\mathfrak{W}) \subset \operatorname{im}(A_{\mathfrak{W}})$ *and* $\mathfrak{W} + \mathfrak{W}^{[\perp]} \subset \mathcal{L}(\mathfrak{W})$*.*

(ii) $\mathcal{D}^0(\mathfrak{W})$ *is a dense subspace of* $\mathcal{D}(\mathfrak{W})$*.*

(iii) *If* $w \in \mathcal{L}(\mathfrak{W})$*,* $z \in \mathfrak{W}$*, and* $z^\dagger \in \mathfrak{W}^{[\perp]}$*, then*

$$(Qw, Qz)_{\mathcal{D}(\mathfrak{W})} = (Q_-\pi_-w, Q_-\pi_-z)_{\mathcal{H}_-} = [\pi_-w, \pi_-z]_{K^2_-(\mathcal{W})} \tag{5.22}$$

$$(Qw, Qz^\dagger)_{\mathcal{D}(\mathfrak{W})} = (Q_+\pi_+w, Q_+\pi_+z^\dagger)_{\mathcal{H}_+} = -[\pi_+w, \pi_+z^\dagger]_{K^2_+(\mathcal{W})}. \tag{5.23}$$

In particular,

$$\|Qz\|^2_{\mathcal{D}(\mathfrak{W})} = \|Q_-\pi_-z\|^2_{\mathcal{H}_-} = [\pi_-z, \pi_-z]_{K^2_-(\mathcal{W})}, \qquad z \in \mathfrak{W}, \tag{5.24}$$

$$\|Qz^\dagger\|^2_{\mathcal{D}(\mathfrak{W})} = \|Q_+\pi_+z^\dagger\|^2_{\mathcal{H}_+} = -[\pi_+z^\dagger, \pi_+z^\dagger]_{K^2_+(\mathcal{W})}, \qquad z^\dagger \in \mathfrak{W}^{[\perp]}. \tag{5.25}$$

Proof. The proof is essentially the same as the proof of [AS10, Lemma 3.3] (see Remark 3.17). □

Lemma 5.14. (i) *If* $w \in \mathcal{L}(\mathfrak{W})$*, then* $\tau^t w \in \mathcal{L}(\mathfrak{W})$ *for all* $t \in \mathbb{R}$*, and*

$$\|Q\tau^t w\|^2_{\mathcal{D}(\mathfrak{W})} = \int_0^t [w(s), w(s)]_{\mathcal{W}}\, \mathrm{d}s + \|Qw\|^2_{\mathcal{D}(\mathfrak{W})}, \quad t \in \mathbb{R}. \tag{5.26}$$

(ii) *If* w_1*,* $w_2 \in \mathcal{L}(\mathfrak{W})$*, then for all* $t \in \mathbb{R}$*,*

$$(Qw_1, Q\tau^t w_2)_{\mathcal{D}(\mathfrak{W})} = \int_0^t [w_1(s-t), w_2(s)]_{\mathcal{W}}\, \mathrm{d}s + (Q\tau^{-t}w_1, Qw_2)_{\mathcal{D}(\mathfrak{W})}. \tag{5.27}$$

Proof. The proof is essentially the same as the proof of [AS10, Lemma 3.4] (see Remark 3.17). □

6 The Output and Input maps

6.1 The output map $\mathfrak{C}_\Sigma$

We begin by presenting the output map $\mathfrak{C}_\Sigma$ of a passive s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ with future behavior $\mathfrak{W}_+$. This is an operator from the state space $\mathcal{X}$ into the Hilbert space $\mathcal{H}(\mathfrak{W}_+)$ which was defined in (5.2). As in Theorem 5.1 we denote the quotient map $K^2_+(\mathcal{W}) \mapsto K^2_+(\mathcal{W})/\mathfrak{W}_+$ by Q_+, and the inverse image of $\mathcal{H}(\mathfrak{W}_+)$ under Q_+ by $\mathcal{K}(\mathfrak{W}_+)$.

Lemma 6.1. *Let* $\Sigma = (V; \mathcal{X}; \mathcal{W})$ *be a passive s/s system with future behavior* $\mathfrak{W}_+$*. If* $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ *is a stable future trajectory of* Σ*, then*

$$w \in \mathcal{K}(\mathfrak{W}_+) \text{ and } \|Q_+w\|_{\mathcal{H}(\mathfrak{W}_+)} \le \|x(0)\|_{\mathcal{X}}. \tag{6.1}$$

Proof. Let $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ be a stable future trajectory of Σ, let $z \in \mathfrak{W}_+$, and let $\left[\begin{smallmatrix} x_1 \\ z \end{smallmatrix}\right]$ be the corresponding externally generated stable future trajectory of Σ. Then $\left[\begin{smallmatrix} x+x_1 \\ w+z \end{smallmatrix}\right]$ is a stable future trajectory of Σ, and by (1.9),

$$-[w+z, w+z]_{K^2_+(\mathcal{W})} \leq \|x(0) + x_1(0)\|^2_{\mathcal{X}} = \|x(0)\|^2_{\mathcal{X}}.$$

Taking the supremum over all $z \in \mathfrak{W}_+$ we find that (6.1) holds. □

Lemma 6.2. *Let $\Sigma = (V; \mathcal{X}; \mathcal{W})$ be a passive s/s system with future behavior $\mathfrak{W}_+$. Then the formula*

$$\mathfrak{C}_\Sigma x_0 = \left\{ Q_+ w \;\middle|\; \begin{array}{l} w \text{ is the signal part of some stable future} \\ \text{trajectory } \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \text{ of } \Sigma \text{ with } x(0) = x_0 \end{array} \right\} \tag{6.2}$$

defines a linear contraction $\mathfrak{C}_\Sigma \colon \mathcal{X} \to \mathcal{H}(\mathfrak{W}_+)$.

Proof. Let $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ be a stable future trajectory of Σ. If $\left[\begin{smallmatrix} x_1 \\ w_1 \end{smallmatrix}\right]$ is another stable future trajectory of Σ with the same initial state $x_1(0) = x(0)$, then $w_1 - w \in \mathfrak{W}_+$, and conversely, if $w_1 - w \in \mathfrak{W}_+$, then there exist a stable future trajectory $\left[\begin{smallmatrix} x_1 \\ w_1 \end{smallmatrix}\right]$ with $x_1(0) = x(0)$. Thus, the set of all signal parts w of the stable future trajectories $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ of Σ with fixed initial state $x(0) = x_0$ is an equivalence class in $K^2_+(\mathcal{W})/\mathfrak{W}_+$. By (6.1), the map $\mathfrak{C}_\Sigma$ from x_0 to this equivalence class is a contraction $\mathcal{X} \to \mathcal{H}(\mathfrak{W}_+)$. It is easy to see that this map is linear, and by Lemma 3.4(i), the domain of $\mathfrak{C}_\Sigma$ is all of $\mathcal{X}$. □

Definition 6.3. The contraction $\mathfrak{C}_\Sigma$ in Lemma 6.2 above is called the *output map* of Σ.

In our next lemma we need the inverse image of $\operatorname{im}(\mathfrak{C}_\Sigma)$ under Q_+ which we denote by

$$\mathfrak{S}^\Sigma_+ = Q_+^{-1} \operatorname{im}(\mathfrak{C}_\Sigma). \tag{6.3}$$

Thus, $\mathfrak{S}^\Sigma_+$ consists of the signal parts w of all stable future trajectories $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ of Σ. By Lemma 6.1, $\mathfrak{S}^\Sigma_+ \subset \mathcal{K}(\mathfrak{W}_+)$, where $\mathcal{K}(\mathfrak{W}_+)$ is the space defined in (5.5).

Lemma 6.4. *Let $\Sigma = (V; \mathcal{X}; \mathcal{W})$ be a passive s/s system with future behavior $\mathfrak{W}_+$ and output map $\mathfrak{C}_\Sigma$, and define $\mathfrak{S}^\Sigma_+$ by (6.3). Then every stable future trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ of Σ satisfies*

$$w \in \mathfrak{S}^\Sigma_+, \; \tau^t_+ w \in \mathfrak{S}^\Sigma_+, \; \text{and } \mathfrak{C}_\Sigma x(t) = Q_+ \tau^t_+ w, \quad t \in \mathbb{R}^+. \tag{6.4}$$

Proof. That $w \in \mathfrak{S}^\Sigma_+$ follows immediately from (6.3). To get (6.4) we simply shift the trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ to the left by the amount t and apply (6.2) with x_0 replaced by $x(t)$. □

Definition 6.5. By an *unobservable future trajectory* of a passive s/s system Σ we mean a future trajectory of Σ of the type $\left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right]$ (i.e., the signal part is identically zero). The *unobservable* subspace $\mathfrak{U}_\Sigma$ of Σ consists of all the initial states $x(0)$ of all unobservable future trajectories of Σ. The system Σ is *observable* if $\mathfrak{U}_\Sigma = \{0\}$.

By Lemma 3.2, every unobservable future trajectory of a passive s/s system is stable.

Lemma 6.6. *The unobservable subspace $\mathfrak{U}_\Sigma$ of a passive s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is equal to the null space of its output map $\mathfrak{C}_\Sigma$.*

Proof. It follows directly from Definition 6.5 and Lemma 6.4 that if $x_0 \in \mathfrak{U}_\Sigma$, then $\mathfrak{C}_\Sigma x_0 = \mathfrak{W}_+$, and hence $\mathfrak{C}_\Sigma x_0$ is the zero element in $\mathcal{H}(\mathfrak{W}_+)$. Conversely, suppose that $x_0 \in \ker\big(\mathfrak{C}_\Sigma\big)$, i.e., $\mathfrak{C}_\Sigma x_0 = \mathfrak{W}_+$. By Lemma 3.4(i), there exists a stable future trajectory $\left[\begin{smallmatrix} x_1 \\ w_1 \end{smallmatrix}\right]$ of Σ with $x_1(0) = x_0$, and by Lemma 6.4, $w_1 \in \mathfrak{C}_\Sigma x_0 = \mathfrak{W}_+$. Let $\left[\begin{smallmatrix} x_2 \\ w_1 \end{smallmatrix}\right]$ be the externally generated future trajectory of Σ whose signal part is w_1 (cf. Lemma 3.11), and define $x = x_1 - x_2$. Then $\left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right]$ is an unobservable future trajectory of Σ with $x(0) = x_0$, and hence $x_0 \in \mathfrak{U}_\Sigma$. □

Lemma 6.7. *If the passive s/s system $\Sigma = (V; \mathcal{X}; \mathcal{W})$ is observable, then $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is a stable future trajectory of Σ if and only if* (6.4) *holds.*

Proof. The necessity of (6.4) follows from Lemma 6.4 and (6.3). Conversely, suppose that (6.4) holds. According to (6.3) there exists at least one stable future trajectory $\left[\begin{smallmatrix} x_1 \\ w \end{smallmatrix}\right]$ of Σ, and by Lemma 6.4, (6.4) holds with x replaced by x_1. By Lemma 6.6 and the observability assumption on Σ, $\mathfrak{C}_\Sigma$ is injective, and hence (6.4) implies that $x(t) = x_1(t)$ for all $t \in \mathbb{R}^+$. This implies that $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is a stable future trajectory of Σ. □

Lemma 6.8. *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive s/s system with output map $\mathfrak{C}_\Sigma$. Then $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is a stable future trajectory of Σ if and only if $x = x_1 + x_2$, where $\left[\begin{smallmatrix} x_1 \\ 0 \end{smallmatrix}\right]$ is an unobservable future trajectory of Σ and $\left[\begin{smallmatrix} x_2 \\ w \end{smallmatrix}\right]$ is a stable future trajectory of Σ with $x_2(0) \in (\ker\big(\mathfrak{C}_\Sigma\big))^\perp$. This decomposition is unique, and* (6.4) *also holds with x replaced by x_2.*

Proof. Trivially, if x has a decomposition of the type described in the lemma, then $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is a stable future trajectory of Σ.

Conversely, let $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ be a stable future trajectory of Σ. Define $x_1(0) = P_{\mathfrak{U}_\Sigma} x(0)$ and $x_2(0) = P_{\mathfrak{U}_\Sigma^\perp} x(0)$. Then $x(0) = x_1(0) + x_2(0)$ and $x_1(0) \in \mathfrak{U}_\Sigma$. The latter condition implies that $x_1(0)$ is the initial state of some unobservable trajectory $\left[\begin{smallmatrix} x_1 \\ 0 \end{smallmatrix}\right]$ of Σ. Define $x_2 = x - x_1$. Then $\left[\begin{smallmatrix} x_2 \\ w \end{smallmatrix}\right]$ is a stable future trajectory of Σ and $x = x_1 + x_2$. That (6.4) also holds with x replaced by x_2 follows from the fact that $\left[\begin{smallmatrix} x_2 \\ w \end{smallmatrix}\right]$ is a stable future trajectory of Σ. □

6.2 The input map $\mathfrak{B}_\Sigma$

We now proceed to the construction of the input map $\mathfrak{B}_\Sigma$ of a passive s/s system Σ. This is an operator from the Hilbert space $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$ defined in (5.9) into the state space $\mathcal{X}$. As in Theorem 5.4 we denote the quotient map $K_-^2(\mathcal{W}) \mapsto K_-^2(\mathcal{W})/\mathfrak{W}_-^{[\perp]}$ by Q_-, the image of $\mathfrak{W}_-$ under Q_- by $\mathcal{H}^0(\mathfrak{W}_-^{[\perp]})$, and the inverse image of $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$ under Q_- by $\mathcal{K}(\mathfrak{W}_-)$.

Before giving the formal definition of the input map of a passive s/s system, let us explain the underlying idea. Let $\Sigma = (V; \mathcal{X}; \mathcal{W})$ be a passive s/s system with past behavior $\mathfrak{W}_-$. By Lemma 3.11, to each $w \in \mathfrak{W}_-$ there exists a unique stable externally generated past trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ of Σ. It is easy to see that the map $\widetilde{\mathfrak{B}}_\Sigma$ from $w \in \mathfrak{W}_-$ to $x(0)$ is a linear operator. By Lemma 3.2, this operator is a contraction with respect to the semi-norm in $\mathfrak{W}_-$ inherited from $K_-^2(\mathcal{W})$. In particular, if $w \in \mathfrak{W}_-$ is neutral, i.e., if $[w, w]_{K_-^2(\mathcal{W})} = 0$, then $\widetilde{\mathfrak{B}}_\Sigma w = 0$. After factoring out the maximal neutral subspace $\mathfrak{W}_0 := \mathfrak{W}_- \cap \mathfrak{W}_-^{[\perp]}$ from $\mathfrak{W}_-$, the space $\mathfrak{W}_-/\mathfrak{W}_0$ becomes a unitary space (the noncomplete version of a Hilbert space), and the operator $\widetilde{\mathfrak{B}}_\Sigma$ becomes a contraction $\mathfrak{W}_-/\mathfrak{W}_0 \to \mathcal{X}$. It follows from Theorem 5.4(ii) that the space $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$ has a natural interpretation as a completion of the unitary space $\mathfrak{W}_-/\mathfrak{W}_0$. Therefore, the contraction $\widetilde{\mathfrak{B}}_\Sigma$ from $\mathfrak{W}_-/\mathfrak{W}_0$ to $\mathcal{X}$ has a natural extension to a contraction $\mathcal{H}(\mathfrak{W}_-^{[\perp]}) \to \mathcal{X}$. In the following lemma this extension of $\widetilde{\mathfrak{B}}_\Sigma$ has been denoted by $\mathfrak{B}_\Sigma$.

Lemma 6.9. *Let $\Sigma = (V; \mathcal{X}; \mathcal{W})$ be a passive s/s system with past behavior $\mathfrak{W}_-$. Then there exist a unique linear contraction $\mathfrak{B}_\Sigma \colon \mathcal{H}(\mathfrak{W}_-^{[\perp]}) \to \mathcal{X}$ whose restriction to $\mathcal{H}^0(\mathfrak{W}_-^{[\perp]})$ is given by*

$$\mathfrak{B}_\Sigma Q_- w = x(0), \quad w \in \mathfrak{W}_-, \tag{6.5}$$

where $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is the unique stable externally generated past trajectory of Σ whose signal part is w (cf. Lemma 3.11).

Proof. Let $w \in \mathfrak{W}_-$, and let $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ be the externally generated stable past trajectory of Σ with signal part w. Then by (3.2) and (5.11)

$$\|x(0)\|_{\mathcal{X}}^2 \le [w, w]_{K_-^2(\mathcal{W})} = \|Q_- w\|^2_{\mathcal{H}(\mathfrak{W}_-^{[\perp]})}.$$

This implies that the mapping $Q_- w \to x(0)$ is a linear contraction $\mathcal{H}^0(\mathfrak{W}_-^{[\perp]}) \to \mathcal{X}$. Since $\mathcal{H}^0(\mathfrak{W}_-^{[\perp]})$ is dense in $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$, this mapping has a unique extension to a linear contraction $\mathfrak{B}_\Sigma \colon \mathcal{H}(\mathfrak{W}_-^{[\perp]}) \to \mathcal{X}$. □

Definition 6.10. The contraction $\mathfrak{B}_\Sigma$ in Lemma 6.9 is called the *input map* of Σ.

In the construction of our three canonical realizations in Sections 8–10 we shall make crucial use of the following lemma.

Lemma 6.11. *Let $\Sigma = (V; \mathcal{X}; \mathcal{W})$ be a passive s/s system with past behavior $\mathfrak{W}_-$, future behavior $\mathfrak{W}_+$, full behavior $\mathfrak{W}$, input map $\mathfrak{B}_\Sigma$, and output map $\mathfrak{C}_\Sigma$.*

(i) *A pair of functions $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is an externally generated stable past trajectory of Σ if and only if*

$$w \in \mathfrak{W}_- \text{ and } x(t) = \mathfrak{B}_\Sigma Q_- \pi_- \tau^t w, \quad t \in \mathbb{R}^-. \tag{6.6}$$

(ii) *A pair of functions $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is an externally generated stable full trajectory of Σ if and only if*

$$w \in \mathfrak{W} \text{ and } x(t) = \mathfrak{B}_\Sigma Q_- \pi_- \tau^t w, \quad t \in \mathbb{R}. \tag{6.7}$$

In this case

$$\mathfrak{C}_\Sigma x(t) = Q_+ \pi_+ \tau^t w, \quad t \in \mathbb{R}. \tag{6.8}$$

(iii) *A pair of functions $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is an externally generated stable future trajectory of Σ if and only if*

$$w \in \mathfrak{W}_+ \text{ and } x(t) = \mathfrak{B}_\Sigma Q_- \pi_- \tau^t w, \quad t \in \mathbb{R}^+. \tag{6.9}$$

In this case

$$\mathfrak{C}_\Sigma x(t) = Q_+ \pi_+ \tau^t w, \quad t \in \mathbb{R}^+. \tag{6.10}$$

Proof. The proof of (i) is an easy modification of the proof of the first half of (ii), and (iii) is a special case of (ii), so let us only give the proof of (ii).

Let $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ be an externally generated stable full trajectory of Σ. Then $w \in \mathfrak{W}$, and (6.5) implies that (6.7) holds with $t = 0$. By shifting the trajectory to the left or right by the amount $|t|$ and applying (6.5) to the shifted trajectory we get (6.6) for all values of $t \in \mathbb{R}$.

Conversely, let $w \in \mathfrak{W}$. By Lemma 3.11, there exists a unique stable externally generated full trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ of Σ, and by the first part of the proof, the function x is given by (6.7).

That also (6.8) holds follows from Lemma 6.4 and the fact that the restriction to $\mathbb{R}^+$ of any left- or right-shifted externally generated stable full trajectory of Σ is a stable future trajectory of Σ. □

Definition 6.12. Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive s/s system.

(i) The *finite time reachable subspace* of Σ is the set

$$\left\{ x_0 \in \mathcal{X} \;\middle|\; \begin{array}{l} x_0 = x(0) \text{ for some (stable) past} \\ \text{trajectory of } \Sigma \text{ with compact support} \end{array} \right\}.$$

(ii) The *infinite time exactly reachable subspace* of Σ is the set

$$\left\{ x_0 \in \mathcal{X} \;\middle|\; \begin{array}{l} x_0 = x(0) \text{ for some stable externally} \\ \text{generated past trajectory of } \Sigma \end{array} \right\}.$$

(iii) The $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$*-exactly reachable subspace* of Σ is the range of the input map $\mathfrak{B}_\Sigma$ of Σ.

(iv) Σ is *exactly reachable in one of the above senses* if the corresponding reachable subspace is all of $\mathcal{X}$.

(v) The closure of the subspace in (i) is called the *(approximately) reachable subspace.*

(vi) The system Σ is *controllable* if the approximately reachable subspace is all of $\mathcal{X}$.

Lemma 6.13. *All the different types of exactly reachable subspaces in Definition 6.12 have the same closure, equal to the approximately reachable subspace.*

Proof. The three different types of exactly reachable subspaces defined in Definition 6.12 are (in the order that they appear) the range of the restriction of $\mathfrak{B}_\Sigma$ to the space $\mathcal{H}_0^0(\mathfrak{W}_-^{[\perp]})$ defined in Lemma 5.2, the range of the restriction of $\mathfrak{B}_\Sigma$ to the space $\mathcal{H}^0(\mathfrak{W}_-^{[\perp]})$, and the full range of $\mathfrak{B}_\Sigma$. That these three subspaces have the same closure follows from the fact that when one restricts the bounded linear operator $\mathfrak{B}_\Sigma$ to a dense subset of its domain, then the closure of its range remains the same. □

Lemma 6.14. *All the different types of exactly reachable subspaces in Definition 6.12 and also the approximately reachable subspace are strongly invariant in the sense that if* $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ *is a future trajectory of* Σ *whose initial state* $x(0)$ *belongs to one of these subspaces, then* $x(t)$ *stays in this subspace for all* $t \in \mathbb{R}^+$.

Proof. In the case of the finite time reachable subspace and the infinite time exactly reachable subspace the strong invariance follows from the general properties of trajectories (see Lemmas 3.3 and 3.4). In the case of the the $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$-exactly reachable subspace the strong invariance follows from Lemma 7.8 below. Finally, in the case of the approximately reachable subspace the strong invariance follows from the density and the strong invariance of the finitely reachable subspace. □

Lemma 6.15. *If Σ is a passive energy preserving s/s system, then the input map $\mathfrak{B}_\Sigma$ of Σ is an isometry. If, in addition, Σ is controllable, then $\mathfrak{B}_\Sigma$ is unitary.*

Proof. That $\mathfrak{B}_\Sigma$ is an isometry follows from the fact that we have equality in (3.2) whenever Σ is energy preserving (because then (1.9) holds as an equality). In particular, $\operatorname{im}(\mathfrak{B}_\Sigma)$ is closed. If, in addition, Σ is controllable, then $\operatorname{im}(\mathfrak{B}_\Sigma)$ is dense in $\mathcal{X}$, and hence equal to $\mathcal{X}$. □

A partial converse to Lemma 6.15 is given in Corollary 9.9 below.

6.3 The adjoints of $\mathfrak{C}_\Sigma$ and $\mathfrak{B}_\Sigma$

The rest of this section is devoted to the study of the adjoints of the input and output maps of a passive s/s system.

Lemma 6.16. *Let $\Sigma^\dagger = (V^\dagger; \mathcal{X}, \mathcal{W})$ be an anti-passive s/s system with past and future behaviors $\mathfrak{W}_-^\dagger$ and $\mathfrak{W}_+^\dagger$, respectively. Let $\mathfrak{W}_- := (\mathfrak{W}_-^\dagger)^{[\perp]}$ and $\mathfrak{W}_+ := (\mathfrak{W}_+^\dagger)^{[\perp]}$ be the corresponding passive past and future behaviors (see Lemma 4.11), and let Q_- and Q_+ be the quotient maps $K_-^2(\mathcal{W}) \to K_-^2(\mathcal{W})/\mathfrak{W}_-^{[\perp]}$ and $K_+^2(\mathcal{W}) \to K_+^2(\mathcal{W})/\mathfrak{W}_+$, respectively.*

(i) *There exists a unique contraction $\mathfrak{B}_{\Sigma^\dagger}^\dagger : \mathcal{H}(\mathfrak{W}_+) \to \mathcal{X}$ such that $\left[\begin{smallmatrix} x^\dagger \\ w^\dagger \end{smallmatrix}\right]$ is an externally generated stable future trajectory of $\Sigma^\dagger$ if and only if*

$$w^\dagger \in \mathfrak{W}_+^\dagger \text{ and } x^\dagger(t) = \mathfrak{B}_{\Sigma^\dagger}^\dagger Q_+ \tau^t w^\dagger, \qquad t \in \mathbb{R}^+. \tag{6.11}$$

(ii) *There exists a unique contraction $\mathfrak{C}_{\Sigma^\dagger}^\dagger : \mathcal{X} \to \mathcal{H}(\mathfrak{W}_-^\dagger)$ satisfying*

$$\mathfrak{C}_{\Sigma^\dagger}^\dagger x^\dagger(-t) = Q_- \tau_-^{*t} w^\dagger, \qquad t \in \mathbb{R}^+, \tag{6.12}$$

for every stable past trajectory $\left[\begin{smallmatrix} x^\dagger \\ w^\dagger \end{smallmatrix}\right]$ of $\Sigma^\dagger$.

Proof. This follows from Lemmas 6.11 and Lemma 6.4 and Remark 4.2. □

Definition 6.17. The contractions $\mathfrak{B}^{\dagger}_{\Sigma^{\dagger}}$ and $\mathfrak{C}^{\dagger}_{\Sigma^{\dagger}}$ are called the backward input and backward output maps of $\Sigma^{\dagger}$, respectively.

Lemma 6.18. *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive s/s system with input map $\mathfrak{B}_{\Sigma}$ and output map $\mathfrak{C}_{\Sigma}$, and let $\Sigma^{[\perp]}$ be the anti-passive dual of Σ, with the backward input map $\mathfrak{B}^{\dagger}_{\Sigma^{[\perp]}}$ and backward output map $\mathfrak{C}^{\dagger}_{\Sigma^{[\perp]}}$. Then $\mathfrak{B}^{\dagger}_{\Sigma^{[\perp]}} = \mathfrak{C}^{*}_{\Sigma}$ and $\mathfrak{C}^{\dagger}_{\Sigma^{[\perp]}} = \mathfrak{B}^{*}_{\Sigma}$.*

Proof. The proof of this lemma is essentially the same as the proof of [AS09b, Lemma 5.19] (see Remark 3.17). □

Lemma 6.19. *If Σ is a co-energy preserving passive s/s system, then the output map $\mathfrak{C}_{\Sigma}$ of Σ is a co-isometry. If, in addition, Σ is observable, then $\mathfrak{C}_{\Sigma}$ is unitary.*

Proof. The first claim follows from the fact that if Σ is co-energy preserving, then the anti-passive dual $\Sigma^{[\perp]}$ is energy preserving (in the backward time direction), and hence its input map $\mathfrak{B}_{\Sigma^{[\perp]}} = \mathfrak{C}^{*}_{\Sigma}$ is an isometry (see Lemma 6.15 and Remark 4.2). The second claim follows from the first claim since $\mathfrak{C}_{\Sigma}$ is injective iff Σ is observable. □

A partial converse to Lemma 6.19 is given in Corollary 8.8 below.

6.4 The backward reachable and unobservable subspaces

Our definitions of the reachable and unobservable subspaces $\mathfrak{R}_{\Sigma}$ and $\mathfrak{U}_{\Sigma}$ have a built-in direction of time. If $\Sigma^{\dagger} = (V^{\dagger}; \mathcal{X}, \mathcal{W})$ is an anti-passive system, then we denote the backward counterparts of $\mathfrak{R}_{\Sigma}$ and $\mathfrak{U}_{\Sigma}$ by $\mathfrak{R}^{\dagger}_{\Sigma^{\dagger}}$ and $\mathfrak{U}^{\dagger}_{\Sigma^{\dagger}}$, respectively. Thus, $\mathfrak{R}^{\dagger}_{\Sigma^{\dagger}}$ is the closure in $\mathcal{X}$ of all states $x(t)$ that appear in backward externally generated past trajectories $\left[\begin{smallmatrix} x^{\dagger} \\ w^{\dagger} \end{smallmatrix}\right]$ of $\Sigma^{\dagger}$, and $\mathfrak{U}^{\dagger}_{\Sigma^{\dagger}}$ consists of all $x^{\dagger}_0 \in \mathcal{X}$ with the property that there exists some future trajectory $\left[\begin{smallmatrix} x^{\dagger} \\ w^{\dagger} \end{smallmatrix}\right]$ of $\Sigma^{\dagger}$ which with $x^{\dagger}(0) = x^{\dagger}_0$ for which $w^{\dagger}$ vanishes identically. We call $\mathfrak{R}^{\dagger}_{\Sigma^{\dagger}}$ the (approximately) *backward reachable subspace* and $\mathfrak{U}^{\dagger}_{\Sigma^{\dagger}}$ the *backward unobservable subspace.*[1] By an *backward unobservable trajectory* we mean a past trajectory $\left[\begin{smallmatrix} x^{\dagger} \\ w^{\dagger} \end{smallmatrix}\right]$ of $\Sigma^{\dagger}$ for which $w^{\dagger}$ vanishes identically.

If Σ is conservative, then it is both passive and anti-passive, and hence both the forward reachable and unobservable subspaces $\mathfrak{R}_{\Sigma}$ and $\mathfrak{U}_{\Sigma}$ as well as the backward reachable and unobservable subspaces $\mathfrak{R}^{\dagger}_{\Sigma}$ and $\mathfrak{U}^{\dagger}_{\Sigma}$ are well-defined. A *full* trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ of a conservative system Σ whose signal part w

[1] In stochastic realization theory $\mathfrak{R}^{\dagger}$ is called the controllable subspace and $\mathfrak{U}^{\dagger}$ the *unconstructable subspace.*

vanishes identically will be called a *bilaterally unobservable trajectory.* The restriction of such a trajectory to $\mathbb{R}^+$ is unobservable, and the restriction to $\mathbb{R}^-$ is backward unobservable.

Definition 6.20. A conservative system is *simple* if it does not have any nontrivial bilaterally unobservable trajectories.

It follows from Lemmas 6.6 and 6.13 and their anti-passive counterparts combined with Lemma 6.18 that $\mathfrak{R}^\dagger_{\Sigma[\perp]} = \mathfrak{U}^{[\perp]}_\Sigma$ and $\mathfrak{U}^\dagger_{\Sigma[\perp]} = \mathfrak{R}^{[\perp]}_\Sigma$. Moreover, a conservative system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is simple if and only if $\mathfrak{U} \cap \mathfrak{U}^\dagger = \{0\}$, or equivalently, if and only if $\mathfrak{R} \vee \mathfrak{R}^\dagger = \mathcal{X}$.

7 The Past/Future Map of a Passive System

We here take a closer look at the past/future map $\Gamma_{\mathfrak{W}}$ introduced in Definition 5.8 in the case where $\mathfrak{W}$ is the full behavior of a passive s/s system Σ.

Definition 7.1. In the case where $\mathfrak{W}$ is the full behavior of a passive s/s system Σ, then we call the past/future map $\Gamma_{\mathfrak{W}}$ introduced in Definition 5.8 the *past/future map of* Σ, and denote it alternatively by Γ_Σ.

Lemma 7.2. *The past/future map Γ_Σ of a passive s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ factors into the product*

$$\Gamma_\Sigma = \mathfrak{C}_\Sigma \mathfrak{B}_\Sigma \tag{7.1}$$

of the input map $\mathfrak{B}_\Sigma$ and the output map $\mathfrak{C}_\Sigma$ of Σ. In particular, if Σ_i, $i = 1, 2$, are two passive s/s systems which have the same behaviors, and if we denote the input and output maps of Σ_i by $\mathfrak{B}_{\Sigma_i}$ and $\mathfrak{C}_{\Sigma_i}$, respectively, then $\mathfrak{C}_{\Sigma_1}\mathfrak{B}_{\Sigma_1} = \mathfrak{C}_{\Sigma_2}\mathfrak{B}_{\Sigma_2}$.

Proof. Let $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ be an externally generated stable full trajectory of Σ. Then the restriction of $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ to $\mathbb{R}^-$ is an externally generated stable past trajectory and the restriction of $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ to $\mathbb{R}^+$ is a stable future trajectory of Σ. Thus, by (6.6), $x(0) = \mathfrak{B}_\Sigma \pi_- w$ and by (6.4), $\mathfrak{C}_\Sigma x(0) = \pi_+ w + \mathfrak{W}_+$. Thus, the two operators Γ_Σ and $\mathfrak{C}_\Sigma \mathfrak{B}_\Sigma$ coincide on the dense subspace $\mathcal{H}^0(\mathfrak{W}^{[\perp]}_-)$ of $\mathcal{H}(\mathfrak{W}^{[\perp]}_-)$, and hence on all of $\mathcal{H}(\mathfrak{W}^{[\perp]}_-)$. If the systems Σ_i, $i = 1, 2$ have the same full behavior $\mathfrak{W}$, then they also have the same past/future map $\Gamma_{\mathfrak{W}}$. Thus $\mathfrak{C}_{\Sigma_1}\mathfrak{B}_{\Sigma_1} = \Gamma_{\mathfrak{W}} = \mathfrak{C}_{\Sigma_2}\mathfrak{B}_{\Sigma_2}$. □

We next turn to the corresponding operator induced by an anti-passive full behavior $\mathfrak{W}^\dagger$.

Lemma 7.3. *Let $\mathfrak{W}^\dagger$ be an anti-passive full behavior with the corresponding anti-passive past behavior $\mathfrak{W}^\dagger_- = \mathfrak{W}^\dagger \cap K^2_-(\mathcal{W})$ and future behavior $\mathfrak{W}^\dagger_+ = \pi_+\mathfrak{W}^\dagger$. Let $\mathfrak{W}_- := (\mathfrak{W}^\dagger_-)^{[\perp]}$ and $\mathfrak{W}_+ := (\mathfrak{W}^\dagger_+)^{[\perp]}$ be the passive past and future behaviors, respectively, induced by the passive full behavior $\mathfrak{W} = (\mathfrak{W}^\dagger)^{[\perp]}$. Then $\mathfrak{W}_- = \pi_-\mathfrak{W}$ and $\mathfrak{W}_+ = \mathfrak{W} \cap K2_+(\mathcal{W})$.*

Let Q_- and Q_+ be the quotient maps $K^2_-(\mathcal{W}) \to K^2_-(\mathcal{W})/\mathfrak{W}^{[\perp]}_-$ and $K^2_+(\mathcal{W}) \to K^2_+(\mathcal{W})/\mathfrak{W}_+$, respectively. Then there is a unique contraction $\Gamma_{\mathfrak{W}^\dagger} : \mathcal{H}(\mathfrak{W}_+) \to \mathcal{H}(\mathfrak{W}^{[\perp]}_-)$ satisfying

$$\Gamma_{\mathfrak{W}^\dagger} Q_+\pi_+ w^\dagger = Q_-\pi_- w^\dagger, \qquad w^\dagger \in \mathfrak{W}^\dagger. \tag{7.2}$$

Proof. The first claim follows from Lemmas 2.3, 3.18(iii), and 4.11, where we consider the orthogonal decomposition $K^2(\mathcal{W}) = K^2_-(\mathcal{W}) \boxplus K^2_+(\mathcal{W})$ in Lemma 2.3. The second claim follows from Remark 4.12 and Lemma 5.7. □

Definition 7.4. The contraction $\Gamma_{\mathfrak{W}^\dagger} : \mathcal{H}(\mathfrak{W}_+) \to \mathcal{H}(\mathfrak{W}^\dagger_-)$ in Lemma 7.3 is called the *future/past map* of the anti-passive full behavior $\mathfrak{W}^\dagger$. If $\mathfrak{W}^\dagger$ is the full behavior of a passive anti-causal s/s system $\Sigma^\dagger$, then we also call $\Gamma_{\mathfrak{W}^\dagger}$ the future/past map of $\Sigma^\dagger$ and denote it by $\Gamma_{\Sigma^\dagger}$.

Lemma 7.5. *The future/past map $\Gamma_{\Sigma^\dagger}$ of the anti-passive full behavior $\mathfrak{W}^\dagger$ induced by an anti-passive reflected s/s system $\Sigma^\dagger$ factors into the product*

$$\Gamma_{\Sigma^\dagger} = \mathfrak{C}^\dagger_{\Sigma^\dagger}\mathfrak{B}^\dagger_{\Sigma^\dagger} \tag{7.3}$$

of the backward input map $\mathfrak{B}_{\Sigma^\dagger}$ of $\Sigma^\dagger$ and the backward output map $\mathfrak{C}_{\Sigma^\dagger}$ of $\Sigma^\dagger$.

Proof. This follows from Lemma 7.2 and Remarks 4.2 and 4.12. □

Lemma 7.6. *The adjoint of the past/future map $\Gamma_{\mathfrak{W}}$ of the full behavior $\mathfrak{W}$ is the future/past map $\Gamma_{\mathfrak{W}^{[\perp]}}$ of the dual behavior $\mathfrak{W}^{[\perp]}$.*

Proof. This follows from Lemmas 7.2, 6.18, and 7.5. □

Lemma 7.7. *Let $\mathfrak{W}$ be a passive full behavior with the corresponding passive past behavior $\mathfrak{W}_- = \pi_-\mathfrak{W}$ and passive future behavior $\mathfrak{W}_+ = \mathfrak{W} \cap K^2_+(\mathcal{W})$. Let $w \in K^2(\mathcal{W})$, and suppose that $\pi_- w \in \mathcal{K}(\mathfrak{W}^{[\perp]}_-)$, $\pi_+ w \in \mathcal{K}(\mathfrak{W}_+)$, and that*

$$Q_+\pi_+ w = \Gamma_{\mathfrak{W}}\, Q_-\pi_- w, \tag{7.4}$$

where $\Gamma_{\mathfrak{W}}$ is the past/future map of $\mathfrak{W}$.

(i) *For all* $t \in \mathbb{R}^+$, $\pi_-\tau^t w \in \mathcal{K}(\mathfrak{W}_-^{[\perp]})$, $\pi_+\tau^t w \in \mathcal{K}(\mathfrak{W}_+)$,

$$Q_+\pi_+\tau^t w = \Gamma_{\mathfrak{W}}\, Q_-\pi_-\tau^t w, \quad t \in \mathbb{R}^+, \tag{7.5}$$

$$\|Q_-\pi_-\tau^t w\|^2_{\mathcal{H}(\mathfrak{W}_-^{[\perp]})} = \|Q_-\pi_- w\|^2_{\mathcal{H}(\mathfrak{W}_-^{[\perp]})} + \int_0^t [w(s), w(s)]_{\mathcal{W}}\, \mathrm{d}s, \quad t \in \mathbb{R}^+. \tag{7.6}$$

(ii) *There exists a sequence* $w^k \in \mathfrak{W}$ *such that*

$$Q_+\pi_+\tau^t w^k \to Q_+\pi_+\tau^t w \text{ in } \mathcal{H}(\mathfrak{W}_+), \qquad t \in \mathbb{R}^+, \tag{7.7}$$

$$Q_-\pi_-\tau^t w^k \to Q_-\pi_-\tau^t w \text{ in } \mathcal{H}(\mathfrak{W}_-^{[\perp]}), \qquad t \in \mathbb{R}^+, \tag{7.8}$$

$$\pi_+ w^k \to w_+ \text{ in } K^2_+(\mathcal{W}), \tag{7.9}$$

as $k \to \infty$, *where the convergence in* (7.7) *and* (7.8) *is uniform in* t.

Proof. The proof is essentially the same as the proof of [AS09b, Lemma 6.8] (see Remark 3.17). □

Lemma 7.8. *Let* $\Sigma = (V; \mathcal{X}, \mathcal{W})$ *be a passive s/s system with input map* $\mathfrak{B}_\Sigma$, *past behavior* $\mathfrak{W}_-$, *future behavior* $\mathfrak{W}_+$, *and past/future map* Γ_Σ. *Then the following two conditions are equivalent:*

(i) $\left[\begin{smallmatrix} x \\ w_+ \end{smallmatrix}\right]$ *is a stable future trajectory of* Σ *satisfying* $x(0) \in \operatorname{im}(\mathfrak{B}_\Sigma)$;

(ii) $w_+ \in \mathcal{K}(\mathfrak{W}_+)$, *and there exists some* $w_- \in \mathcal{K}(\mathfrak{W}_-^{[\perp]})$ *such that*

$$\begin{aligned} Q_+w_+ &= \Gamma_\Sigma\, Q_-w_- \text{ and} \\ x(t) &= \mathfrak{B}_\Sigma Q_-\pi_-\tau^t(w_- + w_+), \quad t \in \mathbb{R}^+. \end{aligned} \tag{7.10}$$

The function w_- *in* (7.10) *can be chosen to be any* $w_- \in \mathcal{K}(\mathfrak{W}_-^{[\perp]})$ *satisfying* $x(0) = \mathfrak{B}_\Sigma Q_- w_-$.

Proof. The proof of the implication (i) ⇒ (ii) and the proof of the final claim are essentially the same as the first part of the proof of [AS09b, Lemma 6.9] (see Remark 3.17).

Conversely, suppose that (ii) holds. Then, in particular, $x(0) = \mathfrak{B}_\Sigma Q_- w_- \in \operatorname{im}(\mathfrak{B}_\Sigma)$ and $Q_+w_+ = \Gamma_\Sigma Q_- w_-$. By Lemma 7.2, $Q_+w_+ = \mathfrak{C}_\Sigma x_0 \in \operatorname{im}(\mathfrak{C}_\Sigma)$. By (6.2), there exists a stable future trajectory $\left[\begin{smallmatrix} x_1 \\ w_+ \end{smallmatrix}\right]$ of Σ. By Lemma 6.4, $\mathfrak{C}_\Sigma x_1(t) = Q_+\tau^t_+ w_+$ for all $t \in \mathbb{R}^+$. On the other hand, by assumption, $x(t) = \mathfrak{B}_\Sigma Q_-\pi_-\tau^t(w_- + w_+)$ for all $t \in \mathbb{R}^+$, and hence by Lemma 7.7,

$$\begin{aligned} \mathfrak{C}_\Sigma x(t) &= \mathfrak{C}_\Sigma \mathfrak{B}_\Sigma Q_-\pi_-\tau^t(w_- + w_+) = \Gamma_\Sigma Q_-\pi_-\tau^t(w_- + w_+) \\ &= Q_+\pi_+\tau^t w. \end{aligned}$$

Thus, $\mathfrak{C}_\Sigma(x(0) - x_1(0)) = 0$. By Lemma 6.6, there exists an unobservable future trajectory $\left[\begin{smallmatrix} x_2 \\ 0 \end{smallmatrix}\right]$ of Σ with $x_2(0) = x(0) - x_1(0)$. Define $x_3 = x_1 + x_2$. Then $\left[\begin{smallmatrix} x_3 \\ w_+ \end{smallmatrix}\right]$ is a stable future trajectory of Σ, $x_3(0) = x(0) = \mathfrak{B}_\Sigma Q_- w_-$, and by the implication (i) ⇒ (ii),

$$x_3(t) = \mathfrak{B}_\Sigma Q_- \pi_- \tau^t (w_- + w_+) = x(t), \qquad t \in \mathbb{R}^+.$$

Thus, $\left[\begin{smallmatrix} x \\ w_+ \end{smallmatrix}\right]$ is a stable future trajectory of Σ. □

8 The Canonical Observable Co-Energy Preserving Model

In this section we shall construct a canonical model $\Sigma_{\text{oce}}^{\mathfrak{W}_+} = (V_{\text{oce}}^{\mathfrak{W}_+}; \mathcal{H}(\mathfrak{W}_+), \mathcal{W})$ of a passive observable co-energy preserving s/s system with a given passive future behavior $\mathfrak{W}_+$.

Theorem 8.1. *Let $\mathfrak{W}_+$ be a passive future behavior on the Kreĭn space $\mathcal{W}$ with the corresponding full behavior $\mathfrak{W}$ defined by* (3.19) *and past behavior $\mathfrak{W}_- := \pi_- \mathfrak{W}$. Define $\mathcal{H}(\mathfrak{W}_+)$ as in Theorem 5.4 and $\mathcal{K}(\mathfrak{W}_+)$ as in* (5.12).

(i) *Define*

$$\mathcal{T}_+ := \left\{ \left[\begin{smallmatrix} x \\ w_+ \end{smallmatrix}\right] \in \begin{bmatrix} C(\mathbb{R}^+; \mathcal{H}(\mathfrak{W}_+)) \\ \mathcal{K}(\mathfrak{W}_+) \end{bmatrix} \,\middle|\, x(t) = Q_+ \tau_+^t w_+,\ t \in \mathbb{R}^+ \right\}. \tag{8.1}$$

Then $\mathcal{T}_+$ is the set of all stable future trajectories of a passive observable co-energy preserving s/s system $\Sigma_{\text{oce}}^{\mathfrak{W}_+}$ with state space $\mathcal{H}(\mathfrak{W}_+)$ whose future behavior is equal to $\mathfrak{W}_+$ and full behavior is equal to $\mathfrak{W}$.

(ii) *The input map of $\Sigma_{\text{oce}}^{\mathfrak{W}_+}$ is the past/future map $\Gamma_{\mathfrak{W}}$ of $\mathfrak{W}$, and the output map of $\Sigma_{\text{oce}}^{\mathfrak{W}_+}$ is the identity on $\mathcal{H}(\mathfrak{W}_+)$.*

(iii) *A pair of functions $\left[\begin{smallmatrix} x \\ w_- \end{smallmatrix}\right]$ is an externally generated stable past trajectory of $\Sigma_{\text{oce}}^{\mathfrak{W}_+}$ if and only if*

$$w_- \in \mathfrak{W}_- \text{ and } x(-t) = \Gamma_{\mathfrak{W}} Q_- \tau_-^{*t} w_-, \quad t \in \mathbb{R}^+. \tag{8.2}$$

Proof. We define the Kreĭn spaces $\mathfrak{K}_{0,t}$ and $\mathfrak{L}_{0,\infty}$ as in the paragraph before Theorem 3.5 with $\mathcal{X}$ replaced by $\mathcal{H}(\mathfrak{W}_+)$, and the subspaces $\mathcal{T}_{0,t}$ and $\mathcal{S}_{0,\infty}$ by (3.5) with $\mathcal{T}_+$ defined by (8.1).

Step 1: $\mathcal{T}_{0,t}$ is a maximal nonnegative subspace of $\mathfrak{K}_{0,t}$. That $\mathcal{T}_{0,t}$ is a nonnegative subspace of $\mathfrak{K}_{0,t}$ follows from Lemma 5.3. The maximality of $\mathcal{T}_{0,t}$ follows from Lemma 3.20 and Lemma 2.2(i) with $\mathcal{Y} = \mathcal{X} = \mathcal{H}(\mathfrak{W}_+)$.

Step 2: $(\mathcal{T}_{0,t})^{[\perp]} \subset \mathcal{T}_{0,t}$. Fix $t \in \mathbb{R}^+$ and define $\mathring{\mathcal{T}}_{[0,t]}$ by

$$\mathring{\mathcal{T}}_{[0,t]} := \left\{ \begin{bmatrix} Q_+\tau_+^t z^\dagger \\ Q_+ z^\dagger \\ \pi_{[0,t]} z^\dagger \end{bmatrix} \middle| \, z^\dagger \in \mathfrak{W}_+^{[\perp]} \right\}. \tag{8.3}$$

Then $\mathring{\mathcal{T}}_{[0,t]} \subset \mathcal{T}_{0,t}$ since $\mathcal{H}^0(\mathfrak{W}_+) \subset \mathcal{H}(\mathfrak{W}_+)$. We claim that $(\mathring{\mathcal{T}}_{[0,t]})^{[\perp]} = \mathcal{T}_{0,t}$. Clearly, this implies that $(\mathcal{T}_{0,t})^{[\perp]} \subset \mathcal{T}_{0,t}$ since $(\mathcal{T}_{0,t})^{[\perp]} = ((\mathring{\mathcal{T}}_{[0,t]})^{[\perp]})^{[\perp]}$ is the closure of $\mathring{\mathcal{T}}_{[0,t]}$.

A vector $k = \left[\begin{smallmatrix} x_1 \\ x_0 \\ w \end{smallmatrix}\right]$ belongs to $\mathring{\mathcal{T}}_{[0,t]}^{[\perp]}$ if and only if $x_1,\ x_0 \in \mathcal{H}(\mathfrak{W}_+)$, $w \in K^2([0,t];\mathcal{W})$, and

$$-(x_1, Q_+\tau_+^t z^\dagger)_{\mathcal{H}(\mathfrak{W}_+)} + (x_0, Q_+ z^\dagger)_{\mathcal{H}(\mathfrak{W}_+)} + [w, z^\dagger]_{K^2([0,t]);\mathcal{W})} = 0, \quad z^\dagger \in \mathfrak{W}_+^{[\perp]}. \tag{8.4}$$

Since $\mathfrak{W}_+$ is τ_+^*-invariant, its orthogonal companion $\mathfrak{W}_+^{[\perp]}$ is τ_+-invariant, i.e., $\tau_+^t z^\dagger \in \mathfrak{W}_+^{[\perp]}$ whenever $z^\dagger \in \mathfrak{W}_+^{[\perp]}$ and $t \in \mathbb{R}^+$. By (5.6), for every $v_1 \in x_1$ and $v_0 \in x_0$, (8.4) can therefore be rewritten in the form

$$[v_1, \tau_+^t z^\dagger]_{K_+^2(\mathcal{W})} - [v_0, z^\dagger]_{K_+^2(\mathcal{W})} + [w, z^\dagger]_{K^2([0,t]);\mathcal{W})} = 0, \quad z^\dagger \in \mathfrak{W}_+^{[\perp]}. \tag{8.5}$$

Extend w to a function in $K_+^2(\mathcal{W})$ by defining $w(s) = 0$ for $s > t$. Then (8.5) can be rewritten as

$$[\tau_+^{*t} v_1 - v_0 + w, z^\dagger]_{K_+^2(\mathcal{W})} = 0, \qquad z^\dagger \in \mathfrak{W}_+^{[\perp]}.$$

Since $(\mathfrak{W}_+^{[\perp]})^{[\perp]} = \mathfrak{W}_+$, this is equivalent to

$$\tau_+^{*t} v_1 - v_0 + w = z$$

for some $z \in \mathfrak{W}_+$. Define $v = v_0 + z$. Then $v \in x_0$, and

$$\tau_+^{*t} v_1 - v + w = 0.$$

This is equivalent to the pair of equations

$$w = \pi_{[0,t]} v \text{ and } v_1 = \tau_+^t v.$$

Thus, $\left[\begin{smallmatrix} x_1 \\ x_0 \\ w_0 \end{smallmatrix}\right] \in (\mathring{\mathcal{T}}_{[0,t]})^{[\perp]}$ if and only if $x_0 = Q_+ v$, $x_1 = Q_+\tau_+^t v$, and $w = \pi_{[0,t]} v$ for some $v \in \mathcal{K}(\mathfrak{W}_+)$, or equivalently, if and only if $k \in \mathcal{T}_{0,t}$.

Step 3: $\mathcal{T}_+$ is the set of stable future trajectories of a passive co-energy preserving s/s system $\Sigma_{\rm oce}^{\mathfrak{W}_+}$. By (8.1) and Lemma 5.3, $\mathcal{T}_+$ is left-shift invariant, and by Steps 1 and 2, $\mathcal{T}_{0,t}$ is a maximal nonnegative co-neutral subspace of $\mathfrak{K}_{0,t}$ for all $t \in \mathbb{R}^+$. By (5.1), $\mathcal{S}_{0,\infty}$ is nonnegative in $\mathfrak{L}_{0,\infty}$, and by the definition of $\mathcal{K}(\mathfrak{W}_+)$, the maximal nonnegativity of $\mathfrak{W}_+$, and Lemma 2.2(i) with $\mathcal{Y} = \{0\}$ and $\mathcal{X} = \mathcal{H}(\mathfrak{W}_+)$, $\mathcal{S}_{0,\infty}$ is maximal nonnegative in $\mathfrak{L}_{0,\infty}$. By Theorem 3.5, $\mathcal{T}_+$ is the set of stable trajectories of a passive s/s system, and by Lemma 4.9, this system is co-energy preserving.

Step 4: The future behavior of $\Sigma_{\mathrm{oce}}^{\mathfrak{W}_+}$ *is equal to* $\mathfrak{W}_+$. This follows from the definition (8.1) of $\mathcal{T}_+$.

Step 5: $\mathfrak{B}_{\Sigma_{\mathrm{oce}}^{\mathfrak{W}_+}} = \Gamma_{\mathfrak{W}}$ *and* $\mathfrak{C}_{\Sigma_{\mathrm{oce}}^{\mathfrak{W}_+}} = 1_{\mathcal{H}(\mathfrak{W}_+)}$. By (8.1), if $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is a stable future trajectory of $\Sigma_{\mathrm{oce}}^{\mathfrak{W}_+}$, then $Q_+ w = x(0)$. On the other hand, by Lemma 6.4, $Q_+ w = \mathfrak{C}_{\Sigma_{\mathrm{oce}}^{\mathfrak{W}_+}} x(0)$. Thus, $\mathfrak{C}_{\Sigma_{\mathrm{oce}}^{\mathfrak{W}_+}} = 1_{\mathcal{H}(\mathfrak{W}_+)}$. By Lemma 7.2, $\Gamma_{\mathfrak{W}} = \mathfrak{C}_{\Sigma_{\mathrm{oce}}^{\mathfrak{W}_+}} \mathfrak{B}_{\Sigma_{\mathrm{oce}}^{\mathfrak{W}_+}} = \mathfrak{B}_{\Sigma_{\mathrm{oce}}^{\mathfrak{W}_+}}$.

Step 6: $\Sigma_{\mathrm{oce}}^{\mathfrak{W}_+}$ *is observable.* This follows from Step 5 and Lemma 6.6.

Step 7: (8.2) *holds.* This follows from Step 5 and Lemma 6.11. □

Theorem 8.2. *The generating subspace of the s/s system* $\Sigma_{\mathrm{oce}}^{\mathfrak{W}_+}$ *in Theorem 8.1 is given by*

$$V_{\mathrm{oce}}^{\mathfrak{W}_+} = \left\{ \begin{bmatrix} Q_+\dot{w}_+ \\ Q_+ w_+ \\ w_+(0) \end{bmatrix} \in \begin{bmatrix} \mathcal{H}(\mathfrak{W}_+) \\ \mathcal{H}(\mathfrak{W}_+) \\ \mathcal{W} \end{bmatrix} \;\middle|\; \begin{array}{l} w_+ \in \mathcal{K}(\mathfrak{W}_+) \textit{ is locally absolutely} \\ \textit{continuous with } \dot{w} \in K_+^2(\mathcal{W}), \textit{ and} \\ \displaystyle\lim_{t\to 0+} \frac{1}{t} Q_+(\tau_+^t w_+ - w_+) \textit{ exists in } \mathcal{H}(\mathfrak{W}_+). \end{array} \right\} \tag{8.6}$$

Proof. Before proving (8.6) we first show that (8.6) is equivalent to the formula

$$V_{\mathrm{oce}}^{\mathfrak{W}_+} = \left\{ \begin{bmatrix} \lim_{t\to 0+} \frac{1}{t} Q_+(\tau_+^t w_+ - w_+) \\ Q_+ w_+ \\ w_+(0) \end{bmatrix} \in \begin{bmatrix} \mathcal{H}(\mathfrak{W}_+) \\ \mathcal{H}(\mathfrak{W}_+) \\ \mathcal{W} \end{bmatrix} \;\middle|\; \begin{array}{l} w_+ \in \mathcal{K}(\mathfrak{W}_+) \text{ is locally} \\ \text{absolutely continuous with} \\ \dot{w} \in K_+^2(\mathcal{W}), \text{ and} \\ \displaystyle\lim_{t\to 0+} \frac{1}{t} Q_+(\tau_+^t w_+ - w_+) \\ \text{exists in } \mathcal{H}(\mathfrak{W}_+). \end{array} \right\} \tag{8.7}$$

Indeed, as $t \to 0+$, the function $\frac{1}{t}(\tau_+^t w_+ - w_+)$ tends to $\dot{w}_+$ in $K_+^2(\mathcal{W})$, and since $Q_+|_{\mathcal{K}(\mathfrak{W}_+)}$ is closed as an operator $K_+^2(\mathcal{W}) \to \mathcal{H}(\mathfrak{W}_+)$, this implies that $\lim_{t\to 0+} \frac{1}{t} Q_+(\tau_+^t w_+ - w_+) = \lim_{t\to 0+} Q_+ \frac{1}{t}(\tau_+^t w_+ - w_+) = Q_+\dot{w}_+$. Thus,(8.6) and (8.7) are equivalent.

Let $w_+ \in \mathcal{K}(\mathfrak{W}_+) \cap W^{1,2}(\mathbb{R}^+; \mathcal{W})$, and suppose that $\lim_{t\to 0+} \frac{1}{t} Q_+(\tau_+^t w_+ - w_+)$ exists in $\mathcal{H}(\mathfrak{W}_+)$. Define $x(t) := Q_+ \tau_+^t w_+$ for $t \in \mathbb{R}^+$. Then $\left[\begin{smallmatrix} x \\ w_+ \end{smallmatrix}\right]$ is a generalized stable future trajectory of $\Sigma_{\mathrm{oce}}^{\mathfrak{W}_+}$. By Proposition 3.8, this trajectory is even classical. In particular, $\left[\begin{smallmatrix} \dot{x}(0) \\ x(0) \\ w_+(0) \end{smallmatrix}\right] \in V_{\mathrm{oce}}^{\mathfrak{W}_+}$, where $\dot{x}(0) =$

$\lim_{t\to 0+} \frac{1}{t} Q_+(\tau_+^t w_+ - w_+)$. Thus the right-hand side of (8.7) is contained in $V_{\text{oce}}^{\mathfrak{W}_+}$. The opposite inclusion follows from Proposition 3.7(ii). □

Corollary 8.3. *The system $\Sigma_{\text{oce}}^{\mathfrak{W}_+}$ is approximately null-controllable in the sense that the set of all the initial states $x(0)$ of all future trajectories of $\Sigma_{\text{oce}}^{\mathfrak{W}_+}$ with compact support is dense in $\mathcal{X}_{\text{oce}} = \mathcal{H}(\mathfrak{W}_+)$.*

Proof. This follows from Theorem 8.1 and Lemma 5.2, since $\tau_+^t w_+ = 0$ for all sufficiently large t whenever $w_+ \in \mathcal{H}_0^0(\mathfrak{W}_+)$. □

Theorem 8.4. *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive s/s system with output map $\mathfrak{C}_\Sigma$ and future behavior $\mathfrak{W}_+$. Then Σ and $\Sigma_{\text{oce}}^{\mathfrak{W}_+} = (V_{\text{oce}}^{\mathfrak{W}_+}; \mathcal{H}(\mathfrak{W}_+), \mathcal{W})$ are contractively intertwined by $\mathfrak{C}_\Sigma$.*

Proof. Let $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ be a stable future trajectory of Σ. By Lemmas 6.1 and 6.4, $w \in \mathcal{K}(\mathfrak{W}_+)$ and $\mathfrak{C}_\Sigma x(t) = Q_+ \tau_+^t w$, $t \in \mathbb{R}^+$. Define $x_o(t) = Q_+ \tau_+^t w$, $t \in \mathbb{R}^+$. Then $\left[\begin{smallmatrix} x_o \\ w \end{smallmatrix}\right]$ is a stable future trajectory of $\Sigma_{\text{oce}}^{\mathfrak{W}_+}$, and $x_o(t) = \mathfrak{C}_\Sigma x(t)$, $t \in \mathbb{R}^+$.

Conversely, let $\left[\begin{smallmatrix} x_o \\ w \end{smallmatrix}\right]$ be a stable future trajectory of $\Sigma_{\text{oce}}^{\mathfrak{W}_+}$ satisfying $x_o(0) \in \operatorname{im}\big(\mathfrak{C}_\Sigma\big)$. Then $Q_+ w = x_o(0)$, and hence $w \in Q_+^{-1} \operatorname{im}\big(\mathfrak{C}_\Sigma\big) = \mathfrak{S}_+^\Sigma$. By (6.2), there exists a stable future trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ of Σ (whose signal part is equal to the given signal w). By Lemma 6.4, $\mathfrak{C}_\Sigma x(t) = Q_+ \tau_+^t w = x_o(t)$, $t \in \mathbb{R}^+$. □

Theorem 8.5. *Every observable and co-energy preserving passive s/s system Σ with future behavior $\mathfrak{W}_+$ is unitarily similar to the system $\Sigma_{\text{oce}}^{\mathfrak{W}_+}$. The unitary similarity transformation is the output map $\mathfrak{C}_\Sigma$ of Σ.*

Proof. By Lemma 6.19, the output map $\mathfrak{C}_\Sigma$ is unitary, and by Theorem 8.4, $\mathfrak{C}_\Sigma$ intertwines Σ and $\Sigma_{\text{oce}}^{\mathfrak{W}_+^\Sigma}$. □

Definition 8.6. We call the system $\Sigma_{\text{oce}}^{\mathfrak{W}_+}$ the *canonical model* of an observable passive co-energy preserving s/s system with future behavior $\mathfrak{W}_+$.

Corollary 8.7. *Any two observable and co-energy preserving realizations of a given passive future behavior $\mathfrak{W}_+$ are unitarily similar to each other.*

Proof. This follows from Theorem 8.5. □

Corollary 8.8. *A passive s/s system Σ is observable and co-energy preserving if and only if its output map $\mathfrak{C}_\Sigma$ is unitary.*

Proof. This follows from Lemma 6.19 and Theorem 8.4. □

9 The Canonical Controllable Energy Preserving Model

In this section we shall construct a canonical model $\Sigma_{\rm cep}^{\mathfrak{W}_-} = (V_{\rm cep}^{\mathfrak{W}_-}; \mathcal{X}_{\rm cep}^{\mathfrak{W}_-}, \mathcal{W})$ of a passive controllable energy preserving s/s system with a given passive past behavior $\mathfrak{W}_-$. The results for this model are analogous to the results on the model $\Sigma_{\rm oce}^{\mathfrak{W}_+}$ obtained in the preceding section. The state space of $\Sigma_{\rm cep}^{\mathfrak{W}_-}$ is the Hilbert space $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$ presented in Theorem 5.4.

Theorem 9.1. *Let $\mathfrak{W}_-$ be a passive past behavior on the Kreĭn space $\mathcal{W}$ with the corresponding full behavior $\mathfrak{W}$ defined by (3.18) and future behavior $\mathfrak{W}_+ := \mathfrak{W} \cap K_+^2(\mathcal{W})$. Define $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$ as in Theorem 5.4 and define $\mathcal{K}(\mathfrak{W}_+)$ and $\mathcal{K}(\mathfrak{W}_-^{[\perp]})$ as in (5.5) and (5.12), respectively.*

(i) *Define*

$$\mathcal{T}_+ := \left\{ \begin{bmatrix} x \\ w_+ \end{bmatrix} \in \begin{bmatrix} C(\mathbb{R}^+;\mathcal{X}) \\ \mathcal{K}(\mathfrak{W}_+) \end{bmatrix} \;\middle|\; \begin{array}{l} Q_+ w_+ = \Gamma_{\mathfrak{W}}\, Q_- w_- \text{ and} \\ x(t) = Q_- \pi_- \tau^t (w_- + w_+) \\ \text{for some } w_- \in \mathcal{K}(\mathfrak{W}_-^{[\perp]}) \text{ satisfying} \\ x(0) = Q_- w_-, \end{array} \right\} \tag{9.1}$$

Then $\mathcal{T}_+$ is the set of all stable future trajectories of a a passive controllable energy preserving s/s system $\Sigma_{\rm cep}^{\mathfrak{W}_-}$ with state space $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$ whose past behavior is equal to $\mathfrak{W}_-$.

(ii) *The input map of $\Sigma_{\rm cep}^{\mathfrak{W}_-}$ is the identity on $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$ and the output map of $\Sigma_{\rm cep}^{\mathfrak{W}_-}$ is the past/future map $\Gamma_{\mathfrak{W}}$ of $\mathfrak{W}$.*

(iii) *A pair of functions $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is an externally generated stable past trajectory of $\Sigma_{\rm cep}^{\mathfrak{W}_-}$ if and only if*

$$w \in \mathfrak{W}_- \text{ and } x(-t) = Q_- \tau_-^{*t} w, \quad t \geq 0. \tag{9.2}$$

Proof. We define the Kreĭn spaces $\mathfrak{K}_{0,t}$ and $\mathfrak{L}_{0,\infty}$ as in the paragraph before Theorem 3.5 with $\mathcal{X}$ replaced by $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$, and the subspaces $\mathcal{T}_{0,t}$ and $\mathcal{S}_{0,\infty}$ by (3.5) with $\mathcal{T}_+$ defined by (9.1).

Step 1: $\mathcal{T}_{0,t}$ is a maximal nonnegative neutral subspace of $\mathfrak{K}_{0,t}$. That $\mathcal{T}_{0,t}$ is a neutral subspace of $\mathfrak{K}_{0,t}$ follows from (7.6). It follows from (9.1) that to every $x_0 \in \mathcal{H}(\mathfrak{W}_-^{[\perp]})$ there exists some $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathcal{T}_+$ such that $x(0) = x_0$. Moreover, if $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathcal{T}_+$ with $x(0) = 0$, then $w \in \mathfrak{W}_+$. These two facts together with Lemma 2.2 with $\mathcal{Y} = \mathcal{X} = \mathcal{H}(\mathfrak{W}_-^{[\perp]})$ imply that $\mathcal{T}_{0,t}$ is maximal nonnegative.

Step 2: $\mathcal{T}_+$ is the set of stable future trajectories of a passive energy preserving s/s system $\Sigma_{\rm cep}^{\mathfrak{W}_-}$. By (9.1) and Lemma 7.7, $\mathcal{T}_+$ is left-shift invariant, and by Step 1, each $\mathcal{T}_{0,t}$ is a maximal nonnegative and neutral subspace of $\mathfrak{K}_{0,t}$. By letting $t \to \infty$ and using the nonnegativity of $\mathcal{T}_{0,t}$ we find that $\mathcal{S}_{0,\infty}$ is nonnegative, and the maximality of $\mathcal{S}_{0,\infty}$ is proved in the same way as the maximality of $\mathcal{T}_{0,t}$. By Theorem 3.5, $\mathcal{T}_+$ is the set of stable trajectories of a passive s/s system, and by Lemma 4.9, this system is energy preserving.

Step 3: The past, full, and future behaviors of $\Sigma_{\rm cep}^{\mathfrak{W}_-}$ are equal to $\mathfrak{W}_-$, $\mathfrak{W}$, and $\mathfrak{W}_+$, respectively. That the future behavior of $\Sigma_{\rm cep}^{\mathfrak{W}_-}$ is equal to $\mathfrak{W}_+$ follows from the definition (9.1) of $\mathcal{T}_+$, and the remaining claims then follow from Proposition 3.15.

Step 4: $\mathfrak{B}_{\Sigma_{\rm cep}^{\mathfrak{W}_-}} = 1_{\mathcal{H}(\mathfrak{W}_-^{[\perp]})}$ and $\mathfrak{C}_{\Sigma_{\rm cep}^{\mathfrak{W}_-}} = \Gamma_{\mathfrak{W}}$. Take $w_- = 0$ and $w_+ \in \mathfrak{W}_+$ in (9.1), define $w = w_- + w_+$, and define $x(t) = Q_-\pi_-\tau^t w$ for $t \in \mathbb{R}$. Then by Step 2 and Lemma 3.9(ii), $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is a full trajectory of $\Sigma_{\rm cep}^{\mathfrak{W}_-}$ supported on $\mathbb{R}^+$. If we left-translate this trajectory by the amount $s > 0$, then we get another stable full trajectory $\left[\begin{smallmatrix} x_s \\ w_s \end{smallmatrix}\right] := \left[\begin{smallmatrix} \tau^s x \\ \tau^s w \end{smallmatrix}\right]$ of $\Sigma_{\rm cep}^{\mathfrak{W}_-}$ supported on $[-s, \infty)$. This trajectory satisfies $(x_s)(0) = Q_-\pi_-w_s$. On the other hand, by Lemma 6.9, $x_s(0) = \mathfrak{B}_{\Sigma_{\rm cep}^{\mathfrak{W}_-}} Q_-\pi_-w_s$. Thus, $\mathfrak{B}_{\Sigma_{\rm cep}^{\mathfrak{W}_-}} Q_-\pi_-w_s = Q_-\pi_-w_s$. By varying w_+ and s we can in this way generate all the stable full trajectories of $\Sigma_{\rm cep}^{\mathfrak{W}_-}$ whose support are bounded to the left, and consequently, the restriction of $\mathfrak{B}_{\Sigma_{\rm cep}^{\mathfrak{W}_-}}$ to the space $\mathcal{H}_0^0(\mathfrak{W}_-^{[\perp]})$ defined in Lemma 5.5 is the identity. By Lemma 5.5, $\mathcal{H}_0^0(\mathfrak{W}_-^{[\perp]})$ is dense in $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$, and thus $\mathfrak{B}_{\Sigma_{\rm cep}^{\mathfrak{W}_-}} = 1_{\mathcal{H}(\mathfrak{W}_-^{[\perp]})}$. By Lemma 7.2, $\Gamma_{\mathfrak{W}} = \mathfrak{C}_{\Sigma_{\rm cep}^{\mathfrak{W}_-}} \mathfrak{B}_{\Sigma_{\rm cep}^{\mathfrak{W}_-}} = \mathfrak{C}_{\Sigma_{\rm cep}^{\mathfrak{W}_-}}$.

Step 5: $\Sigma_{\rm cep}^{\mathfrak{W}_-}$ is controllable. This follows from Step 4 and Lemma 6.13.

Step 6: A pair of functions $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is an externally generated stable past trajectory of $\Sigma_{\rm cep}^{\mathfrak{W}_-}$ if and only if (9.2) *holds.* This follows from Step 4 and Lemma 6.11(i). □

Corollary 9.2. *Every stable past trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ of $\Sigma_{\rm cep}^{\mathfrak{W}_-}$ (not necessarily externally generated) satisfies*

$$w \in \mathcal{K}(\mathfrak{W}_-^{[\perp]}) \text{ and } x(-t) = Q_-\tau_-^{*t} w,\ t \geq 0. \tag{9.3}$$

Proof. Since $\Sigma_{\rm cep}^{\mathfrak{W}_-}$ is energy preserving, it follows from Lemma 4.8 that every stable past trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ of $\Sigma_{\rm cep}^{\mathfrak{W}_-}$ is also a stable past trajectory of the anti-passive dual $\Sigma^\dagger$ of $\Sigma_{\rm cep}^{\mathfrak{W}_-}$. By applying the reflected version of Theorem 8.1 to the system $\Sigma^\dagger$ we find that (9.3) holds. □

Corollary 9.3. *The system $\Sigma_{\rm cep}^{\mathfrak{W}_-}$ is both $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$-exactly controllable and observable in backward time in the sense that if the signal part w of a stable past trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ of $\Sigma_{\rm cep}^{\mathfrak{W}_-}$ is zero, then also the state part x is zero.*

Proof. The first claim follows from the fact that the input map is the identity, and the second claim follows from (9.3). □

Lemma 9.4. *Let $\mathfrak{W}_-$ be a passive past behavior on the Kreĭn space $\mathcal{W}$. Then the generating subspace $V_{\rm cep}^{\mathfrak{W}_-}$ of the controllable and energy preserving s/s system $\Sigma_{\rm cep}^{\mathfrak{W}_-}$ in Theorem 9.1 is a closed subspace of the subspace*

$$(V_{\rm cep}^{\mathfrak{W}_-})^{[\perp]} = \left\{ \begin{bmatrix} Q_-\dot{w}_- \\ Q_-w_- \\ w_-(0) \end{bmatrix} \in \begin{bmatrix} \mathcal{H}(\mathfrak{W}_-) \\ \mathcal{H}(\mathfrak{W}_-) \\ \mathcal{W} \end{bmatrix} \,\middle|\, \begin{array}{l} w_- \in \mathcal{H}(\mathfrak{W}_-^{[\perp]}) \text{ is locally absolutely} \\ \text{continuous with } \dot{w} \in K_-^2(\mathcal{W}) \text{ and} \\ \lim\limits_{t\to 0+} \dfrac{1}{t} Q_-(\tau_-^{*t} w - w) \text{ exists in } \mathcal{H}(\mathfrak{W}_-). \end{array} \right\}. \tag{9.4}$$

Proof. The above subspace is the generating subspace of the anti-causal dual of $\Sigma_{\rm cep}^{\mathfrak{W}_-}$. That system is a co-energy preserving anti-passive s/s realization of the anti-passive past behavior $\mathfrak{W}_-^{[\perp]}$, and its generating subspace is obtained from (8.6) through a time reflection and the replacement of $\mathfrak{W}_+$ by $\mathfrak{W}_-^{[\perp]}$. Since $\Sigma_{\rm cep}^{\mathfrak{W}_-}$ is energy-preserving, $V_{\rm cep}^{\mathfrak{W}_-}$ is a closed subspace of its orthogonal companion. □

The exact description of $V_{\rm cep}^{\mathfrak{W}_-}$ will be given in Theorem 10.9 below.

Theorem 9.5. *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive s/s system with input map $\mathfrak{B}_\Sigma$ and past behavior $\mathfrak{W}_-$. Then $\Sigma_{\rm cep}^{\mathfrak{W}_-}$ and Σ are intertwined by $\mathfrak{B}_\Sigma$.*

Proof. This follows from Lemma 7.8 and Theorem 9.1. □

Theorem 9.6. *Every controllable and energy preserving passive s/s system Σ with past behavior $\mathfrak{W}_-$ is unitarily similar to the system $\Sigma_{\rm cep}^{\mathfrak{W}_-}$. The unitary similarity transformation is the inverse of the input map $\mathfrak{B}_\Sigma$ of Σ.*

Proof. By Lemma 6.15, the input map $\mathfrak{B}_\Sigma$ is unitary, and by Theorem 9.5, $\mathfrak{B}_\Sigma^{-1}$ intertwines Σ and $\Sigma_{\rm cep}^{\mathfrak{W}_-^\Sigma}$. □

Definition 9.7. We call the system $\Sigma_{\rm cep}^{\mathfrak{W}_-}$ the *canonical model* of a passive controllable energy preserving s/s system with past behavior $\mathfrak{W}_-$.

Corollary 9.8. *Any two controllable and energy preserving realizations of a given passive past behavior $\mathfrak{W}_-$ are unitarily similar to each other.*

Proof. This follows from Theorem 9.6. □

Corollary 9.9. *A passive s/s system Σ is controllable and energy preserving if and only if its input map $\mathfrak{B}_\Sigma$ is unitary.*

Proof. This follows from Lemma 6.15 and Theorem 9.5. □

Theorem 9.10. *The operator $\Gamma_{\mathfrak{W}}$ intertwines the two s/s systems $\Sigma_{\rm cep}^{\mathfrak{W}_-}$ and $\Sigma_{\rm oce}^{\mathfrak{W}_+}$.*

Proof. This follows from Theorem 8.4 and also from Theorem 9.5. □

10 The Canonical Simple Conservative Model.

We finally develop a canonical model for a conservative simple state/signal system with a given passive full behavior $\mathfrak{W}$ (see Definition 6.20 for the notion of a simplicity of a conservative system).

Theorem 10.1. *Let $\mathfrak{W}$ be a passive full behavior on the Kreĭn space $\mathcal{W}$, and let $\mathfrak{W}_- = \pi_-\mathfrak{W}$ and $\mathfrak{W}_+ = \mathfrak{W} \cap K_+^2(\mathcal{W})$ be the corresponding passive past and future behaviors. Let $\mathcal{D}(\mathfrak{W})$ be the range space of the operator $A_{\mathfrak{W}}^{1/2}$, where $A_{\mathfrak{W}}$ is the nonnegative self-adjoint operator on $\mathcal{H}_+ \oplus \mathcal{H}_-$ defined by* (5.18)*, and define $\mathcal{L}(\mathfrak{W})$ by* (5.20)*.*

(i) *Define*

$$\mathcal{T} := \left\{ \begin{bmatrix} x \\ w \end{bmatrix} \in \begin{bmatrix} BUC(\mathbb{R};\mathcal{X}) \\ \mathcal{L}(\mathfrak{W}) \end{bmatrix} \;\middle|\; x(t) = Q\tau^t w,\ t \in \mathbb{R} \right\}. \qquad (10.1)$$

Then $\mathcal{T}$ is the set of all stable full trajectories of a simple conservative s/s system $\Sigma_{\rm sc}^{\mathfrak{W}}$ with state space $\mathcal{D}(\mathfrak{W})$ whose full behavior is equal to $\mathfrak{W}$.

(ii) *The input map of $\Sigma_{\rm sc}^{\mathfrak{W}}$ is $\mathfrak{B}_{\Sigma_{\rm sc}^{\mathfrak{W}}} = \begin{bmatrix} \Gamma_{\mathfrak{W}} \\ 1_{\mathcal{H}_-} \end{bmatrix}$ with $(\mathfrak{B}_\Sigma^{\mathfrak{W}})^* = \Pi_-|_{\mathcal{D}(\mathfrak{W})}$, the output map of $\Sigma_{\rm sc}^{\mathfrak{W}}$ is $\mathfrak{C}_{\Sigma_{\rm sc}^{\mathfrak{W}}} = \Pi_+|_{\mathcal{D}(\mathfrak{W})}$ with $\mathfrak{C}^*_{\Sigma_{\rm sc}^{\mathfrak{W}}} = \begin{bmatrix} 1_{\mathcal{H}_+} \\ \Gamma_{\mathfrak{W}}^* \end{bmatrix}$.*

Proof. We define the Kreĭn spaces $\mathfrak{K}_{0,t}$ and $\mathfrak{L}_{0,\infty}$ as in the paragraph before Theorem 3.5 with $\mathcal{X}$ replaced by $\mathcal{D}(\mathfrak{W})$, and the subspaces $\mathcal{T}_{0,t}$ and $\mathcal{S}_{0,\infty}$ by (3.5), with $\mathcal{T}_+ := \pi_+\mathcal{T}$, with $\mathcal{T}$ defined by (10.1).

Step 1: $\mathcal{T}_{0,t}$ is a Lagrangian subspace of $\mathfrak{K}_{0,t}$. By (5.26), $\mathcal{T}_{0,t}$ is a neutral subspace of $\mathfrak{K}_{0,t}$. To prove that $\mathcal{T}_{0,t}$ is a Lagrangian we shall use Lemma 2.2 with $\mathcal{Y} = \mathcal{X} = \mathcal{D}(\mathfrak{W})$. Clearly condition (a) in that lemma holds because of the definition of $\mathcal{L}(\mathfrak{W})$, and (c) holds because of Lemma 5.14. The set described in condition (b) is equal to the section $\mathfrak{W}_{[0,t]} = \pi_{[0,t]}\mathfrak{W}_+$, which according to Lemma 3.20 is maximal nonnegative, and and the set described in condition (d) is equal to $\tau_+^{*t}\pi_{[-t,0]}\mathfrak{W}_-$, which according to Lemma 3.20 and Remark 4.2 is maximal nonpositive.

Step 2: $\mathcal{S}_{0,\infty}$ is a maximal nonnegative subspace of $\mathfrak{L}_{0,\infty}$. By dropping the term $\|Q\tau^t w\|^2_{\mathcal{D}(\mathfrak{W})}$ in (5.26) and letting $t \to \infty$ we find that $\mathcal{S}_{0,\infty}$ is a nonnegative subspace of $\mathfrak{L}_{0,\infty}$.. The proof of the maximality of $\mathcal{S}_{0,\infty}$ is analogous to (but simpler than) the proof of the maximality of $\mathcal{T}_{0,t}$ given in Step 1.

Step 3: $\mathcal{T}_+ := \pi_+\mathcal{T}$ is the set of stable future trajectories of a conservative s/s system $\Sigma^{\mathfrak{W}}_{\rm sc}$. By (10.1) and Lemma 5.14, $\mathcal{T}$ is shift invariant in $\left[\begin{smallmatrix} BUC(\mathbb{R};\mathcal{X}) \\ K^2(\mathcal{W}) \end{smallmatrix}\right]$. In particular, $\mathcal{T}_+ := \pi_+\mathcal{T}$ is then left-shift invariant. By Steps 1 and 2, $\mathcal{T}_{0,t}$ is a Lagrangian subspace of $\mathfrak{K}_{0,t}$ for all $t \in \mathbb{R}^+$ and $\mathcal{S}_{0,\infty}$ is a maximal nonnegative subspace of $\mathfrak{L}_{0,\infty}$. By Theorem 3.5, $\mathcal{T}_+$ is the set of stable trajectories of a passive s/s system, and by Lemma 4.9, this system is conservative.

Step 4: $\mathcal{T}$ is the set of stable full trajectories of $\Sigma^{\mathfrak{W}}_{\rm sc}$. This follows from Step 3 and Remark 4.2.

Step 5: The behavior of $\Sigma^{\mathfrak{W}}_{\rm sc}$ is equal to $\mathfrak{W}$. If $w \in \mathfrak{W}_+$, then $Qw \in \mathcal{D}(\mathfrak{W})$, and it follows from (10.1) that $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$, where $x(t) = Q\tau^t w$, $t \in \mathbb{R}^+$, is an externally generated stable future trajectory of $\Sigma^{\mathfrak{W}}_{\rm sc}$. This implies that $\mathfrak{W}_+ \subset \mathfrak{W}_+^{\Sigma^{\mathfrak{W}}_{\rm sc}}$. Since $\mathfrak{W}_+$ is maximal nonnegative and $\mathfrak{W}_+^{\Sigma^{\mathfrak{W}}_{\rm sc}}$ is nonnegative, this implies that $\mathfrak{W}_+ = \mathfrak{W}_+^{\Sigma^{\mathfrak{W}}_{\rm sc}}$. From this follows that also $\mathfrak{W}_- = \mathfrak{W}_-^{\Sigma^{\mathfrak{W}}_{\rm sc}}$ and $\mathfrak{W} = \mathfrak{W}^{\Sigma^{\mathfrak{W}}_{\rm sc}}$.

Step 6: The input map of $\Sigma^{\mathfrak{W}}_{\rm sc}$ is $\left[\begin{smallmatrix} \Gamma_{\mathfrak{W}} \\ 1_{\mathcal{H}_-} \end{smallmatrix}\right]$. According to Lemma 6.9, the operator $\mathfrak{B}_{\Sigma^{\mathfrak{W}}_{\rm sc}}$ is the unique operator $\mathcal{H}_- \to \mathcal{D}(\mathfrak{W})$ which satisfies $\mathfrak{B}_{\Sigma^{\mathfrak{W}}_{\rm sc}} Q_-\pi_- w = x(0)$ for every $w \in \mathfrak{W}$, where x is the state component of the unique externally generated trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ whose signal part is w. Let $w \in \mathfrak{W}$. By (10.1),

$$x(0) = Qw = \begin{bmatrix} Q_+\pi_+ w \\ Q_-\pi_- w \end{bmatrix} = \begin{bmatrix} \Gamma_{\mathfrak{W}} \\ 1_{\mathcal{H}_-} \end{bmatrix} Q_-\pi_- w.$$

Thus, $\mathfrak{B}_{\Sigma^{\mathfrak{W}}_{\rm sc}} = \left[\begin{smallmatrix} \Gamma_{\mathfrak{W}} \\ 1_{\mathcal{H}_-} \end{smallmatrix}\right]$.

Step 7: The output map of Σ is $\Pi_+|_{\mathcal{D}(\mathfrak{W})}$. According to Lemma 6.2, $\mathfrak{C}_{\Sigma^{\mathfrak{W}}_{\rm sc}}$ is the operator which maps $x_0 \in \mathcal{D}(\mathfrak{W})$ into the equivalence class consisting of all the signal parts w of all stable future trajectories $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ of $\Sigma^{\mathfrak{W}}_{\rm sc}$ satisfying $x(0) = x_0$. Let $x_0 \in \mathcal{D}(\mathfrak{W})$, and choose some $w_0 \in \mathcal{L}(\mathfrak{W})$ such that $Qw_0 = x_0$. It follows from (10.1) that $\left[\begin{smallmatrix} x \\ w_0 \end{smallmatrix}\right]$, where $x(t) = Q\tau^t w_0$, $t \in \mathbb{R}^+$, is a stable future trajectory of $\Sigma^{\mathfrak{W}}_{\rm sc}$ satisfying $x(0) = x_0$. If $\left[\begin{smallmatrix} x_1 \\ w_1 \end{smallmatrix}\right]$ is another stable future trajectory of $\Sigma^{\mathfrak{W}}_{\rm sc}$ satisfying $x_1(0) = x(0) = x_0$, then $\left[\begin{smallmatrix} x-x_1 \\ w_0-w_1 \end{smallmatrix}\right]$ is an externally generated stable future trajectory of $\Sigma^{\mathfrak{W}}_{\rm sc}$, and hence $w_1 - w_0 \in \mathfrak{W}_+$. Thus, the equivalence class of all the signal parts w of all stable future trajectories $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ of $\Sigma_{\rm sc}^{\mathfrak{W}}$ satisfying $x(0) = x_0$ is equal to $Q_+\pi_+w_0 = \Pi_+x_0$. Consequently, $\mathfrak{C}_{\Sigma_{\rm sc}^{\mathfrak{W}}} = \Pi_+|_{\mathcal{D}(\mathfrak{W})}$.

Step 8: $\Sigma_{\rm sc}^{\mathfrak{W}}$ is simple. According to Lemma 5.10, the linear span of the ranges of $\mathfrak{B}_{\Sigma_{\rm sc}^{\mathfrak{W}}} = \begin{bmatrix} \Gamma_{\mathfrak{W}} \\ 1_{\mathcal{W}_-} \end{bmatrix}$ and $\mathfrak{C}^*_{\Sigma_{\rm sc}^{\mathfrak{W}}} = \begin{bmatrix} 1_{\mathcal{W}_+} \\ \Gamma^*_{\mathfrak{W}} \end{bmatrix}$ is dense in the state space $\mathcal{D}(\mathfrak{W})$, and hence $\Sigma_{\rm sc}^{\mathfrak{W}}$ is simple. □

Theorem 10.2. *The generating subspace of the s/s system $\Sigma_{\rm sc}^{\mathfrak{W}}$ in Theorem 10.1 is given by*

$$V_{\rm sc}^{\mathfrak{W}} = \left\{ \begin{bmatrix} Q\dot{w} \\ Qw \\ w(0) \end{bmatrix} \in \begin{bmatrix} \mathcal{D}(\mathfrak{W}) \\ \mathcal{D}(\mathfrak{W}) \\ \mathcal{W} \end{bmatrix} \;\middle|\; \begin{array}{l} w \in \mathcal{L}(\mathfrak{W}) \textit{ is locally absolutely} \\ \textit{continuous with } \dot{w} \in K^2(\mathcal{W}), \textit{ and} \\ \lim_{t\to 0} \frac{1}{t} Q(\tau^t w - w) \textit{ exists in } \mathcal{D}(\mathfrak{W}). \end{array} \right\} \tag{10.2}$$

Proof. The proof is essentially the same as the proof of Theorem 8.2 with $\mathbb{R}^+$ replaced by $\mathbb{R}$, Q_+ replaced by Q, and $\mathcal{K}(\mathfrak{W}_+)$ replaced by $\mathcal{L}(\mathfrak{W})$. For the converse direction one needs the fact that for a conservative system part (ii) of Proposition 3.7 holds in the following modified form:

(ii') For each $\begin{bmatrix} z_0 \\ x_0 \\ w_0 \end{bmatrix} \in V$ there exists a stable full classical trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ satisfying $\begin{bmatrix} \dot{x}(0) \\ x(0) \\ w(0) \end{bmatrix} = \begin{bmatrix} z_0 \\ x_0 \\ w_0 \end{bmatrix}$ with the additional property that w is locally absolutely continuous and $\left[\begin{smallmatrix} \dot{x} \\ \dot{w} \end{smallmatrix}\right]$ is a stable full trajectory of Σ. In particular,

$$V = \left\{ \begin{bmatrix} \dot{x}(0) \\ x(0) \\ w(0) \end{bmatrix} \;\middle|\; \begin{bmatrix} x \\ w \end{bmatrix} \text{ is a full classical trajectory of } \Sigma \right\}. \tag{10.3}$$

That (ii') holds for conservative systems follows from Proposition 3.7 and Remark 4.2. □

Let $\mathfrak{R}$ be the reachable subspace, $\mathfrak{U}$ the unobservable subspace, $\mathfrak{R}^\dagger$ the backward reachable subspace, and $\mathfrak{U}^\dagger$ the backward unobservable subspace of $\Sigma_{\rm sc}^{\mathfrak{W}}$. As we noticed earlier, $\mathfrak{R}^\dagger = \mathfrak{U}^\perp$ and $\mathfrak{U}^\dagger = \mathfrak{R}^\perp$. By Lemma 5.12 and Theorem 10.1,

$$\begin{aligned} \mathfrak{R} &= \operatorname{im}\left(\mathfrak{B}_{\Sigma_{\rm sc}^{\mathfrak{W}}}\right) = \operatorname{im}\left(\begin{bmatrix} \Gamma_{\mathfrak{W}} \\ 1_{\mathcal{H}_-} \end{bmatrix}\right) = \left\{ \left[\begin{smallmatrix} \Gamma_{\mathfrak{W}} x_- \\ x_- \end{smallmatrix}\right] \mid x_- \in \mathcal{H}_- \right\}, \\ \mathfrak{U}^\dagger &= \ker\left(\mathfrak{B}^*_{\Sigma_{\rm sc}^{\mathfrak{W}}}\right) = \ker\left(\Pi_-|_{\mathcal{D}(\mathfrak{W})}\right) = \{Qw \mid w \in \mathcal{L}(\mathfrak{W}) \cap K^2_+(\mathcal{W})\}, \\ \mathfrak{R}^\dagger &= \operatorname{im}\left(\mathfrak{C}^*_{\Sigma_{\rm sc}^{\mathfrak{W}}}\right) = \operatorname{im}\left(\begin{bmatrix} 1_{\mathcal{H}_+} \\ \Gamma^*_{\mathfrak{W}} \end{bmatrix}\right) = \left\{ \left[\begin{smallmatrix} x_+ \\ \Gamma^*_{\mathfrak{W}} x_+ \end{smallmatrix}\right] \mid x_+ \in \mathcal{H}_+ \right\}, \\ \mathfrak{U} &= \ker\left(\mathfrak{C}_{\Sigma_{\rm sc}^{\mathfrak{W}}}\right) = \ker\left(\Pi_+|_{\mathcal{D}(\mathfrak{W})}\right) = \{Qw \mid w \in \mathcal{L}(\mathfrak{W}) \cap K^2_-(\mathcal{W})\}. \end{aligned} \tag{10.4}$$

The orthogonal projections onto these subspaces are given by

$$\begin{aligned}
P_{\mathfrak{R}} &= \mathfrak{B}_{\Sigma_{\rm sc}^{\mathfrak{W}}}\mathfrak{B}_{\Sigma_{\rm sc}^{\mathfrak{W}}}^* = \begin{bmatrix} \Gamma_{\mathfrak{W}}\Pi_-|_{\mathcal{D}(\mathfrak{W})} \\ \Pi_-|_{\mathcal{D}(\mathfrak{W})} \end{bmatrix}, \\
P_{\mathfrak{U}^\dagger} &= 1_{\mathcal{D}(\mathfrak{W})} - P_{\mathfrak{R}} = \Pi_+|_{\mathcal{D}(\mathfrak{W})} - \Gamma_{\mathfrak{W}}\Pi_-|_{\mathcal{D}(\mathfrak{W})}, \\
P_{\mathfrak{R}^\dagger} &= \mathfrak{C}_{\Sigma_{\rm sc}^{\mathfrak{W}}}^*\mathfrak{C}_{\Sigma_{\rm sc}^{\mathfrak{W}}} = \begin{bmatrix} \Pi_+|_{\mathcal{D}(\mathfrak{W})} \\ \Gamma_{\mathfrak{W}}^*\Pi_+|_{\mathcal{D}(\mathfrak{W})} \end{bmatrix}, \\
P_{\mathfrak{U}} &= 1_{\mathcal{D}(\mathfrak{W})} - P_{\mathfrak{R}^\dagger} = \Pi_-|_{\mathcal{D}(\mathfrak{W})} - \Gamma_{\mathfrak{W}}^*\Pi_+|_{\mathcal{D}(\mathfrak{W})},
\end{aligned} \tag{10.5}$$

Theorem 10.3. *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a conservative s/s system with behavior $\mathfrak{W}$, input map $\mathfrak{B}_\Sigma$, output map $\mathfrak{C}_\Sigma$, reachable subspace $\mathfrak{R} = \operatorname{im}(\mathfrak{B}_\Sigma)$, unobservable subspace $\mathfrak{U}_\Sigma = \ker(\mathfrak{C}_\Sigma)$, backward reachable subspace $\mathfrak{R}_\Sigma^\dagger = \operatorname{im}(\mathfrak{C}_\Sigma^*)$ and backward unobservable subspace $\mathfrak{U}_\Sigma^\dagger = \ker(\mathfrak{B}_\Sigma^*)$.*

(i) *The operator*

$$\mathfrak{C}_\Sigma^{\rm bil} := \begin{bmatrix} \mathfrak{C}_\Sigma \\ \mathfrak{B}_\Sigma^* \end{bmatrix} \tag{10.6}$$

is a co-isometry from $\mathcal{X}$ onto $\mathcal{D}(\mathfrak{W})$, with kernel $\mathcal{X}_0 := \ker\left(\mathfrak{C}_\Sigma^{\rm bil}\right) = \mathfrak{U} \cap \mathfrak{U}^\dagger$. Thus, Σ is simple if and only if $\mathfrak{C}_\Sigma^{\rm bil}$ is injective.

(ii) *Define $\mathfrak{B}_\Sigma^{\rm bil} := (\mathfrak{C}_\Sigma^{\rm bil})^*$. Then $\mathfrak{B}_\Sigma^{\rm bil}$ is an isometry $\mathcal{D}(\mathfrak{W}) \to \mathcal{X}$ with range $\mathcal{X}_0^\perp = \overline{\mathfrak{R} + \mathfrak{R}^\dagger}$, which is uniquely determined by the fact that*

$$\begin{bmatrix} \mathfrak{C}_\Sigma^* & \mathfrak{B}_\Sigma \end{bmatrix} = \mathfrak{B}_\Sigma^{\rm bil} A_{\mathfrak{W}}, \tag{10.7}$$

where $A_{\mathfrak{W}}$ is the operator defined in (5.18)*. In particular, $\mathfrak{B}_\Sigma^{\rm bil}$ is surjective if and only if Σ is simple.*

(iii) *A full trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ of Σ is stable if and only if $w \in K^2(\mathcal{W})$.*

(iv) *If $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is a stable full trajectory of Σ, then $w \in \mathcal{L}(\mathfrak{W})$, $Q\tau^t w = \mathfrak{C}_\Sigma^{\rm bil} x(t)$, and $P_{\mathcal{X}_0^\perp} x(t) = \mathfrak{B}_\Sigma^{\rm bil} Q\tau^t w$ for all $t \in \mathbb{R}$.*

(v) *Conversely, let $w \in \mathcal{L}(\mathfrak{W})$, and define $x(t) = \mathfrak{B}_\Sigma^{\rm bil} Q\tau^t w$, $t \in \mathbb{R}$. Then $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is a stable full trajectory of Σ.*

(vi) *The state component x of a stable full trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ of Σ is determined uniquely by the signal component w if and only if Σ is simple.*

Proof. The proof of this theorem is essentially the same as the proof of parts 1)–6) of [AS10, Theorem 4.1] (see Remark 3.17). □

Corollary 10.4. *Let $\mathfrak{W}$ be a full behavior on the Kreĭn space $\mathcal{W}$. Then the pair of functions $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is a stable full trajectory of $\Sigma_{\rm sc}^{\mathfrak{W}}$ if and only if*

$$w \in \mathcal{L}(\mathfrak{W}) \text{ and } x(t) = Q\tau^t w,\ t \in \mathbb{R}.$$

Proof. This follows from Theorem 10.3. $\square$

Theorem 10.5. *Every simple conservative s/s system Σ with full behavior $\mathfrak{W}$ is unitarily similar to the system $\Sigma_{\rm sc}^{\mathfrak{W}}$. The unitary similarity transformation is the map $\mathfrak{C}_\Sigma^{\rm bil}$ defined in* (10.6).

Proof. Let $\left[\begin{smallmatrix} x_+ \\ w_+ \end{smallmatrix}\right]$ be a stable future trajectory of Σ. By Lemma 3.4 and Remark 4.2, this trajectory can be extended to a stable full trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ of Σ. For each $t \in \mathbb{R}$, define $x_1(t) = Q\tau^t w$. Then $\left[\begin{smallmatrix} x_1 \\ w \end{smallmatrix}\right]$ is a stable full trajectory of $\Sigma_{\rm sc}^{\mathfrak{W}^\Sigma}$, and by Theorem 10.3(iv), $x_1(t) = \mathfrak{C}_\Sigma^{\rm bil} x(t)$ for all $t \in \mathbb{R}$, and hence, in particular, for all $t \in \mathbb{R}^+$.

Conversely, let $\left[\begin{smallmatrix} x_{1+} \\ w_+ \end{smallmatrix}\right]$ be a stable future trajectory of $\Sigma_{\rm sc}^{\mathfrak{W}^\Sigma}$. This trajectory can be extended to a stable full trajectory $\left[\begin{smallmatrix} x_1 \\ w \end{smallmatrix}\right]$, after which it satisfies $x_1(t) = Q\tau^t w$ for all $t \in \mathbb{R}$. For each $t \in \mathbb{R}$ we can define $x(t) = \mathfrak{B}_\Sigma^{\rm bil}\tau^t Q w$. Then by Theorem 10.3(v), $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is a stable full trajectory of Σ and $x(t) = \mathfrak{B}_\Sigma^{\rm bil} Q\tau^t w$ for all $t \in \mathbb{R}$, and hence, in particular, for all $t \in \mathbb{R}^+$.

Since $\mathfrak{C}_\Sigma^{\rm bil}$ is unitary we conclude that Σ is unitarily similar to $\Sigma_{\rm sc}^{\mathfrak{W}^\Sigma}$ with similarity operator $\mathfrak{S}_\Sigma^{\rm bil}$. $\square$

Definition 10.6. We call the system $\Sigma_{\rm sc}^{\mathfrak{W}}$ the *canonical model* of a simple conservative s/s system with full behavior $\mathfrak{W}$.

Corollary 10.7. *Any two simple conservative realizations of a given passive full behavior $\mathfrak{W}$ are unitarily similar to each other.*

Proof. This follows from Theorem 10.5. $\square$

Definition 10.8. We call the operators $\mathfrak{C}_\Sigma^{\rm bil}$ and $\mathfrak{B}_\Sigma^{\rm bil}$ defined in Theorem 10.3 the *bilateral output and input maps*, respectively, of the conservative s/s system Σ.

As we shall show below, the two models in Sections 8 and 9 can be obtained from $\Sigma_{\rm sc}^{\mathfrak{W}}$ by first performing an orthogonal compression, and then applying a unitary similarity transform.

By Theorem 8.4, the output map $\mathfrak{C}_{\Sigma_{\rm sc}^{\mathfrak{W}}}$ intertwines $\Sigma_{\rm sc}^{\mathfrak{W}}$ and the co-energy preserving system $\Sigma_{\rm oce}^{\mathfrak{W}_+}$. Since $\Sigma_{\rm sc}^{\mathfrak{W}}$ is conservative, it follows from Lemma 6.19 that $\mathfrak{C}_{\Sigma_{\rm sc}^{\mathfrak{W}}} = \Pi_+|_{\mathcal{D}(\mathfrak{W})}$ is a co-isometry, and the restriction of $\mathfrak{C}_{\Sigma_{\rm sc}^{\mathfrak{W}}}$ to $\mathfrak{R}^\dagger$ is a unitary map of $\mathfrak{R}^\dagger$ onto $\mathcal{H}(\mathfrak{W}_+)$. Clearly $\Sigma_{\rm oce}^{\mathfrak{W}_+}$ is unitarily similar to the system $\Sigma_{\rm sc}^o := (V_{\rm sc}^o; \mathfrak{R}^\dagger, \mathcal{W})$ that we get by applying $\mathfrak{C}_{\Sigma_{\rm sc}^{\mathfrak{W}}}^* = \left[\begin{smallmatrix} 1_{\mathcal{H}(\mathfrak{W}_+)} \\ \Gamma_{\mathfrak{W}}^* \end{smallmatrix}\right]$ to the state of the state of $\Sigma_{\rm oce}^{\mathfrak{W}_+}$. The set of all future stable trajectories of $\Sigma_{\rm sc}^o$ is given by

$$\Big\{\left[\begin{smallmatrix} x \\ w_+ \end{smallmatrix}\right] \in \left[\begin{smallmatrix} C(\mathbb{R}^+;\mathfrak{R}^\dagger) \\ \mathcal{K}(\mathfrak{W}_+) \end{smallmatrix}\right] \;\Big|\; x(t) = \left[\begin{smallmatrix} 1_{\mathcal{H}(\mathfrak{W}_+)} \\ \Gamma_{\mathfrak{W}}^* \end{smallmatrix}\right] Q_+\tau_+^t w_+,\ t \in \mathbb{R}^+\Big\}. \tag{10.8}$$

Since $w_+ \in \mathcal{K}(\mathfrak{W}_+)$ if and only if it can be written in the form $w_+ = \pi_+ w$ for some $w \in \mathcal{L}(\mathfrak{W})$, we can replace the parameter w_+ by $\pi_+ w$ with $w \in \mathcal{L}(\mathfrak{W})$, after which (10.8) can be written in the equivalent form

$$\left\{ \left[\begin{smallmatrix} x \\ w_+ \end{smallmatrix}\right] \in \left[\begin{smallmatrix} C(\mathbb{R}^+;\mathfrak{R}^\dagger) \\ \mathcal{K}(\mathfrak{W}_+) \end{smallmatrix}\right] \;\middle|\; x(t) = P_{\mathfrak{R}^\dagger} Q \tau^t w,\ w_+ = \pi_+ w,\ w \in \mathcal{L}(\mathfrak{W}),\ t \in \mathbb{R}^+ \right\}. \tag{10.9}$$

Comparing this to (10.1) we find that $\Sigma_{\rm sc}^o$ is the orthogonal outgoing compression of $\Sigma_{\rm sc}^{\mathfrak{W}}$ onto the backward reachable subspace $\mathfrak{R}^\dagger$ of $\Sigma_{\rm sc}^{\mathfrak{W}}$.

By Theorem 9.5, the input map $\mathfrak{B}_{\Sigma_{\rm sc}^{\mathfrak{W}}}$ intertwines the energy preserving system $\Sigma_{\rm cep}^{\mathfrak{W}_-}$ with $\Sigma_{\rm sc}^{\mathfrak{W}}$. Since $\Sigma_{\rm sc}^{\mathfrak{W}}$ is conservative, it follows from Lemma 6.15 that $\mathfrak{B}_{\Sigma_{\rm sc}^{\mathfrak{W}}} = \left[\begin{smallmatrix} \Gamma_{\mathfrak{W}} \\ 1_{\mathcal{H}(\mathfrak{W}_-)} \end{smallmatrix}\right]$ is an isometry whose range is the reachable subspace $\mathfrak{R}$ of $\Sigma_{\rm sc}^{\mathfrak{W}}$. Clearly $\Sigma_{\rm cep}^{\mathfrak{W}_-}$ is unitarily similar to the system $\Sigma_{\rm sc}^c := (V_{\rm sc}^c; \mathfrak{R}, \mathcal{W})$ that we get by applying $\mathfrak{B}_{\Sigma_{\rm sc}^{\mathfrak{W}}}$ to the state of $\Sigma_{\rm cep}^{\mathfrak{W}_-}$. The set of all future stable trajectories of $\Sigma_{\rm sc}^c$ is given by

$$\left\{ \left[\begin{smallmatrix} x \\ w_+ \end{smallmatrix}\right] \in \left[\begin{smallmatrix} C(\mathbb{R}^+;\mathfrak{R}) \\ \mathcal{K}(\mathfrak{W}_+) \end{smallmatrix}\right] \;\middle|\; x = Q\tau^t w,\ w_+ = \pi_+ w,\ w \in Q^{-1}\mathfrak{R} \right\}. \tag{10.10}$$

Comparing this to (10.1) we find that $\Sigma_{\rm sc}^c$ is an orthogonal incoming compression of $\Sigma_{\rm sc}^{\mathfrak{W}}$ onto the reachable subspace $\mathfrak{R}$ of $\Sigma_{\rm sc}^{\mathfrak{W}}$.

Theorem 10.9. *Let $\mathfrak{W}_-$ be a passive past behavior on the Kreĭn space $\mathcal{W}$. Then the generating subspace $V_{\rm cep}^{\mathfrak{W}_-}$ of the canonical model $\Sigma_{\rm cep}^{\mathfrak{W}_-}$ in Theorem 9.1 is given by*

$$V_{\rm cep}^{\mathfrak{W}_-} = \left\{ \left[\begin{smallmatrix} Q_-\pi_-\dot w \\ Q_-\pi_- w \\ w(0) \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{H}(\mathfrak{W}_-) \\ \mathcal{H}(\mathfrak{W}_-) \\ \mathcal{W} \end{smallmatrix}\right] \;\middle|\; \begin{array}{l} w \in \operatorname{im}\left(\left[\begin{smallmatrix} \Gamma_{\mathfrak{W}} \\ 1_{\mathcal{H}_-} \end{smallmatrix}\right]\right) \text{ is locally absolutely} \\ \text{continuous with } \dot w \in K^2(\mathcal{W}), \text{ and} \\ \lim_{t\to 0+} \frac{1}{t} Q_-\pi_-(\tau^t w - w) \text{ exists in } \mathcal{H}(\mathfrak{W}_-). \end{array} \right\} \tag{10.11}$$

Proof. As we established above, $\Sigma_{\rm cep}^{\mathfrak{W}_-}$ is unitarily similar to the orthogonal incoming compression $\Sigma_{\rm sc}^c$ of $\Sigma_{\rm sc}^{\mathfrak{W}}$ onto its reachable subspace $\mathfrak{R}$. That subspace is strongly invariant in the sense that if $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is a future trajectory of $\Sigma_{\rm sc}^{\mathfrak{W}}$ satisfying $x(0) \in \mathfrak{R}$, then $x(t) \in \mathfrak{R}$ for all $t \geq 0$ (see Lemma 6.14). In particular, $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is a smooth future trajectory of $\Sigma_{\rm sc}^c$ if and only if $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is a smooth future trajectory of $\Sigma_{\rm sc}^{\mathfrak{W}}$ and $x(0) \in \mathfrak{R}$. This, combined with Proposition 3.7 and Theorem 10.2 implies that the generating subspace $V_{\rm sc}^c$ of $\Sigma_{\rm sc}^c$ is given by

$$V_{\rm sc}^c = V_{\rm sc}^{\mathfrak{W}} \cap \begin{bmatrix} \mathcal{D}(\mathfrak{W}) \\ \mathfrak{R} \\ \mathcal{W} \end{bmatrix} = \left\{ \left[\begin{smallmatrix} Q\dot w \\ Qw \\ w(0) \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathfrak{R} \\ \mathfrak{R} \\ \mathcal{W} \end{smallmatrix}\right] \;\middle|\; \begin{array}{l} w \in Q^{-1}\mathfrak{R} \text{ is locally absolutely} \\ \text{continuous with } \dot w \in K^2(\mathcal{W}), \text{ and} \\ \lim_{t\to 0+} \frac{1}{t} Q(\tau^t w - w) \text{ exists in } \mathfrak{R}. \end{array} \right\} \tag{10.12}$$

From here we get the generating subspace $V_{\mathrm{cep}}^{\mathfrak{W}_-}$ of $\Sigma_{\mathrm{cep}}^{\mathfrak{W}_-}$ by applying the unitary operator $\left(\mathfrak{B}_{\mathrm{sc}}^{\mathfrak{W}}\right)^* = \Pi_-$ to the two state components. This leads to formula (10.11). □

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