A Complete Model of a Finite-Dimensional Impedance-Passive System

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Abstract

We extend the classes of standard discrete- and continuous-time input/state/output matrix systems by adding reverse internal and/or external channels. The reverse internal channel permits the impulse response to contain a differentiating part, and the reverse external channel allows us to include inputs which are forced to be zero and outputs which are undetermined. The purpose of this extension is obtaining a class of state-space matrix systems that can be used to realise all right-coprime positive-real rational relations – in particular non-proper positive-real rational transfer functions can be realised. We generalise the notions of impedance and scattering passivity to extended systems. When we restrict our attention to passive systems, the new class of extended impedance-passive systems is closed under the operations of interchanging the input and the output, as well as frequency inversion and duality. We generalise two system Cayley transformations to extended systems. The first transformation that we consider is the internal Cayley transformation, which maps an impedance- or scattering-passive continuous-time system into a discrete-time approximation of the original system that is again impedance passive or scattering passive, respectively. The second transformation is the external Cayley transformation that maps a contiuous- or discrete-time impedance-passive system into a scattering-passive system with the same time axis. In our extended setting, the two Cayley transformations become bijections between the respective classes of extended passive systems.

Keywords: Energy-based modelling, passive system, realisation, non-proper transfer function, Cayley transform, linear time-invariant system, positive-real function, bounded-real function, rational relation.

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1 Introduction

A typical example of a finite-dimensional continuous-time impedance-passive system (or shorter just continuous-time impedance system) is an electrical circuit made up from the standard non-active components resistors, capacitors and inductors. These are connected through a finite number of external terminals, where we regard the voltages over these terminals as *inputs* and the corresponding currents as *outputs*, or the other way around. If this system is proper, i.e., if the outputs depend continuously on the inputs, then, after a suitable normalisation, the system can be modelled by a standard i/s/o (input/state/output) system of the type:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t). \end{cases} \tag{1}$$

Normalisation in this case means that we divide each voltage by $\sqrt{R}$ and multiply each current by $\sqrt{R}$ where R is a fixed resistance, in order to give them both the physical dimension $\sqrt{W}$. Please see [B] for details on this.

In (1), we let $x(t)$, $u(t)$ and $y(t)$ denote finite-dimensional vectors (with real or complex entries), $x(t)$ takes its values in the *state space* X, and both the input $u(t)$ and the output $y(t)$ take their values in the common *input/output space* U.

However, not all passive electrical circuits with voltage inputs and current outputs are proper. Because of the internal structure of the circuit, it may be that the output is proportional to a derivative of the input. It is also possible that some of the inputs are forced to be zero (e.g. a short-circuited terminal with voltage input) and some of the outputs may be undetermined (e.g. a short-circuited terminal with current output).

Especially the case of a differentiating circuit is important in practise. The ignition coil of an old-fashioned car engine and a PID (Proportional-Integral-Differential) controller are examples of this. For more information on PID controllers, see Example 3.8 later in this paper or [Å]. It would be useful to have a class of state systems that can be used to represent arbitrary (proper or non-proper) impedance-passive systems. The main purpose of this work is to present such a class of extended state-space systems.

The class we present contains exactly the necessary ingredients, and the systems resemble standard systems so much that they are relatively simple to understand and to use in computations. The key feature is the inclusion of certain internal and external *reverse channels* in the system, and we always recover the class of standard systems by simply removing these reverse channels. The idea of reverse channels was utilised in the input/output setting already in [B].

The complete model that we use is of the following type. We split the state space X into a direct part X_1 and a reverse part X_0, so that the full state is given by $x = \left[\begin{smallmatrix} x_0 \\ x_1 \end{smallmatrix}\right]$ where $x_0 \in X_0$ and $x_1 \in X_1$. Likewise, we split the input/output space U into a direct part U_1 and a reverse part U_0, so that the full input is $\left[\begin{smallmatrix} u_1 \\ u_0 \end{smallmatrix}\right]$ and the full output is $\left[\begin{smallmatrix} y_1 \\ y_0 \end{smallmatrix}\right]$, where u_1, $y_1 \in U_1$ and u_0, $y_0 \in U_0$. We do allow the direct or reverse channels to be absent. Our general continuous-time model is of the type:

$$\left[\begin{array}{c} x_0(t) \\ \dot{x}_1(t) \\ \hline y_1(t) \\ u_0(t) \end{array}\right] = \left[\begin{array}{cc|cc} 0 & 0 & B_0 & 0 \\ 0 & A_1 & B_1 & 0 \\ \hline C_0 & C_1 & D_1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{c} \dot{x}_0(t) \\ x_1(t) \\ \hline u_1(t) \\ y_0(t) \end{array}\right]. \tag{2}$$

Note that the reverse parts of the state, input, and output appear on the wrong side of the equation compared to a standard system, and that there is no direct coupling between the reverse internal and external channels. A more detailed discussion of the particular choice of the system (2) is given in Section 2.

The reverse external channel in (2) has a trivial nature in the sense that it is completely decoupled from the rest of the system. If B_0 is not surjective and C_0 is not injective then a part of the reverse internal channel can also be trivial in the sense that the corresponding row and column in the system matrix (the matrix on the right-hand side of (2)) is identically zero. After the removal of these trivial channels the situation becomes the following. Either the remaining system is of standard type, in which case we have nothing more to say, or a part of the output depends on some derivative of the input. The latter case means that the transfer function of the system has a pole at infinity. To represent this pole at infinity we need a reverse internal channel, whose dimension is determined by the rank of the pole at infinity.

It is also possible to consider the dual situation, where one uses a nontrivial reverse external channel with dimension given by the rank of the pole at infinity and no reverse internal channel, but in this work we study only the system (2).

A central part of the article is a detailed study of the connection between the continuous-time system (2) described above, and a particular discrete-time approximation. We study this relationship, the internal Cayley transformation, in Section 5.

In the discrete-time setting no reverse internal channel is needed and the reverse external channel is trivial. Therefore our discrete-time models are throughout a trivial modification of standard systems, described by

$$\left[\begin{array}{c} \mathbf{x}(n+1) \\ \hline \mathbf{y}_1(n) \\ \mathbf{u}_0(n) \end{array}\right] = \left[\begin{array}{c|cc} \mathbf{A} & \mathbf{B}_1 & 0 \\ \hline \mathbf{C}_1 & \mathbf{D}_1 & 0 \\ 0 & 0 & 0 \end{array}\right] \left[\begin{array}{c} \mathbf{x}(n) \\ \hline \mathbf{u}_1(n) \\ \mathbf{y}_0(n) \end{array}\right]. \tag{3}$$

A trivial reverse channel is needed in the discrete-time impedance-passive case, if we want the external Cayley transformation to be a bijection. The reverse channel is absent in the discrete-time scattering setting. These claims are justified in Section 3.

The paper is organised as follows. Fundamental theory of solutions, the transfer function and realisation performance of the systems (2) and (3) are given in Section 2. Energy properties are discussed in Section 3. The internal Cayley transformation is presented in Section 5 and the external Cayley transformation is studied in Section 6.

2 Basic Theory of Extended I/S/O Systems

We begin with a treatment of the basic theory of the continuous-time system (2), presented in Section 1, ending the section with some short comments on the discrete-time system (3).

Letting $\mathbb{C}$ denote the set of complex numbers, we define $\mathbb{C}_+ = \{s \in \mathbb{C} : \operatorname{Re} s > 0\}$ and $\mathbb{D}_+ = \{z \in \mathbb{C} : |z| > 1\}$. Let $U = \mathbb{C}^{\dim U}$ and $X = \mathbb{C}^{\dim X}$ both be finite-dimensional. We equip the vector spaces with the Cartesian inner product $\langle v_1, v_2\rangle_V = v_2^* v_1$ and the induced norm $\|v\|_V = \sqrt{\langle v, v\rangle_V}$. We moreover denote the space $\mathbb{C}^{\dim X\times\dim U}$ of matrices mapping U into X by $\mathcal{L}(U; X)$ and abbreviate $\mathcal{L}(U) = \mathcal{L}(U; U)$.

Definition 2.1 *An* extended linear continuous-time time-invariant input/state/output system Σ, *or shorter* an extended continuous-time i/s/o system Σ, *with* state space X *and* input/output space U *consists of a* splitting *of X into a* direct part X_1 *and a* reverse part X_0, *a splitting of U into a direct part U_1 and U_0, and a set of equations of the type (we denote $\dot{x} = \frac{\mathrm{d}}{\mathrm{d}t} x$)*

$$\left[\begin{array}{c} x_0(t) \\ \dot{x}_1(t) \\ \hline y_1(t) \\ u_0(t) \end{array}\right] = \left[\begin{array}{cc|cc} 0 & 0 & B_0 & 0 \\ 0 & A_1 & B_1 & 0 \\ \hline C_0 & C_1 & D_1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{c} \dot{x}_0(t) \\ x_1(t) \\ \hline u_1(t) \\ y_0(t) \end{array}\right], \tag{4}$$

with $x_0(t) \in X_0$, $x_1(t) \in X_1$, $u_1(t), y_1(t) \in U_1$ and $u_0(t), y_0(t) \in U_0$ for all $t \geq 0$.

We call $x = \left[\begin{smallmatrix} x_0 \\ x_1 \end{smallmatrix}\right]$ the state, *$u = \left[\begin{smallmatrix} u_1 \\ u_0 \end{smallmatrix}\right]$ the (formal)* input *and $y = \left[\begin{smallmatrix} y_1 \\ y_0 \end{smallmatrix}\right]$ the (formal)* output. *Together they form the* trajectory (u, x, y) *on* Σ.

The matrix on the right-hand side of (4) *is called the* system matrix. *As indicated in* (4), *we sometimes write this matrix as $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$, where $A = \left[\begin{smallmatrix} 0 & 0 \\ 0 & A_1 \end{smallmatrix}\right]$ is the* main matrix, *$B = \left[\begin{smallmatrix} B_0 & 0 \\ B_1 & 0 \end{smallmatrix}\right]$ is the* control matrix, *$C = \left[\begin{smallmatrix} C_0 & C_1 \\ 0 & 0 \end{smallmatrix}\right]$ is the* observation matrix, *and $D = \left[\begin{smallmatrix} D_1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ is the* feed-through matrix.

The dual system Σ^* *of* (4) *is (with $x_i^d(t) \in X_i$ and $u_i^d(t), y_i^d(t) \in U_i$ for all $t \geq 0$)*

$$\left[\begin{array}{c} x_0^d(t) \\ \dot{x}_1^d(t) \\ \hline u_1^d(t) \\ y_0^d(t) \end{array}\right] = \left[\begin{array}{cc|cc} 0 & 0 & C_0^* & 0 \\ 0 & A_1^* & C_1^* & 0 \\ \hline B_0^* & B_1^* & D_1^* & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{c} \dot{x}_0^d(t) \\ x_1^d(t) \\ \hline y_1^d(t) \\ u_0^d(t) \end{array}\right]. \tag{5}$$

We say that $\left[\begin{smallmatrix} u_1 \\ u_0 \end{smallmatrix}\right]$ is the *formal* input of (4), since we interprete it as an input because of its physical dimension, but mathematically u_0 plays the role of an output, while y_0 plays the role of an input. Instead of talking about inputs and outputs, we could be talking about currents and voltages in the network theory setting in [B], or more generally, about efforts and flows, as in the more recent theory of port-Hamiltonian systems. See. e.g. [Sc, Section 4.4].

Remark 2.2 The adjoint system matrix in (5) yields a system matrix of the same type as the system matrix in (4). Therefore the class of extended continuous-time systems (4) is closed under the operation of taking adjoints.

We leave the easy proof of the following lemma to the reader. Note that we here treat the reverse part of the formal input as an output, and the reverse part of the formal output as an input.

Lemma 2.3 *For each $u_1 \in C(\mathbb{R}_+; U_1)$ satisfying $B_0 u_1 \in C^1(\mathbb{R}_+; U)$ and each $y_0 : \mathbb{R}_+ \to U_0$, the system (4) has a unique state trajectory $x \in C^1(\mathbb{R}_+; X)$, given an arbitrary initial value $x_1(0) \in X_1$. The internal and external signals are for all $t \in \mathbb{R}_+$ related by*

$$\begin{aligned} x_0(t) &= B_0 u_1(t), \\ x_1(t) &= e^{A_1 t} x_1(0) + \int_0^t e^{A_1(t-s)} B_1 u_1(s)\,\mathrm{d}s, \\ y_1(t) &= C_0 \dot{x}_0(t) + C_1 x_1(t) + D_1 u_1(t), \\ u_0(t) &= 0. \end{aligned} \tag{6}$$

Formally Laplace transforming (4) we obtain

$$\begin{aligned} \widehat{x}_0(s) &= B_0 \widehat{u}_1(s), \\ \widehat{x}_1(s) &= (s - A_1)^{-1} x_1(0) + (s - A_1)^{-1} B_1 \widehat{u}_1(s), \\ \widehat{y}_1(s) &= C_0[s\widehat{x}_0(s) - x_0(0)] + C_1 \widehat{x}_1(s) + D_1 \widehat{u}_1(s), \\ \widehat{u}_0(s) &= 0. \end{aligned} \tag{7}$$

In particular, taking $x_1(0) = 0$ and $x_0(0) = B_0 u_1(0) = 0$, we find the frequency domain input/output relationship

$$\begin{bmatrix} \widehat{y}_1(s) \\ \widehat{u}_0(s) \end{bmatrix} = \begin{bmatrix} \widehat{\mathfrak{D}}_1(s) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \widehat{u}_1(s) \\ \widehat{y}_0(s) \end{bmatrix}, \tag{8}$$

where

$$\widehat{\mathfrak{D}}_1(s) = sC_0 B_0 + C_1 (s - A_1)^{-1} B_1 + D_1. \tag{9}$$

We call $\widehat{\mathfrak{D}} = \left[\begin{smallmatrix} \widehat{\mathfrak{D}}_1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ the *transfer function* of the extended continuous-time i/s/o system Σ, given by (4), and we call Σ a *realisation* of $\widehat{\mathfrak{D}}$.

Definition 2.4 *The matrix-valued function $F : \mathbb{C} \supset \mathrm{dom}\,(F) \to \mathcal{L}(U)$, $\dim U < \infty$, is* rational *if every element of F is described by a rational scalar function.*

The function F is positive real *on $\mathbb{C}_+ \cap \mathrm{dom}\,(F)$ if $\langle u, (F(s) + F(s)^*)u\rangle \geq 0$ for all $u \in U$ and $s \in \mathbb{C}_+ \cap \mathrm{dom}\,(F)$, i.e., the matrix $F(s) + F(s)^*$ is positive semi-definite for all $s \in \mathbb{C}_+ \cap \mathrm{dom}\,(F)$.*

If F is bounded on some right-half plane $\mathbb{C}_\omega = \{s \in \mathbb{C} | \mathrm{Re}\, s > \omega\}$, i.e., $\exists \omega \in \mathbb{R}, M \in \mathbb{R}_+ : s \in \mathbb{C}_\omega \Longrightarrow \|F(s)\|_{\mathcal{L}(U)} \leq M$, then F is said to be proper*. Otherwise F is* improper*.*

The transfer function $\widehat{\mathfrak{D}}$, as given in (8), is always rational, since X_1 is finite-dimensional. The *spectrum* of A_1 consists of the *finite spectrum* $\sigma(A_1) = \{s \in \mathbb{C} :$

$s - A_1$ not invertible}, plus the point at infinity if $\dim X_0 \neq 0$ and $C_0 B_0 \neq 0$. The complement of the spectrum is called the *resolvent set* of A_1. Thus $\mathrm{dom}\,(\widehat{\mathfrak{D}}) = \rho(A_1)$.

In the absence of the reverse external channel, i.e., when $\dim U_0 = 0$, the transfer function is reduced to $\widehat{\mathfrak{D}}_1$, given by (9).

Observe that the existence of the reverse internal channel permits the transfer function to have a first order pole at infinity. However, a *proper* transfer function has no pole at infinity. In order to realise a proper rational matrix-valued function we do not need any internal reverse channel, as is well-known, see e.g. [W]. Such a function can always be realised in the form

$$\widehat{\mathfrak{D}}(s) = C(s - A)^{-1} B + D \tag{10}$$

by a continuous-time standard system (1), possibly adding a trivial external reverse channel.

We now illustrate the realisation capabilities of the system (4).

Theorem 2.5 *Any rational positive-real function $F(s)$ can be realised by an extended system Σ of the type*

$$\begin{bmatrix} x_0(t) \\ \dot{x}_1(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & B_0 \\ 0 & A_1 & B_1 \\ C_0 & C_1 & D_1 \end{bmatrix} \begin{bmatrix} \dot{x}_0(t) \\ x_1(t) \\ u(t) \end{bmatrix}. \tag{11}$$

The product $C_0 B_0 = B_0^* C_0^*$ is positive semi-definite, and it can be recovered as

$$C_0 B_0 = \lim_{s \to \infty} \frac{1}{s} F(s). \tag{12}$$

Proof: Every rational scalar entry $f_{i,j}$ of F can be written in the form

$$f_{i,j}(s) = (f_p)_{i,j}(s) + \sum_{k=1}^{m_{i,j}} (q_k)_{i,j} s^k, \quad q_{m_{i,j}} \neq 0,$$

where $(f_p)_{i,j}$ is proper. By identifying the coefficients of s^k, we can then also write F in the form

$$F(s) = F_p(s) + \sum_{k=1}^{m} Q_k s^k, \quad Q_m \neq 0, \tag{13}$$

with $m = \max_{i,j} m_{i,j}$, F_p proper and $Q_k \in \mathcal{L}(U)$ for all k. We now proceed by showing that $m \leq 1$.

We have that $\lim_{s \to \infty} s^{-m} F(s) = Q_m$, since F_p is bounded on some right-half plane $\mathbb{C}_\omega$, and $\langle u, Q_m u \rangle \neq 0$ for some $u \in U$, as $Q_m \neq 0$. It is a simple exercise in complex analysis to show that if $z_n \to z \neq 0$ in $\mathbb{C}$, then also $\arg(z_n) \to \arg(z)$. Therefore

$$\begin{aligned} \arg\left(\langle u, F(s) u \rangle\right) &= \arg\left(\overline{s^m} \langle u, s^{-m} F(s) u \rangle\right) \\ &= -m \arg(s) + \arg(\langle u, s^{-m} F(s) u \rangle) \mod 2\pi \\ &\to -m \arg(s) + \arg(\langle u, Q_m u \rangle) \mod 2\pi \end{aligned} \tag{14}$$

if $s \to \infty$ along a straight half line. Regardless of the value of $\arg(\langle u, Q_m u\rangle)$, it is always possible to choose $\arg(s)$ so that the limit on the last line of (14) has absolute value greater than $\pi/2$ if $m \geq 2$. This means that the $\lim_{s\to\infty}\langle u, F(s)u\rangle$ lies in the complex open left-half plane $\mathbb{C}_-$ and Re $(\langle u, F(s), u\rangle) < 0$ for sufficiently large s, which contradicts the positive realness of F. Thus $m \leq 1$.

If Q_1 is not self-adjoint and positive semi-definite, then $\langle u, Q_1 u\rangle \notin \mathbb{R}_+$ for some $u \in U$ and $\arg(\langle u, Q_1 u\rangle) \neq 0$. In this case it is again possible to choose s in such a way that Re $(\langle u, F(s), u\rangle) < 0$.

Now obviously $Q_1 = C_0 B_0$ is given by (12) and $F_p(s) = F(s) - sC_0B_0$ is proper. Let the standard system $\Sigma_1(A_1, B_1, C_1, D_1)$ realise F_p. Then the system (4), where C_0B_0 is an arbitrary factorisation of Q_1, realises $F(s)$. (Such a factorisation always exists, as can easily be seen by taking e.g. $C_0 = 1$ and $B_0 = Q_1$.) ∎

Remark 2.6 An analogous computation shows that a pole anywhere on the imaginary axis must be simple and have a positive residue.

Similarly, in the discrete-time case we have no need for an inverted internal channel, when realising a positive-real rational function. This is a consequence of the fact that the point at infinity is an internal point of $\rho(\mathbf{A}_1)$. The argument is similar to the proof of Theorem 2.5.

Following Remark 2.6, we arrive at the discrete-time system $\mathbf{\Sigma}$:

$$\left[\begin{array}{c}\mathbf{x}(n+1)\\ \hline \mathbf{y}_1(n)\\ \mathbf{u}_0(n)\end{array}\right] = \left[\begin{array}{c|cc}\mathbf{A} & \mathbf{B}_1 & 0\\ \hline \mathbf{C}_1 & \mathbf{D}_1 & 0\\ 0 & 0 & 0\end{array}\right]\left[\begin{array}{c}\mathbf{x}(n)\\ \hline \mathbf{u}_1(n)\\ \mathbf{y}_0(n)\end{array}\right], \tag{15}$$

with $\mathbf{x}(n) \in X$ and $\mathbf{u}_i(n), \mathbf{y}_i(n) \in U_i$ for all $n \geq 0$.

Arguments analogous, but simpler, to the continuous-time case given above can be made for discrete-time systems as well. The standard theory of discrete-time i/s/o systems applies, with the slight extension that we allow a short-circuited (formal) input u_0 and an arbitrary (formal) output y_0.

The adjoint of this system is defined as ($\forall n \geq 0:\ \mathbf{u}_i^d(n), \mathbf{y}_i^d(n) \in U_i, \mathbf{x}^d(n) \in X$):

$$\left[\begin{array}{c}\mathbf{x}^d(n+1)\\ \hline \mathbf{u}_1^d(n)\\ \mathbf{y}_0^d(n)\end{array}\right] = \left[\begin{array}{c|cc}\mathbf{A}^* & \mathbf{C}_1^* & 0\\ \hline \mathbf{B}_1^* & \mathbf{D}_1^* & 0\\ 0 & 0 & 0\end{array}\right]\left[\begin{array}{c}\mathbf{x}^d(n)\\ \hline \mathbf{y}_1^d(n)\\ \mathbf{u}_0^d(n)\end{array}\right],$$

Thus also the class of systems (15) is closed under the operation of taking adjoints.

The system (15) is written in solved form as

$$\begin{aligned}
&\mathbf{x}(n) = \mathbf{A}^n\mathbf{x}(0) + \sum_{k=0}^{n-1}\mathbf{A}^{n-1-k}\mathbf{B}_1\mathbf{u}_1(k),\\
&\mathbf{y}_1(n) = \mathbf{C}_1\mathbf{A}^n\mathbf{x}(0) + \mathbf{C}_1\sum_{k=0}^{n-1}\mathbf{A}^{n-1-k}\mathbf{B}_1\mathbf{u}_1(k) + \mathbf{D}_1\mathbf{u}_1(n) \quad \text{and}\\
&\mathbf{u}_0(n) = 0.
\end{aligned} \tag{16}$$

Premultiplying the equations of (15) by z^{-n} and taking the sum as $n \in \mathbb{N}_0$, we obtain that the z-transforms of the signals are related by $z\widehat{\mathbf{x}}(z) - z\mathbf{x}(0) = \mathbf{A}\widehat{\mathbf{x}}(z) + \mathbf{B}_1\widehat{\mathbf{u}_1}(z)$, $\widehat{\mathbf{y}_1}(z) = \mathbf{C}_1\widehat{\mathbf{x}}(z) + \mathbf{D}_1\widehat{\mathbf{u}_1}(z)$ and $\widehat{\mathbf{u}_0}(z) = 0$. We thus regard

$$\widehat{\mathbb{D}}(z) = \begin{bmatrix} \mathbf{C}_1(z-\mathbf{A})^{-1}\mathbf{B}_1 + \mathbf{D}_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad z \in \rho(A_1)$$

as the transfer function of $\mathbf{\Sigma}$.

Definition 2.7 *By a* rational relation R *on* $\Omega := \overline{\mathbb{C}_+} \cup \{\infty\}$ *over* $\left[\begin{smallmatrix} U \\ U \end{smallmatrix}\right]$ *we mean the range*

$$\forall s \in \Omega: \quad R(s) := \mathcal{R}\left(\begin{bmatrix} P(s) \\ Q(s) \end{bmatrix}\right) \subset \begin{bmatrix} U \\ U \end{bmatrix}, \tag{17}$$

where $P(s), Q(s) : U \to U$ *are rational matrices with no poles in* Ω*, i.e. the closed complex right-half plane* $\overline{\mathbb{C}_+}$ *or at the point* ∞*.*

The relation R *is* right coprime *on* Ω *if* $\left[\begin{smallmatrix} P(s) \\ Q(s) \end{smallmatrix}\right]$ *is injective for all* $s \in \overline{\mathbb{C}_+}$ *and also* $\lim_{s\to\infty} \left[\begin{smallmatrix} P(s) \\ Q(s) \end{smallmatrix}\right]$ *is injective. We then call* (P, Q)*, given in* (17)*, a* right-coprime factorisation *of* R*.*

Similarly, $R(s)$ *is* positive real *on* Ω *if* $\operatorname{Re}\langle u, y\rangle \geq 0$ *for all* $s \in \Omega$ *and* $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in R(s)$*.*

We remark that a rational relation R on Ω over $\left[\begin{smallmatrix} U \\ U \end{smallmatrix}\right]$ is positive real if and only if its restriction $R_+ := R|_{\mathbb{C}_+}$ to $\mathbb{C}_+$ is positive real. This is easily shown using continuity in both arguments of the inner product on U.

We finalise this section by stating that the class of systems (2) has been chosen to have the following properties:

1. Every frequency domain input/output relationship of the type

$$\begin{bmatrix} \widehat{y}(s) \\ \widehat{u}(s) \end{bmatrix} \in \widehat{\mathfrak{D}}_{rel}(s), \quad s \in \mathbb{C}_+,$$

   where $\widehat{\mathfrak{D}}_{rel}(s)$ is a right-coprime positive-real rational relation on Ω over $\left[\begin{smallmatrix} U \\ U \end{smallmatrix}\right]$, should have a realisation in this class.

2. The class should contain all standard systems (1).

3. The class should be as small as possible.

Item one is an obvious generalisation of the classical case, where $\widehat{\mathfrak{D}}_{rel}(s)$ would be the graph of some transfer *function* $\widehat{\mathfrak{D}}(s)$ evaluated at s. Item one is proven later, in the last theorem of this paper (Theorem 7.2), where we also give a converse.

Item two is trivial. Regarding item three, if the inverse internal channel is removed, then the transfer function of the system can no longer have a pole at infinity, see (9). If the trivial external channel is removed, then the input is free and the output is completely determined by the input, thus excluding both a short-circuited input and an undetermined output.

A nice property of the choice (2) is that it contains exactly what is needed in order to make all Cayley transformations bijective, when we assume that the class of scattering-passive discrete-time systems is the standard class of discrete-time systems. More about this in Remark 6.5..

One main drawback of the class (2) is that it is general enough to cover the general case only if we assume minimality of the inverted channels. That assumption may, indeed, cause unreasonable restriction when considering interconnection in a network. Taking this point of view, it would be better to allow a non-minimal inverted external channel for a more general "well-posed n-port", in the sense of [B, p. 66]. It turns out that all results of the present article have counterparts also in the more general setting. However, we do not want to sacrifice transparency for the, in a sense non-essential, generality obtained by allowing non-minimal inverted channels.

3 Passive Systems

In this section we treat discrete- and continuous-time systems that exchange energy with their environment in special ways. We call these systems impedance and scattering passive, respectively. The notion of an impedance-passive system can be found already in [B, pp. 71–73] and similar theory for infinite-dimensional systems can be found in [St1]. Our contribution here is a generalisation of the standard passivity therory in another direction, namely to extended systems.

We recall our continuous-time and discrete-time models from Section 2. They are

$$\begin{bmatrix} x_0(t) \\ \dot{x}_1(t) \\ y_1(t) \\ u_0(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & B_0 & 0 \\ 0 & A_1 & B_1 & 0 \\ C_0 & C_1 & D_1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_0(t) \\ x_1(t) \\ u_1(t) \\ y_0(t) \end{bmatrix} \tag{18}$$

and

$$\begin{bmatrix} \mathbf{x}(n+1) \\ \mathbf{y}_1(n) \\ \mathbf{u}_0(n) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B}_1 & 0 \\ \mathbf{C}_1 & \mathbf{D}_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(n) \\ \mathbf{u}_1(n) \\ \mathbf{y}_0(n) \end{bmatrix}, \tag{19}$$

respectively. Note that we throughout recover standard i/s/o systems, and thus the standard version of the energy theory, by taking $\dim X_0 = \dim U_0 = 0$. Also note that, for simplicity, we always assume that the input and output spaces coincide.

Definition 3.1 *We make the following definitions.*

1. *The extended continuous-time system Σ in* (18), *is* forward impedance passive *if for any given* $\left[\begin{smallmatrix} u_1 \\ y_0 \end{smallmatrix}\right]$, *the corresponding* $\left[\begin{smallmatrix} x_0 \\ x_1 \end{smallmatrix}\right]$ *and* $\left[\begin{smallmatrix} y_1 \\ u_0 \end{smallmatrix}\right]$ *of* (18), *as they are given in Lemma 2.3, satisfy the inequality*

$$\left\| \begin{bmatrix} x_0(t) \\ x_1(t) \end{bmatrix} \right\|^2 - \left\| \begin{bmatrix} x_0(0) \\ x_1(0) \end{bmatrix} \right\|^2 \leq 2\mathrm{Re} \int_0^t \langle u_1(s), y_1(s) \rangle \, \mathrm{d}s \tag{20}$$

for all $t \geq 0$.

The system is forward impedance conservative *if we have equality, instead of mere inequality, in* (20)*, for all* $t \geq 0$*.*

Furthermore, the system is impedance passive *if not only* Σ*, but also the dual* Σ^**, cf.* (5)*, is forward impedance passive. If* Σ *and* Σ^* *are both forward impedance conservative, then we say that* Σ *is* impedance conservative*.*

2. *The system* (18) *is* forward scattering passive *if for all* u*,* x*,* y *and* $t \geq 0$*:*

$$\left\| \begin{bmatrix} x_0(t) \\ x_1(t) \end{bmatrix} \right\|^2 - \left\| \begin{bmatrix} x_0(0) \\ x_1(0) \end{bmatrix} \right\|^2 \leq \int_0^t \left\| \begin{bmatrix} u_1(s) \\ 0 \end{bmatrix} \right\|^2 - \left\| \begin{bmatrix} y_1(s) \\ y_0(s) \end{bmatrix} \right\|^2 \mathrm{d}s. \quad (21)$$

The system is forward scattering conservative *if we have equality for all* $t \geq 0$*. The system is* scattering passive (conservative) *if both* Σ *and* Σ^* *are forward scattering passive (forward scattering conservative).*

3. *The extended discrete-time system* (19) *is* impedance passive *if for all* $\mathbf{u}$*,* $\mathbf{x}$*,* $\mathbf{y}$ *and* $n \geq 0$ *related as in* (16)*:*

$$\|\mathbf{x}(n+1)\|^2 - \|\mathbf{x}(0)\|^2 \leq 2\mathrm{Re} \sum_{k=0}^{n} \langle \mathbf{u}_1(k), \mathbf{y}_1(k) \rangle. \quad (22)$$

It is impedance conservative *if we have equality for all* $\mathbf{u}$*,* $\mathbf{x}$*,* $\mathbf{y}$ *and* $n \geq 0$*.*

4. *The system* (19) *is* scattering passive *if for all* $n \geq 0$*:*

$$\|\mathbf{x}(n+1)\|^2 - \|\mathbf{x}(0)\|^2 \leq \sum_{k=0}^{n} \left\| \begin{bmatrix} \mathbf{u}_1(k) \\ 0 \end{bmatrix} \right\|^2 - \left\| \begin{bmatrix} \mathbf{y}_1(k) \\ \mathbf{y}_0(k) \end{bmatrix} \right\|^2. \quad (23)$$

It is scattering conservative *if we have equality for all* $n \geq 0$*.*

We could define a system to be *backward passive* if its dual is forward passive, making a system *passive* if and only if it is both forward and backward passive.

Note that there is no need to define forward passivity for discrete-time systems, because (22) holds if and only if the corresponding inequality for the dual system holds. The non-equivalence in the continuous-time case arises from the inverted internal channel, which is absent in the discrete-time system.

Remark 3.2 For (forward) scattering-passive systems, i.e., where we have (21) or (23), necessarily $\dim U_0 = 0$. To see this, let $y_0(t)$ be nonzero on some interval and take all the other signals to be zero. Then $\left(0, 0, \left[\begin{smallmatrix} 0 \\ y_0 \end{smallmatrix}\right]\right)$ is a trajectory of Σ and (21) is violated. We can thus disregard the inverted external channel U_0 in the energy inequalities, since it is a zero-dimensional subspace (of the finite-dimensional input space U).

Thus, continuous-time scattering-passive systems are always of the type

$$\begin{bmatrix} x_0(t) \\ \dot{x}_1(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_1 & B_1 \\ 0 & C_1 & D \end{bmatrix} \begin{bmatrix} \dot{x}_0(t) \\ x_1(t) \\ u(t) \end{bmatrix} \quad (24)$$

and, since $x_0(t) = 0$ for all t, the corresponding energy inequality is

$$\|x_1(t)\|^2 - \|x_1(0)\|^2 \leq \int_0^t \|u(s)\|^2 - \|y(s)\|^2 \, \mathrm{d}s. \tag{25}$$

Analogously, the discrete-time scattering-passive systems have the form

$$\begin{bmatrix} \mathbf{x}(n+1) \\ \mathbf{y}(n) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{x}(n) \\ \mathbf{u}(n) \end{bmatrix}. \tag{26}$$

Instead of considering the total flow of energy over some time interval, we may study the instantaneous power exchange of a given system at any given time instant. From this kind of consideration we obtain the following theorem.

Theorem 3.3 *We have the following results regarding passivity.*

1. *The system Σ in* (18) *is forward impedance passive if and only if $B_0^* B_0 = C_0 B_0$ and the block-matrix inequality*

$$\begin{bmatrix} A_1 + A_1^* & B_1 \\ B_1^* & 0 \end{bmatrix} \leq \begin{bmatrix} 0 & C_1^* \\ C_1 & D_1 + D_1^* \end{bmatrix} \tag{27}$$

   holds on $X \oplus U_1$. In that case $B_0^* B_0 = C_0 B_0 = B_0^* C_0^*$. It is forward impedance conservative if and only if (27) *holds as an equality.*

   The system is impedance passive (conservative) if and only if, in addition to forward passivity (forward conservativity), we have $B_0^* B_0 = C_0 C_0^*$.

   The condition $B_0^* B_0 = C_0 B_0 = B_0^* C_0^* = C_0 C_0^*$ is equivalent to $C_0 = B_0^*$ and (27) *can be referred to as* impedance passivity of the standard part *of Σ.*

2. *The system* (24) *is forward scattering passive if and only if $B_0 = 0$ and the block-matrix inequality*

$$\begin{bmatrix} A_1 + A_1^* & B_1 \\ B_1^* & 0 \end{bmatrix} \leq \begin{bmatrix} -C_1^* C_1 & -C_1^* D_1 \\ -D_1^* C_1 & 1 - D_1^* D_1 \end{bmatrix} \tag{28}$$

   holds on $X_1 \oplus U_1$ (i.e., the "the standard part of Σ" is scattering passive).

   The system is forward scattering conservative if and only if (28) *holds as an equality.*

   A forward scattering-passive (forward scattering-conservative) system is scattering passive (scattering conservative) if and only if $C_0 = 0$.

   In a forward scattering-passive continuous-time system $x_0(t) = 0$ for all $t \geq 0$.

3. *The extended discrete-time system* (19) *is impedance passive if and only if the block-matrix inequality*

$$\begin{bmatrix} \mathbf{A}^* \\ \mathbf{B}_1^* \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{B}_1 \end{bmatrix} \leq \begin{bmatrix} 1 & \mathbf{C}_1^* \\ \mathbf{C}_1 & \mathbf{D}_1 + \mathbf{D}_1^* \end{bmatrix} \tag{29}$$

holds on $X \oplus U_1$*. Then, in particular* $\mathbf{A}$ *is a contraction and* $\mathbf{D}+\mathbf{D}^* \geq \mathbf{B}^*\mathbf{B} \geq 0$*. The system is impedance conservative if and only if we have equality in* (29)*. Then, in particular,* $\mathbf{A}$ *is unitary.*

4. *The system* (26) *is scattering passive if and only if its system matrix* $\Sigma := \left[\begin{smallmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{smallmatrix}\right]$ *is a contraction. Then also both* $\mathbf{A}$ *and* $\mathbf{D}$ *are contractive. The system is scattering conservative if and only if* $\mathbf{\Sigma}$ *is unitary.*

The proof of Theorem 3.3 utilises the following lemmas.

Lemma 3.4 *Let* A *and* B *be finite-dimensional vector spaces with inner product* $\langle \cdot, \cdot \rangle_A$ *and* $\langle \cdot, \cdot \rangle_B$*, respectively. Let* $P \in \mathcal{L}(B; A)$ *and let* $Q = Q^* \in \mathcal{L}(B)$*,* $R = R^* \in \mathcal{L}(A)$*, so that*

$$S = \begin{bmatrix} 0 & P \\ P^* & Q \end{bmatrix} \qquad \text{and} \qquad S' = \begin{bmatrix} R & P \\ P^* & 0 \end{bmatrix}$$

are self-adjoint. Then $S \leq 0$ *if and only if* $P = 0$ *and* $Q \leq 0$*.*

Analogously $S' \leq 0$ *if and only if* $P = 0$ *and* $R \leq 0$*.*

Proof: If $P = 0$ and $Q \leq 0$ then trivially $S \leq 0$. Conversely, for all $t \in \mathbb{C}$, $(a, b)^T \in A \oplus B$:

$$\left\langle \begin{bmatrix} ta \\ b \end{bmatrix}, S \begin{bmatrix} ta \\ b \end{bmatrix} \right\rangle = \langle b, Qb \rangle_B + 2\mathrm{Re}\,(t\langle a, Pb \rangle_A).$$

We have that $\langle b, Qb \rangle \in \mathbb{R}$, since $Q^* = Q$ and a real choice of t yields $\forall t \in \mathbb{R} : \ \langle b, Qb \rangle + 2t\,\mathrm{Re}\,\langle a, Pb \rangle \leq 0$, which is possible only if $\mathrm{Re}\,\langle a, Pb \rangle = 0$. Taking t to be imaginary yields $\mathrm{Im}\,\langle a, Pb \rangle = 0$, i.e., $\forall a, b : \ \langle a, Pb \rangle = 0$ and therefore also $\forall b \in B : \ \langle b, Qb \rangle \leq 0$. Thus $P = 0$ and $Q \leq 0$. ■

Lemma 3.5 *Let* T *be a contraction on a finite-dimensional vector space* V*. Let* $\lambda \in \mathbb{C}$*,* $|\lambda| = 1$ *and split* V *into* $V = \mathcal{N}(\lambda - T)^\perp \oplus \mathcal{N}(\lambda - T)$*. Then* $T = \left[\begin{smallmatrix} T_1 & 0 \\ 0 & \lambda \end{smallmatrix}\right]$*, where* $\lambda - T_1$ *is invertible, is the corresponding splitting of* T*. (If* λ *is not an eigenvalue of* T*, then the splitting becomes trivial, with* $\dim \mathcal{N}(\lambda - T) = 0$*.)*

Proof: From the splitting of V it is clear that $\lambda - T_1$ is injective. Since T_1 is square, it is then also invertible. The splitting also yields that $\lambda - T = \left[\begin{smallmatrix} \lambda - T_1 & 0 \\ -T_0 & 0 \end{smallmatrix}\right]$, i.e., $T = \left[\begin{smallmatrix} T_1 & 0 \\ T_0 & \lambda \end{smallmatrix}\right]$. Then we obtain

$$T^*T = \begin{bmatrix} T_1^* & T_0^* \\ 0 & \overline{\lambda} \end{bmatrix} \begin{bmatrix} T_1 & 0 \\ T_0 & \lambda \end{bmatrix} = \begin{bmatrix} T_1^* T_1 + T_0^* T_0 & \lambda T_0^* \\ \overline{\lambda} T_0 & 1 \end{bmatrix}.$$

Since T is a contraction, necessarily $T^*T - I_V \leq 0$, i.e.,

$$\begin{bmatrix} T_1^* T_1 + T_0^* T_0 - 1 & \lambda T_0^* \\ \overline{\lambda} T_0 & 0 \end{bmatrix} \leq 0.$$

Applying Lemma 3.4 yields $\overline{\lambda}T_0 = 0$, i.e., $T_0 = 0$. ■

Proof of Theorem 3.3:
We focus on the passive cases, as the conservative cases are analogous.

1. If the input u_1 of (18) is differentiable, then $\left[\begin{smallmatrix} x_0 \\ x_1 \end{smallmatrix}\right]$ is differentiable and, differentiating (20), we obtain the equivalent condition

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\| \begin{bmatrix} x_0(t) \\ x_1(t) \end{bmatrix} \right\|^2 = 2\mathrm{Re}\left\langle \begin{bmatrix} x_0(t) \\ \dot{x}_1(t) \end{bmatrix}, \begin{bmatrix} \dot{x}_0(t) \\ x_1(t) \end{bmatrix} \right\rangle \leq 2\mathrm{Re}\left\langle u_1(t), y_1(t) \right\rangle \tag{30}$$

$$\iff \left\langle \begin{bmatrix} x_0(t) \\ \dot{x}_1(t) \\ u_1(t) \end{bmatrix}, \begin{bmatrix} \dot{x}_0(t) \\ x_1(t) \\ -y_1(t) \end{bmatrix} \right\rangle + \left\langle \begin{bmatrix} \dot{x}_0(t) \\ x_1(t) \\ -y_1(t) \end{bmatrix}, \begin{bmatrix} x_0(t) \\ \dot{x}_1(t) \\ u_1(t) \end{bmatrix} \right\rangle \leq 0, \tag{31}$$

for all $t \geq 0$. Using time invariance, it is easy to show that the condition (31) is satisfied for every $t \geq 0$, $\left[\begin{smallmatrix} u_1(t) \\ y_0(t) \end{smallmatrix}\right]$ and corresponding $\left[\begin{smallmatrix} x_0(t) \\ \dot{x}_1(t) \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} \dot{x}_0(t) \\ x_1(t) \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} y_1(t) \\ u_0(t) \end{smallmatrix}\right]$ iff the condition holds for all $u(0)$, $x(0)$, $\dot{x}(0)$ and $y(0)$. We can thus abbreviate our notation by writing e.g. $u_1 := u_1(t)$ and $\dot{x}_0 := \dot{x}_0(t)$, for some arbitrary fixed $t \geq 0$.

Substitute $x_0 = B_0 u_1$, $\dot{x}_0 = B_0 \dot{u}_1$ and $\left[\begin{smallmatrix} x_0 \\ \dot{x}_1 \\ -y_1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 & 0 & B_0 \\ 0 & A_1 & B_1 \\ -C_0 & -C_1 & -D_1 \end{smallmatrix}\right]\left[\begin{smallmatrix} \dot{x}_0 \\ x_1 \\ u_1 \end{smallmatrix}\right]$ into (31) to obtain the equivalent condition $\forall \left[\begin{smallmatrix} \dot{u}_1 \\ x_1 \\ u_1 \end{smallmatrix}\right] \in U_1 \oplus X_1 \oplus U_1$:

$$\left\langle \begin{bmatrix} \dot{u}_1 \\ x_1 \\ u_1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & B_0^* B_0 - B_0^* C_0^* \\ 0 & A_1 + A_1^* & B_1 - C_1^* \\ B_0^* B_0 - C_0 B_0 & B_1^* - C_1 & -D_1 - D_1^* \end{bmatrix} \begin{bmatrix} \dot{u}_1 \\ x_1 \\ u_1 \end{bmatrix} \right\rangle \leq 0. \tag{32}$$

This iimplies (27) and applying Lemma 3.4 to (32) yields that $B_0^* B_0 = C_0 B_0$. The converse is obvious.

It is now trivial that equality in (20) is equivalent to equality in (32). Analogous computations for the dual system yields the condition $\forall \left[\begin{smallmatrix} \dot{y}_1^d \\ -x_1^d \\ y_1^d \end{smallmatrix}\right]$:

$$\left\langle \begin{bmatrix} \dot{y}_1^d \\ -x_1^d \\ y_1^d \end{bmatrix}, \begin{bmatrix} 0 & 0 & C_0 C_0^* - C_0 B_0 \\ 0 & A_1 + A_1^* & B_1 - C_1^* \\ C_0 C_0^* - B_0^* C_0^* & B_1^* - C_1 & -D_1 - D_1^* \end{bmatrix} \begin{bmatrix} \dot{y}_1^d \\ -x_1^d \\ y_1^d \end{bmatrix} \right\rangle \leq 0. \tag{33}$$

Let $B_0^* B_0 = C_0 B_0$. Then (32) reduces to $\left[\begin{smallmatrix} A_1 + A_1^* & B_1 - C_1^* \\ B_1^* - C_1 & -D_1 - D_1^* \end{smallmatrix}\right] \leq 0$, which is equivalent to (33) holding everywhere if and only if also (33) reduces to a condition on the standard part, i.e., $C_0 C_0^* = C_0 B_0$.

If $B_0^* B_0 = C_0 B_0 = C_0 C_0^* = B_0^* C_0^*$, then $(C_0^* - B_0)^*(C_0^* - B_0) = C_0 C_0^* - C_0 B_0 - B_0 C_0^* + B_0^* B_0 = 0$. Moreover $(C_0^* - B_0)^*(C_0^* - B_0) = 0$ iff $(C_0^* - B_0) = 0$. On the other hand, if $C_0 = B_0^*$, then trivially $B_0^* B_0 = C_0 B_0 = C_0 C_0^* = B_0^* C_0^*$.

2. After differentiating (21), disregarding the inverted external signals in accordance with Remark 3.2, we obtain the equivalent condition

$$\begin{bmatrix} B_0^*C_0^*C_0B_0 & B_0^*C_0^*C_1 & B_0^*B_0 + B_0^*C_0^*D_1 \\ C_1^*C_0B_0 & A_1 + A_1^* + C_1^*C_1 & B_1 + C_1^*D_1 \\ B_0^*B_0 + D_1^*C_0B_0 & B_1^* + D_1^*C_1 & D_1^*D_1 - I \end{bmatrix} \leq 0. \tag{34}$$

   Taking $x_1, u_1 = 0$ and letting $\dot{u}_1$ vary, we obtain that $\langle \dot{u}_1, B_0^*C_0^*C_0B_0\dot{u}_1\rangle \leq 0$, i.e., $\|C_0B_0\dot{u}_1\|^2 \leq 0$ for all u_1. Thus $C_0B_0 = 0$ and, applying Lemma 3.4, we obtain that $B_0^*B_0 = 0$. This implies that $\langle u_1, B_0^*B_0u_1\rangle = 0$ for all u_1, i.e., $B_0 = 0$, implying in particular that $x_0(t) = 0$ for all t. In this case the first row and the first column of (34) contain only zeros, and thus, (34) holds if and only if the standard part of the system is forward scattering passive.

   Applying the previous to the dual system, we obtain that the dual Σ^* is forward scattering passive if and only if $C_0^* = 0$ and the standard part of the dual is forward scattering passive. However, it is straightforward to check that a standard system is forward passive (forward conservative) iff it is passive (conservative).

3. The discrete-time analogue of (30) is that (22) holds for all n if and only if it holds for $n = 0$, as we will show. The only if-part is trivial. Conversely, assume that it holds for $n = 0$. Utilising time invariance we then obtain the equivalent condition

$$\forall k \geq 0: \qquad \|\mathbf{x}(k+1)\|^2 - \|\mathbf{x}(k)\|^2 \leq 2\mathrm{Re}\,\langle \mathbf{u}_1(k), \mathbf{y}_1(k)\rangle. \tag{35}$$

   Summing over k from 0 to n proves the if-part. The rest is only substitution and rewriting. Case 4. is analogous. ■

We have the following technical corollary.

Corollary 3.6 *For any forward scattering-passive continuous-time system* (24) *we have that* $\mathcal{N}(\lambda - D) \subseteq \mathcal{N}(B_1 + \lambda C_1^*)$ *for all* $\lambda \in \mathbb{C}$ *with* $|\lambda| = 1$.

Proof: Letting $x_1 = 0$ in (34), we obtain $\langle u, (D_1^*D_1 - I)u\rangle \leq 0$, i.e., $\langle u, D^*D_1u\rangle \leq \langle u, u\rangle$, which means that $D = D_1$ is a contraction. (Recall that $B_0 = 0$, annihilating the first row and column of the block matrix in (34).)

According to Lemma 3.5, if we split $U_1 = \mathcal{N}(\lambda - D)^\perp \oplus \mathcal{N}(\lambda - D)$, where $|\lambda| = 1$, then $D = \left[\begin{smallmatrix} D_{11} & 0 \\ 0 & \lambda \end{smallmatrix}\right]$ (with $\lambda - D_{11}$ invertible). Noting that $I - D^*D = \left[\begin{smallmatrix} I - D_{11}^*D_{11} & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and applying Lemma 3.4 to (34) completes the proof. ■

We now describe the transfer function of a passive system.

Theorem 3.7 *For continuous-time (impedance- and scattering-) passive systems, we have* $\mathbb{C}_+ \subset \rho(A_1)$ *and for discrete-time passive systems* $\mathbb{D}_+ \subset \rho(\mathbf{A})$.

If Σ *is a continuous-time impedance-passive system, then* $\widehat{\mathfrak{D}}_1(s) + \widehat{\mathfrak{D}}_1(s)^* \geq 0$ *for* $s \in \mathbb{C}_+$. *If, moreover,* Σ *is impedance-conservative, then* $\widehat{\mathfrak{D}}_1(s)^* = -\widehat{\mathfrak{D}}_1(s)$ *for* $s \in \rho(A_1) \cap i\mathbb{R}$, *i.e., the non-inverted part* $\widehat{\mathfrak{D}}_1$ *of the transfer function is skew-adjoint on the imaginary axis.*

In a scattering-passive system, $\widehat{\mathfrak{D}}_1$ is a contraction on the open right-half plane. If the system is scattering conservative, then $\widehat{\mathfrak{D}}_1$ is unitary on the imaginary axis.

Corresponding claims hold for the discrete-time case, when we replace the open right-half plane with the complement of the closed complex unit disc $\mathbb{D}_+ = \{s \in \mathbb{C} \mid |s| > 1\}$ and the imaginary axis with the complex unit circle.

Proof: The claim $\mathbb{C}_+ \subset \rho(A_1)$ is proved in Corollary 4.2.

The inequality (27) can be equivalently written as

$$\mathrm{Re}\left\langle \begin{bmatrix} x_1 \\ u_1 \end{bmatrix}, \begin{bmatrix} -A_1 & -B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} x_1 \\ u_1 \end{bmatrix} \right\rangle \geq 0.$$

Let $\mathrm{Re}\, s \geq 0$ and choose in particular $x_1 := (s - A_1)^{-1} B_1 u_1$ to obtain the special case

$$\mathrm{Re}\left\langle \begin{bmatrix} (s-A_1)^{-1}B_1 \\ I \end{bmatrix} u_1, \begin{bmatrix} -A_1 & -B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} (s-A_1)^{-1}B_1 \\ I \end{bmatrix} u_1 \right\rangle =$$

$$\mathrm{Re}\left\langle (s-A_1)^{-1}B_1u_1, (-A_1(s-A_1)^{-1}B_1 - B_1)u_1 \right.$$

$$+\mathrm{Re}\left\langle u_1, \big(\widehat{\mathfrak{D}}_1(s) - sB_0^*B_0\big)u_1\right\rangle \geq 0$$

$$\Longleftrightarrow \mathrm{Re}\left\langle u_1, \widehat{\mathfrak{D}}_1(s)_1 u_1\right\rangle \geq (\mathrm{Re}\, s)\|B_0u_1\|^2 + (\mathrm{Re}\, s)\|(s-A_1)^{-1}B_1u_1\|^2 \quad (\geq 0).$$

Thus $2\mathrm{Re}\left\langle u_1, \widehat{\mathfrak{D}}_1(s)u_1\right\rangle \geq 0$, i.e., $\left\langle u_1, \big(\widehat{\mathfrak{D}}_1(s) + \widehat{\mathfrak{D}}_1(s)^*\big)u_1\right\rangle \geq 0$. Trivially $\widehat{\mathfrak{D}}_1(s) + \widehat{\mathfrak{D}}_1(s)^* \geq 0$ iff $\widehat{\mathfrak{D}}(s) + \widehat{\mathfrak{D}}(s)^* \geq 0$.

If, moreover, we have a conservative system and $\mathrm{Re}\, s = 0$, then, by the computations above, $\left\langle u_1, \big(\widehat{\mathfrak{D}}_1(s) + \widehat{\mathfrak{D}}_1(s)^*\big)u_1\right\rangle = 0$ for all $u_1 \in U_1$, i.e., $\widehat{\mathfrak{D}}(s) = -\widehat{\mathfrak{D}}(s)^*$. The claims on the transfer function of a scattering-passive system is proved later, in Remark 6.5. The claims on the discrete-time transfer functions now follow from Theorem 5.4. ■

We end the section with a simple example.

Example 3.8 The PID (Proportional-Integral-Differential) controller can be written

$$\begin{bmatrix} x_0(t) \\ \dot{x}_1(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & K_D \\ 0 & 0 & K_I \\ K_D^* & K_I^* & K_P \end{bmatrix} \begin{bmatrix} \dot{x}_0(t) \\ x_1(t) \\ u(t) \end{bmatrix}.$$

Here u indicates the instantaneous difference between the value of some measured plant variables and the desired values. The variable y is the output for the plant actuators.

We immediatly see that $y(t) = K_D^* K_D \dot{u}(t) + K_I^* K_I \int_0^t u(s)\,\mathrm{d}s + K_P u(t)$ for all $t \geq 0$, agreeing with the transfer function $\widehat{\mathfrak{D}}(s) = sK_D^*K_D + \frac{1}{s}K_I^*K_I + K_P$. Moreover, the PID controller is impedance passive (conservative) if and only if $K_P + K_P^* \geq 0$ ($K_P^* = -K_P$).

The impedance passivity of the controller implies that

$$\widehat{\mathfrak{D}}(s) + \widehat{\mathfrak{D}}(s)^* = (2\mathrm{Re}\, s)\left(K_D^*K_D + \frac{1}{|s|^2}K_D^*K_D\right) + K_P + K_P^* \geq 0$$

for all $s \in \overline{\mathbb{C}_+} \setminus \{0\}$. For an impedance-conservative controller, we have skew-adjointness of the transfer function on the imaginary axis.

4 A Generalised Matrix Cayley Transformation

The main use of the celebrated matrix Cayley transformation is to map any positive-real matrix into a contractive matrix. In the sections to come we need a slightly generalised version of the matrix Cayley transformation.

Lemma 4.1 *Let $T \in \mathcal{L}(V)$ be a square matrix and $\gamma_1, \gamma_2, \tau_1, \tau_2 \in \mathcal{L}(V)$ be diagonal matrices.*

If $\tau_1 + \tau_2 T$ is invertible, then we define the generalised matrix Cayley transform

$$\mathcal{T} := (\gamma_1 + \gamma_2 T)(\tau_1 + \tau_2 T)^{-1} \tag{36}$$

of T with parameter matrices $\gamma_1, \gamma_2, \tau_1, \tau_2$. *We have the following results on the transform $\mathcal{T}$.*

1. *If $\gamma_1^*\gamma_1 = \tau_1^*\tau_1$, $\gamma_2^*\gamma_2 = \tau_2^*\tau_2$ and $\tau_1^*\tau_2 - \gamma_1^*\gamma_2 = aI_V$ for some $a \in \mathbb{R}_+$, then the transform $\mathcal{T}$ is a contraction if and only if T is positive real. By this we mean that $\|\mathcal{T}\| \leq 1 \Longleftrightarrow T + T^* \geq 0$.*

   Furthermore, $\mathcal{T}$ is unitary if and only if T is skew-adjoint, i.e., $\mathcal{T}^*\mathcal{T} = I_V \Longleftrightarrow T^* = -T$.

2. *If T is positive real, τ_2 is invertible and $\tau_2^{-1}\tau_1 + \tau_1^*\tau_2^{-*} > 0$, then $\tau_1 + \tau_2 T$ is invertible.*

3. *The equation (36) can be solved for T if and only if $\mathcal{T}\tau_2 - \gamma_2$ is invertible. Then T is given by*

$$T = (\mathcal{T}\tau_2 - \gamma_2)^{-1}(\gamma_1 - \mathcal{T}\tau_1). \tag{37}$$

Proof:

1. The matrix $\mathcal{T} = (\gamma_1 + \gamma_2 T)(\tau_1 + \tau_2 T)^{-1}$ is a contraction if and only if

$$\langle (\gamma_1 + \gamma_2 T)(\tau_1 + \tau_2 T)^{-1} v, (\gamma_1 + \gamma_2 T)(\tau_1 + \tau_2 T)^{-1} v\rangle \leq \langle v, v\rangle$$

for every $v \in V$. By mapping V bijectively onto itself by the change $v = (\tau_1 + \tau_2 T)w$ of variables, the equivalent condition

$$\forall w \in V : \qquad \langle (\gamma_1 + \gamma_2 T)w, (\gamma_1 + \gamma_2 T)w\rangle \leq \langle (\tau_1 + \tau_2 T)w, (\tau_1 + \tau_2 T)w\rangle$$

is obtained. This is furthermore equivalent to writing $\forall w \in V$:

$$\langle w, (\gamma_1^*\gamma_1 - \tau_1^*\tau_1)w\rangle + \langle Tw, (\gamma_2^*\gamma_2 - \tau_2^*\tau_2)Tw\rangle \leq 2\mathrm{Re}\,\langle w, (\tau_1^*\tau_2 - \gamma_1^*\gamma_2)Tw\rangle$$

$$\Longleftrightarrow \quad 0 \leq 2a\mathrm{Re}\,\langle w, Tw\rangle \quad \Longleftrightarrow \quad \langle w, (T + T^*)w\rangle \geq 0.$$

2. First of all, $\tau_1 + \tau_2 T$ is trivially invertible iff $\tau_2^{-1}\tau_1 + T$ is invertible. Secondly, the injectivity of the latter (square) matrix is proved by contradiction. If there existed a nonzero $v \in V$, such that $(\tau_2^{-1}\tau_1 + T)v = 0$, then

$$\langle v, (T + T^*)v\rangle = -\langle v, (\tau_2^{-1}\tau_1 + \tau_1^*\tau_2^{-*})v\rangle < 0,$$

which would contradict the positive realness of T.

3. Postmultiply equation (36) by $(\tau_1 + \tau_2 T)$ in order to obtain the equivalent equivalent equation $(\mathcal{T}\tau_2 - \gamma_2)T = \gamma_1 - \mathcal{T}\tau_1$. ■

The standard matrix Cayley transformation is a very important special case of Lemma 4.1. We study this case in the following corollary, omitting the simple proof.

Corollary 4.2 *Given a $T \in \mathcal{L}(V)$, let $\tau_1 \in \mathbb{C}_+ \cap \rho(-T)$ and take $\tau_2 = 1$, $\gamma_1 = \overline{\tau_1}$ and $\gamma_2 = -1$. Then the* standard matrix Cayley transform

$$\mathcal{T} = (\overline{\tau_1} - T)(\tau_1 + T)^{-1} = (\tau_1 + T)^{-1}(\overline{\tau_1} - T) \tag{38}$$

satisfies the following.

1. *The transform $\mathcal{T}$ is a contraction if and only if T is positive real and, moreover, $\mathcal{T}$ is unitary if and only if T is skew-adjoint.*

2. *If T is positive real, then $\mathbb{C}_+ \subseteq \rho(-T)$ and the Cayley transform (38) is defined for every $\tau_1 \in \mathbb{C}_+$.*

3. *If $\mathcal{T}$ is defined by (38), then it directly follows that $1 + \mathcal{T} = (2\mathrm{Re}\,\tau_1)(\tau_1 + T)^{-1}$ and so $1 + \mathcal{T}$ is always invertible.*

   More generally, the equation (38) can be solved for T if and only if $1 + \mathcal{T}$ is invertible and then T is given by

$$T = (\overline{\tau}_1 - \tau_1\mathcal{T})(1 + \mathcal{T})^{-1} = (1 + \mathcal{T})^{-1}(\overline{\tau}_1 - \tau_1\mathcal{T}). \tag{39}$$

In particular, taking $V = \mathbb{C}$ and $\tau \in \mathbb{C}_+$, we obtain that $T \to (\tau_1 + T)/(\overline{\tau_1} - T)$ maps the open complex right-half-plane bijectively onto the open external of the complex unit disc $\mathbb{D}_+ = \{\lambda \in \mathbb{C} \mid |\lambda| > 1\}$. (The point $\overline{\tau_1}$ is mapped into the point at infinity.) The imaginary axis is mapped one-to-one onto the unit circle.

In the following sections we will define two Cayley transforms of the *system* (18). These transforms are again *systems*, so the transforms are indeed not special cases of the generalised matrix Cayley transform, introduced in Lemma 4.1, that maps *matrices into matrices*. However, there is a close relationship between the generalised matrix Cayley transform and the to-be-introduced system Cayley tranforms. Therefore Lemma 4.1 is useful for reference.

5 The Internal Cayley Transform of an Extended System

The Cayley transforms of standard systems are treated in great detail in [St1]. In this section we present a generalised internal Cayley transformation that maps extended continuous-time systems to discrete-time systems. The standard case is, as usual, recovered by disregarding everything connected to U_0 and X_0.

5.1 Motivation and Definition

We review a simple method for approximating the trajectory of a given continuous-time extended system of standard type numerically by transforming the system into a discrete-time system. More background and details on this approximation can be found in [FP, Section 3], where it is referred to as "the Tustin method".

Let $h \in \mathbb{R}_+$ and consider the equidistant discretisation $\{t_n\}_{n\in\mathbb{N}_0}$, $t_n = nh$ of $\overline{\mathbb{R}_+}$. By integrating the standard input equation

$$\dot{x}(t) = Ax(t) + Bu(t)$$

from t_n to t_{n+1} one obtains

$$x(t_{n+1}) - x(t_n) = \int_{nh}^{(n+1)h} \big(Ax(t) + Bu(t)\big)\,\mathrm{d}t.$$

If the continuous integrand is approximated by a straight line, then the approximation

$$x(t_{n+1}) - x(t_n) \approx \frac{h}{2}\big(Ax(t_{n+1}) + Ax(t_n) + Bu(t_{n+1}) + Bu(t_n)\big). \tag{40}$$

is obtained. Introduce $\alpha := 2/h \in \mathbb{R}_+$ and $\mathbf{u}(n) = \big(u(t_{n+1}) + u(t_n)\big)/\sqrt{2\alpha}$. If $\alpha \in \rho(A)$, then (40) can be solved for $x(t_{n+1})$:

$$x(t_{n+1}) \approx (\alpha - A)^{-1}(\alpha + A)x(t_n) + \sqrt{2\alpha}(\alpha - A)^{-1}B\mathbf{u}_n,$$

Defining $\mathbf{x}(n)$ as

$$\mathbf{x}(0) = x(0), \quad \mathbf{x}(n+1) = (\alpha - A)^{-1}(\alpha + A)\mathbf{x}(n) + \sqrt{2\alpha}(\alpha - A)^{-1}B\mathbf{u}(n), \tag{41}$$

we therefore make $\mathbf{x}(n)$ a discrete approximation of $x(t_n)$. If $\mathrm{d}^2u/\mathrm{d}t^2$ exists for all $t \geq 0$, then $\|\mathbf{x}([t/h]) - x(t)\| \leq C(t)h^2$, denoting the integer part of t/h by $[t/h]$. See Figure 1 for an illustration.

The convergence properties of this approximation is studied in much greater detail in [HM]. Another approach to solving a continuous-time system numerically using the internal Cayley transform is taken in [AG].

We now give an abstract definition of the internal Cayley transform. The definition is abstract in the sense that we only assume that the original system is of input/state/output type. Even this is not strictly necessary. The transformation is defined in terms of the internal and external signals of a system, independently of the precise system equations. We provide explicit representations for the transform of a (scattering- or impedance-) passive system in Theorem 5.2. In particular (49) might be familiar to the reader.

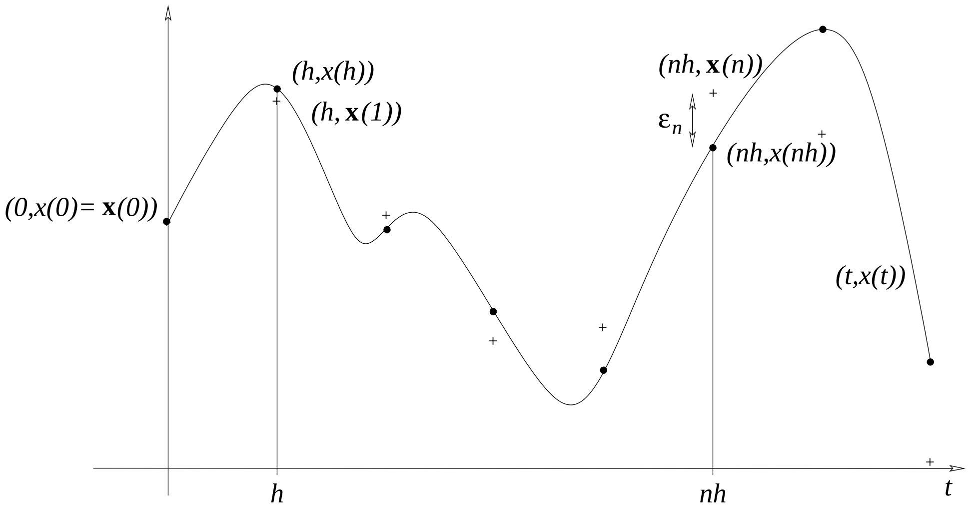

Figure 1: The approximation illustrated. The points $\big(nh, \mathbf{x}(n)\big)$, marked by $+$, do not lie on the curve $\big(t, x(t)\big)$, but the error $\varepsilon_{[t/h]}$ is proportional to h^2.

Definition 5.1 *Let* Σ *be a continuous-time system with (formal) input signal* u*, state trajectory* x *and (formal) output signal* y*. Take* $\alpha \in \mathbb{C}_+$*. The* internal Cayley transform *of* Σ with parameter α *is the discrete-dime system* $\mathbf{\Sigma}$ *that is obtained by performing the text substitutions*

$$\begin{array}{rclcrcl} \dot{x}(t) & \rightarrow & \frac{1}{\sqrt{2\mathrm{Re}\,\alpha}}\big(\alpha\mathbf{x}(n+1) - \overline{\alpha}\mathbf{x}(n)\big) & \qquad & y(t) & \rightarrow & \mathbf{y}(n) \\ x(t) & \rightarrow & \frac{1}{\sqrt{2\mathrm{Re}\,\alpha}}\big(\mathbf{x}(n+1) + \mathbf{x}(n)\big) & & u(t) & \rightarrow & \mathbf{u}(n) \end{array}. \tag{42}$$

The inverse Cayley transform *of the discrete-time system* $\mathbf{\Sigma}$ *with parameter* α *is the continuous-time system obtained by performing the inverse text substitutions, i.e., by reversing the arrows in* (42)*.*

We remark that the state part of (42) can equivalently be written

$$\mathbf{x}(n+1) \leftarrow \frac{1}{\sqrt{2\mathrm{Re}\,\alpha}}\big(\overline{\alpha}x(t) + \dot{x}(t)\big) \quad \text{and} \quad \mathbf{x}(n) \leftarrow \frac{1}{\sqrt{2\mathrm{Re}\,\alpha}}\big(\alpha x(t) - \dot{x}(t)\big). \tag{43}$$

This gives an explicit expression for the inverse internal Cayley transform.

The reason for choosing this particular transformation is that it often preserves passivity:

$$\begin{aligned} (2\mathrm{Re}\,\alpha)\frac{\mathrm{d}}{\mathrm{d}t}\,\|x(t)\|^2 &= (2\mathrm{Re}\,\alpha)2\mathrm{Re}\,\langle \dot{x}(t), x(t)\rangle \\ &\rightarrow 2\mathrm{Re}\,\big(\alpha\|\mathbf{x}(n+1)\|^2 - \overline{\alpha}\|\mathbf{x}(n)\|^2 + i2\mathrm{Im}\,(\alpha\langle\mathbf{x}(n+1), \mathbf{x}(n)\rangle)\big) \\ &= (2\mathrm{Re}\,\alpha)\big(\|\mathbf{x}(n+1)\|^2 - \|\mathbf{x}(n)\|^2\big), \end{aligned}$$

i.e., $\frac{\mathrm{d}}{\mathrm{d}t}\,\|x(t)\|^2 \rightarrow \big(\|\mathbf{x}(n+1)\|^2 - \|\mathbf{x}(n)\|^2\big)$ cf. the proof of Theorem 3.3.

Now we proceed to the next section, where we compute the Cayley transform of a continuous-time impedance-passive system.

5.2 The Internal Cayley Transform of an Impedance-Passive System

We throughout let $\alpha \in \mathbb{C}_+$ and denote $W_\alpha = \operatorname{diag}(\sqrt{2\operatorname{Re}\alpha}\,, \sqrt{2\operatorname{Re}\alpha}\,, 1, 1) \in \mathcal{L}(X_0 \oplus X_1 \oplus U_1 \oplus U_0)$. Trivially W_α is invertible. Now consider the extended i/s/o system Σ :

$$\begin{bmatrix} x_0(t) \\ \dot{x}_1(t) \\ y_1(t) \\ u_0(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & B_0 & 0 \\ 0 & A_1 & B_1 & 0 \\ C_0 & C_1 & D_1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_0(t) \\ x_1(t) \\ u_1(t) \\ y_0(t) \end{bmatrix}. \tag{44}$$

Applying (43) to (44), keeping in mind that $x = \left[\begin{smallmatrix} x_0 \\ x_1 \end{smallmatrix}\right]$, $u = \left[\begin{smallmatrix} u_1 \\ u_0 \end{smallmatrix}\right]$ and $y = \left[\begin{smallmatrix} y_1 \\ y_0 \end{smallmatrix}\right]$, we can write

$$\begin{bmatrix} \mathbf{x}_0(n+1) \\ \mathbf{x}_1(n+1) \\ \mathbf{y}_1(n) \\ \mathbf{u}_0(n) \end{bmatrix} = W_\alpha^{-1}\mathbf{P} \begin{bmatrix} \dot{x}_0(t) \\ x_1(t) \\ u_1(t) \\ y_0(t) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \mathbf{x}_0(n) \\ \mathbf{x}_1(n) \\ \mathbf{u}_1(n) \\ \mathbf{y}_0(n) \end{bmatrix} = W_\alpha^{-1}\mathbf{Q} \begin{bmatrix} \dot{x}_0(t) \\ x_1(t) \\ u_1(t) \\ y_0(t) \end{bmatrix}, \tag{45}$$

where

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & \overline{\alpha}B_0 & 0 \\ 0 & \overline{\alpha} + A_1 & B_1 & 0 \\ C_0 & C_1 & D & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{Q} = \begin{bmatrix} -1 & 0 & \alpha B_0 & 0 \\ 0 & \alpha - A_1 & -B_1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

If we define e.g. $\mathbf{P}_1 = \operatorname{diag}(1, \overline{\alpha}, 0, 0)$, $\mathbf{P}_2 = \operatorname{diag}(\overline{\alpha}, 1, 1, 1)$, $\mathbf{Q}_1 = \operatorname{diag}(-1, \alpha, 1, 1)$ and $\mathbf{Q}_2 = \operatorname{diag}(\alpha, -1, 0, 0)$, then $\mathbf{P} = \mathbf{P}_1 + \mathbf{P}_2\Sigma$ and $\mathbf{Q} = \mathbf{Q}_1 + \mathbf{Q}_2\Sigma$.

From the block-triangularity of $\mathbf{Q}$ it is easy to see that $\mathbf{Q}$ is invertible if and only if the block $\alpha - A_1$ is invertible, i.e., $\alpha \in \rho(A_1)$. In this case (45) implies that

$$\begin{bmatrix} \mathbf{x}_0(n+1) \\ \mathbf{x}_1(n+1) \\ \mathbf{y}_1(n) \\ \mathbf{u}_0(n) \end{bmatrix} = W_\alpha^{-1}\mathbf{P}\mathbf{Q}^{-1}W_\alpha \begin{bmatrix} \mathbf{x}_0(n) \\ \mathbf{x}_1(n) \\ \mathbf{u}_1(n) \\ \mathbf{y}_0(n) \end{bmatrix}. \tag{46}$$

Whenever $\alpha \in \rho(A_1)$, we have that

$$\mathbf{Q}^{-1} = \begin{bmatrix} 1 & 0 & \alpha B_0 & 0 \\ 0 & (\alpha - A_1)^{-1} & (\alpha - A_1)^{-1}B_1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Thus, he system matrix $\mathbf{\Sigma} = W_\alpha^{-1}\mathbf{P}\mathbf{Q}^{-1}W_\alpha$ of (46) is

$$\mathbf{\Sigma} = \begin{bmatrix} -1 & 0 & \sqrt{2\operatorname{Re}\alpha}\,B_0 & 0 \\ 0 & (\overline{\alpha} + A_1)(\alpha - A_1)^{-1} & \sqrt{2\operatorname{Re}\alpha}\,(\alpha - A_1)^{-1}B_1 & 0 \\ -\sqrt{2\operatorname{Re}\alpha}\,C_0 & \sqrt{2\operatorname{Re}\alpha}\,C_1(\alpha - A_1)^{-1} & \widehat{\mathfrak{D}}_1(\alpha) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{47}$$

where $\widehat{\mathfrak{D}}_1(\alpha) = \alpha C_0 B_0 + C_1(\alpha - A_1)^{-1}B_1 + D_1$ is the non-trivial part of the transfer function of Σ.

Thus, the internal Cayley transform of our extended i/s/o system is indeed well defined if $\alpha \in \rho(A_1)$. Furthermore, it is of the familiar type

$$\begin{bmatrix} \mathbf{x}(n+1) \\ \mathbf{y}_1(n) \\ \mathbf{u}_0(n) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B}_1 & 0 \\ \mathbf{C}_1 & \mathbf{D}_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(n) \\ \mathbf{u}_1(n) \\ \mathbf{y}_0(n) \end{bmatrix}. \tag{48}$$

The following theorem gives the main properties of the transform.

Theorem 5.2 *The internal Cayley transform of the system* (44) *with parameter* $\alpha \in \rho(A_1) \cap \mathbb{C}_+$ *is the system*

$$\begin{bmatrix} \mathbf{x}_0(n+1) \\ \mathbf{x}_1(n+1) \\ \mathbf{y}_1(n) \\ \mathbf{u}_0(n) \end{bmatrix} = \mathbf{\Sigma} \begin{bmatrix} \mathbf{x}_0(n) \\ \mathbf{x}_1(n) \\ \mathbf{u}_1(n) \\ \mathbf{y}_0(n) \end{bmatrix},$$

where $\mathbf{\Sigma}$ *is given by* (47). *We always have, for* $\mathbf{A}_1 := (\overline{\alpha}+A_1)(\alpha-A_1)^{-1}$, *that* $1+\mathbf{A}_1$ *is invertible.*

The transform is impedance passive/conservative if and only if the original system is impedance passive/conservative. In this case $\mathbb{C}_+ \subseteq \rho(A_1)$, *i.e., the internal Cayley transform is defined for any* $\alpha \in \mathbb{C}_+$.

In particular, the formula for the internal Cayley transform of a standard system can be recovered from the central 2×2 *blocks of* (47) *as*

$$\mathbf{\Sigma} = \begin{bmatrix} (\overline{\alpha} + A_1)(\alpha - A_1)^{-1} & \sqrt{2\mathrm{Re}\,\alpha}\,(\alpha - A_1)^{-1}B_1 \\ \sqrt{2\mathrm{Re}\,\alpha}\,C_1(\alpha - A_1)^{-1} & C_1(\alpha - A_1)^{-1}B_1 + D_1 \end{bmatrix}. \tag{49}$$

Proof: The expression for the transform follows from the computations above. We easily see that $1 + \mathbf{A}_1 = (2\mathrm{Re}\,\alpha)(\alpha - A_1)^{-1}$, which is invertible, as $\alpha \in \mathbb{C}_+$.

By Theorem 3.3, the untransformed system Σ is impedance passive iff $C_0 = B_0^*$ and for all $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in X_1 \oplus U_1$ we have

$$\left\langle \begin{bmatrix} x \\ u \end{bmatrix}, \begin{bmatrix} A_1 + A_1^* & B_1 - C_1^* \\ B_1^* - C_1 & -D_1 - D_1^* \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \right\rangle \le 0. \tag{50}$$

We make the invertible change $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] = \left[\begin{smallmatrix} \sqrt{2\mathrm{Re}\,\alpha}(\alpha-A_1)^{-1} & (\alpha-A_1)^{-1}B_1 \\ 0 & 1 \end{smallmatrix}\right] \left[\begin{smallmatrix} z \\ u \end{smallmatrix}\right]$ of variables in (50). After a straightforward, but rather lengthy, computation we turn (50) into

$$\left\langle \begin{bmatrix} z \\ u \end{bmatrix}, \begin{bmatrix} \mathbf{A}_1^*\mathbf{A}_1 - 1 & \mathbf{A}_1^*\mathbf{B}_{11} - \mathbf{C}_{11}^* \\ \mathbf{B}_{11}^*\mathbf{A}_1 - \mathbf{C}_{11} & \mathbf{B}_{11}^*\mathbf{B}_{11} - \mathbf{D}_1 - \mathbf{D}_1^* \end{bmatrix} \begin{bmatrix} z \\ u \end{bmatrix} \right\rangle \le 0, \tag{51}$$

where $\mathbf{A} = \left[\begin{smallmatrix} -1 & 0 \\ 0 & \mathbf{A}_1 \end{smallmatrix}\right]$, $\mathbf{B}_1 = \left[\begin{smallmatrix} \mathbf{B}_{01} \\ \mathbf{B}_{11} \end{smallmatrix}\right]$, $\mathbf{C}_1 = \begin{bmatrix} \mathbf{C}_{10} & \mathbf{C}_{11} \end{bmatrix}$ and $\mathbf{D}_1$ are given by (47).

It remains to show that (51) holds iff $\mathbf{\Sigma}$ given in (47) is impedance passive. For systems (47), we always have $\mathbf{A}^*\mathbf{A} - 1 = \left[\begin{smallmatrix} 0 & 0 \\ 0 & \mathbf{A}_1^*\mathbf{A}_1 - 1 \end{smallmatrix}\right]$ and

$$C_0 = B_0^* \quad \Longleftrightarrow \quad \mathbf{C}_{10} = -\mathbf{B}_{01}^* \quad \Longleftrightarrow \quad \mathbf{B}_1^*\mathbf{A} - \mathbf{C}_1 = \begin{bmatrix} 0 & \mathbf{B}_{11}^*\mathbf{A}_1 - \mathbf{C}_{11} \end{bmatrix}.$$

Thus, everything in (29) that is connected to X_0 is trivial, i.e., (29) reduces to (51), iff $C_0 = B_0^*$, or equivalently, $\mathbf{C}_{10} = -\mathbf{B}_{01}^*$. If $\mathbf{\Sigma}$ is impedance passive, then $\mathbf{C}_{10} = -\mathbf{B}_{01}^*$, by Lemma 3.4. Conversely, if Σ is impedance passive, then $C_0 = B_0^*$, by Theorem 3.3.

Replacing inequality by equality above the conservative case is proved. We have now proved that Σ is of the same energy type as $\mathbf{\Sigma}$. If, Σ (or equivalently $\mathbf{\Sigma}$) is impedance passive, then $A_1 + A_1^* \leq 0$, and by Corollary 4.2, $\mathbb{C}_+ \subset \rho(A_1)$. ■

Remark 5.3 Note that the expression (41) coincides with the input equation of the internal Cayley transform $\mathbf{\Sigma}$ of Σ, given in (49), for real $\alpha \in \rho(A)$. Therefore, the parameter α in the general transformation introduced in Definition 5.1 can be thought of as the inverse step length in the discrete approximation, as explained in the motivation.

We proceed by studying how the internal Cayley transformation affects the transfer function. According to Corollary 4.2, the bilinear transformation

$$s \quad \rightarrow \quad z(s) = \frac{\overline{\alpha} + s}{\alpha - s}, \qquad \left(\text{with inverse } s(z) = \frac{\alpha z - \overline{\alpha}}{z + 1}\right) \tag{52}$$

maps the open complex right half plane $\mathbb{C}_+$ one-to-one onto the external of the complex unit disk $\mathbb{D}_+$, while mapping the imaginary axis $i\mathbb{R}$ bijectively onto the unit circle $\mathbb{D}$. The proof of the following theorem furthermore requires the *resolvent identity*:

$$\forall s_1, s_2 \in \rho(A): \quad (s_1 - A)^{-1} - (s_2 - A)^{-1} = (s_2 - s_1)(s_1 - A)^{-1}(s_2 - A)^{-1}. \tag{53}$$

Theorem 5.4 *Let Σ be given by (44), let $\mathbf{\Sigma}$ be given by (47) and let the variables $s(\neq \alpha)$ and $z(\neq -1)$ be related as in (52). Denote the transfer functions of Σ and $\mathbf{\Sigma}$ by $\widehat{\mathfrak{D}}$ and $\widehat{\mathbb{D}}$, respectively. Then $z \in \rho(\mathbf{A}) \setminus \{-1\} \Longleftrightarrow s \in \rho(A_1) \setminus \{\alpha\}$ and*

$$\forall s \in \rho(A_1): \qquad \widehat{\mathfrak{D}}(s) = \widehat{\mathbb{D}}\big(z(s)\big) = \widehat{\mathbb{D}}\left(\frac{\overline{\alpha} + s}{\alpha - s}\right).$$

Proof: We see that, for $s \neq \alpha$ and $z \neq -1$, we have

$$z - \mathbf{A} = \frac{2\mathrm{Re}\,\alpha}{\alpha - s} \begin{bmatrix} 1 & 0 \\ 0 & (s - A_1)(\alpha - A_1)^{-1} \end{bmatrix}.$$

As $2\mathrm{Re}\,\alpha > 0$, it follows that $s \in \rho(A_1) \setminus \{\alpha\}$ if and only if $z \in \rho(\mathbf{A}) \setminus \{-1\}$. Then the nontrivial parts of $\widehat{\mathfrak{D}}$ and $\widehat{\mathbb{D}}$ satisfy

$$\begin{aligned}\widehat{\mathbb{D}}_1\big(z(s)\big) &= \frac{\alpha - s}{2\mathrm{Re}\,\alpha}\begin{bmatrix} -\sqrt{2\mathrm{Re}\,\alpha}\,C_0 & \sqrt{2\mathrm{Re}\,\alpha}\,C_1(\alpha - A_1)^{-1}\end{bmatrix} \\ &\quad \times \begin{bmatrix} 1 & 0 \\ 0 & (\alpha - A_1)(s - A_1)^{-1}\end{bmatrix}\begin{bmatrix} \sqrt{2\mathrm{Re}\,\alpha}\,B_0 \\ \sqrt{2\mathrm{Re}\,\alpha}\,(\alpha - A_1)^{-1}B_1\end{bmatrix} + \widehat{\mathfrak{D}}(\alpha) \\ &= sC_0B_0 + C_1(\alpha - s)(s - A_1)^{-1}(\alpha - A_1)^{-1}B_1 \\ &\quad + C_1(\alpha - A_1)^{-1}B_1 + D_1 = \widehat{\mathfrak{D}}_1(s).\end{aligned}$$

The last equality follows from the resolvent identity (53).

The transfer function $\widehat{\mathfrak{D}}$ is analytic at $\alpha \in \rho(A_1)$ and so it follows that $\widehat{\mathfrak{D}}(\alpha) = \lim_{s\to\alpha}\widehat{\mathfrak{D}}(s) = \lim_{z\to\infty}\widehat{\mathbb{D}}(z) = \mathbf{D}$. ■

Example 5.5 The internal Cayley transform with parameter α of the PID controller in Example 3.8 is the discrete-time system on $X_0 \oplus X_1 \oplus U$, whose system matrix is given by

$$\boldsymbol{\Sigma} = \begin{bmatrix} -1 & 0 & \sqrt{2\mathrm{Re}\,\alpha}\,K_D \\ 0 & \overline{\alpha}/\alpha & \sqrt{2\mathrm{Re}\,\alpha}\,K_I/\alpha \\ -\sqrt{2\mathrm{Re}\,\alpha}\,K_D^* & \sqrt{2\mathrm{Re}\,\alpha}\,K_I^*/\alpha & \alpha K_D^*K_D + K_I^*K_I/\alpha + K_P\end{bmatrix}.$$

The transfer function of $\boldsymbol{\Sigma}$ is

$$\widehat{\mathbb{D}}(z) = \frac{\alpha z - \overline{\alpha}}{1+z}K_D^*K_D + \frac{1+z}{\alpha z - \overline{\alpha}}K_I^*K_I + K_P,$$

agreeing with Theorem 5.4.

We now show that the internal Cayley transformation is surjective by computing the inverse Cayley transform explicitely for an arbitrary discrete-time impedance-passive system.

Consider the general impedance-passive discrete-time system (48). Split the state space into $X = \mathcal{N}(1+\mathbf{A}) \oplus \mathcal{N}(1+\mathbf{A})^\perp$. Then $\mathbf{A}$ splits as

$$\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 0 & \mathbf{A}_1\end{bmatrix}, \qquad \text{with } -1 \in \rho(\mathbf{A}_1), \tag{54}$$

according to Lemma 3.5. Straightforward computations in the fashion of the forward transformation now results in the following theorem.

Theorem 5.6 *The inverse internal Cayley transform of the discrete-time impedance-passive system* $\boldsymbol{\Sigma}$*:*

$$\begin{bmatrix} \mathbf{x}_0(n+1) \\ \mathbf{x}_1(n+1) \\ \mathbf{y}_1(n) \\ \mathbf{u}_0(n)\end{bmatrix} = \begin{bmatrix} -1 & 0 & \mathbf{B}_0 & 0 \\ 0 & \mathbf{A}_1 & \mathbf{B}_1 & 0 \\ \mathbf{C}_0 & \mathbf{C}_1 & \mathbf{D}_1 & 0 \\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix} \mathbf{x}_0(n) \\ \mathbf{x}_1(n) \\ \mathbf{u}_1(n) \\ \mathbf{y}_0(n)\end{bmatrix}, \quad -1 \in \rho(\mathbf{A}_1),$$

is the continuous-time impedance-passive system Σ :

$$\begin{bmatrix} x_0(t) \\ \dot{x}_1(t) \\ y_1(t) \\ u_0(t) \end{bmatrix} = W_\alpha^{-1}(P_1 + P_2\mathbf{\Sigma})(Q_1 + Q_2\mathbf{\Sigma})^{-1}W_\alpha \begin{bmatrix} \dot{x}_0(t) \\ x_1(t) \\ u_1(t) \\ y_0(t) \end{bmatrix},$$

where $P_1 = \operatorname{diag}(1, -\overline{\alpha}, 0, 0)$, $P_2 = \operatorname{diag}(1, \alpha, 1, 1)$, $Q_1 = \operatorname{diag}(-\overline{\alpha}, 1, 1, 1)$ *and* $Q_2 = \operatorname{diag}(\alpha, 1, 0, 0)$.

The explicit expression of the system matrix Σ *is*

$$\Sigma = \begin{bmatrix} 0 & 0 & \mathbf{B}_0/\sqrt{2\operatorname{Re}\alpha} & 0 \\ 0 & (\alpha\mathbf{A}_1 - \overline{\alpha})(1+\mathbf{A}_1)^{-1} & \sqrt{2\operatorname{Re}\alpha}\,(1+\mathbf{A}_1)^{-1}\mathbf{B}_1 & 0 \\ -\mathbf{C}_0/\sqrt{2\operatorname{Re}\alpha} & \sqrt{2\operatorname{Re}\alpha}\,\mathbf{C}_1(1+\mathbf{A}_1)^{-1} & D_1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{55}$$

where $D_1 = \alpha\mathbf{C}_0\mathbf{B}_0/(2\operatorname{Re}\alpha) - \mathbf{C}_1(1+\mathbf{A}_1)^{-1}\mathbf{B}_1 + \mathbf{D}$. *In particular* $\alpha \in \rho(A_1)$.

Note that $1 + \mathbf{A}_1$ is invertible by the splitting (54) of $\mathbf{A}$ and thus the inverse internal Cayley transformation with parameter $\alpha \in \mathbb{C}_+$ is always well-defined. If $1 + \mathbf{A}$ is originally invertible, no splitting of the state space X is required and the standard formulas are recovered from (55):

$$\Sigma = \begin{bmatrix} (\alpha\mathbf{A} - \overline{\alpha})(1+\mathbf{A})^{-1} & \sqrt{2\operatorname{Re}\alpha}\,(1+\mathbf{A})^{-1}\mathbf{B} \\ \sqrt{2\operatorname{Re}\alpha}\,\mathbf{C}(1+\mathbf{A})^{-1} & \mathbf{D} - \mathbf{C}(1+\mathbf{A})^{-1}\mathbf{B} \end{bmatrix}.$$

Remark 5.7 The standard inverse internal Cayley transformation that is given in [St1], assumes that $1 + \mathbf{A}$ is invertible. We replaced this assumption by the weaker assumption that both $\mathcal{N}(1 + \mathbf{A})$ and $\mathcal{N}(1 + \mathbf{A})^\perp$ are invariant under $\mathbf{A}$, which we showed always to be the case for discrete-time impedance-passive systems.

Moreover, by computing both the forward and inverse internal Cayley transforms explicitely, utilising only the impedance-passivity assumption, we showed that there is a one-to-one relationship between our classes of continuous- and discrete-time impedance-passive systems.

5.3 The Scattering-Passive Case

The results of section 5.2 apply also to the scattering case after suitable reformulations. We briefly present the results for the scattering case. The proof techniques are the same as in the impedance-passive case.

If we start from a scattering-passive continuous-time system $\Sigma^\times$:

$$\begin{bmatrix} x_0(t) \\ \dot{x}_1(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_1^\times & B_1^\times \\ 0 & C_1^\times & D^\times \end{bmatrix} \begin{bmatrix} \dot{x}_0(t) \\ x_1(t) \\ u(t) \end{bmatrix} \tag{56}$$

and perform an internal Cayley transformation, by making the change (42) of signals, the result is obtained as a special case of (47):

$$\begin{bmatrix} \mathbf{x}_0^\times(n+1) \\ \mathbf{x}_1^\times(n+1) \\ \mathbf{y}^\times(n) \end{bmatrix} = \mathbf{\Sigma}^\times \begin{bmatrix} \mathbf{x}_0^\times(n) \\ \mathbf{x}_1^\times(n) \\ \mathbf{u}^\times(n) \end{bmatrix}, \tag{57}$$

where

$$\mathbf{\Sigma}^\times = \begin{bmatrix} -1 & 0 & 0 \\ 0 & (\overline{\alpha}+A_1^\times)(\alpha - A_1^\times)^{-1} & \sqrt{2\mathrm{Re}\,\alpha}\,(\alpha - A_1^\times)^{-1}B_1^\times \\ 0 & \sqrt{2\mathrm{Re}\,\alpha}\,C_1^\times(\alpha - A_1^\times)^{-1} & C_1^\times(\alpha - A_1^\times)^{-1}B_1^\times + D^\times \end{bmatrix}.$$

The transform (57) is also scattering passive, by an argument analogous to the impedance case. The transfer function is given by $\widehat{\mathbb{D}}^\times(z) = \widehat{\mathfrak{D}}^\times\big(s(z)\big) = \widehat{\mathfrak{D}}^\times\left(\frac{\alpha z - \overline{\alpha}}{z+1}\right)$ for every $z \in \rho(\mathbf{A})$.

For the inverse internal Cayley transformation, consider a general discrete-time scattering-passive system $\mathbf{\Sigma}^\times$:

$$\begin{bmatrix} \mathbf{x}^\times(n+1) \\ \mathbf{y}^\times(n) \end{bmatrix} = \begin{bmatrix} \mathbf{A}^\times & \mathbf{B}^\times \\ \mathbf{C}^\times & \mathbf{D}^\times \end{bmatrix} \begin{bmatrix} \mathbf{x}^\times(n) \\ \mathbf{u}^\times(n) \end{bmatrix} \tag{58}$$

and again split X into $X_0 \oplus X_0^\perp$, where $X_0 = \mathcal{N}(1 + \mathbf{A}^\times)$ to obtain

$$\mathbf{\Sigma}^\times = \begin{bmatrix} -1 & 0 & \mathbf{B}_0^\times \\ 0 & \mathbf{A}_1^\times & \mathbf{B}_1^\times \\ \mathbf{C}_0^\times & \mathbf{C}_1^\times & \mathbf{D}^\times \end{bmatrix}.$$

Since $\mathbf{\Sigma}^\times$ is a contraction in the discrete-time scattering case, we necessarily have $\mathbf{C}_0^\times = 0$ and $\mathbf{B}_0^\times = 0$, by Lemma 3.5. The inverse internal Cayley transform of (58) is then given by

$$\begin{bmatrix} x_0(t) \\ \dot{x}_1(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & (\alpha\mathbf{A}_1^\times - \overline{\alpha})(1+\mathbf{A}_1^\times)^{-1} & \sqrt{2\mathrm{Re}\,\alpha}\,(1+\mathbf{A}_1^\times)^{-1}\mathbf{B}_1^\times \\ 0 & \sqrt{2\mathrm{Re}\,\alpha}\,\mathbf{C}_1^\times(1+\mathbf{A}_1^\times)^{-1} & \mathbf{D}^\times - \mathbf{C}_1^\times(1+\mathbf{A}_1^\times)^{-1}\mathbf{B}_1^\times \end{bmatrix} \begin{bmatrix} \dot{x}_0(t) \\ x_1(t) \\ y(t) \end{bmatrix}, \tag{59}$$

as a special case of (55).

Again we remark that the internal Cayley transformation maps the class of continuous-time scattering-passive systems bijectively onto the class of discrete-time scattering-passive systems. From the computations above it is quite obvious that the class of systems (24) is the simplest extension of the class of standard systems, which makes the internal Cayley transformation bijective in the scattering case. We illustrate the theoretical bijectivity result with a corollary on partial frequency inversion.

Definition 5.8 *Consider the* continuous-time *extended system* Σ*, in which we have* *pre-split both the forward and the inverse internal channels of the state space:*

$$\begin{bmatrix} x_0(t) \\ x_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ y_1(t) \\ u_0(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & B_0 & 0 \\ 0 & 0 & 0 & 0 & B_1 & 0 \\ 0 & 0 & A_{22} & A_{23} & B_2 & 0 \\ 0 & 0 & A_{32} & A_{33} & B_3 & 0 \\ C_0 & C_1 & C_2 & C_3 & D_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_0(t) \\ \dot{x}_1(t) \\ x_2(t) \\ x_3(t) \\ u_1(t) \\ y_0(t) \end{bmatrix}.$$

The partial frequency inverse Σ_f *of* Σ *is the system that we obtain by swapping* $x_1 \leftrightarrow \dot{x}_1$ *and* $x_2 \leftrightarrow \dot{x}_2$*, whenever this makes sense.*

Let $x_i, \dot{x}_i \in X_i$*. If* $\dim X_0 = \dim X_3 = 0$*, then we call* Σ_f *the* (full) frequency inverse *of* Σ.

The name frequency inversion is justified in the following way. The transfer function of a continuous-time standard system Σ is $\widehat{\mathfrak{D}}(s) = C(s-A)^{-1}B+D$, whereas the transfer function of the full frequency inverse Σ_f is $\widehat{\mathfrak{D}}_f(s) = C(1/s-A)^{-1}B+D$. It is well-known that the transfer function $\widehat{\mathfrak{D}}(s)$ evaluated at $s = if \in i\mathbb{R}$ gives the amplification of the system Σ when we input a periodic signal with frequency f. Then changing from s to $1/s$ means that we change from frequency f to frequency $-1/f$.

Corollary 5.9 *For any continuous-time (impedance- or scattering-) passive system, the (partial) frequency inverse is well-defined, and it is of the same form as the original system.*

Proof: For ease of reading we give the proof only in the case of full frequency inversion. The proof can be extended to the more general case of partial frequency inversion in a straightforward manner.

Consider the internal Cayley transform $\mathbf{\Sigma}$ with parameter $\alpha = 1$ of the continuous-time (scattering or impedance) system Σ. Frequency inversion means exchanging $\left[\begin{smallmatrix} x_1 \\ \dot{x}_2 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} \dot{x}_1 \\ x_2 \end{smallmatrix}\right]$, or equivalently, x and $\dot{x}$. Studying (43):

$$\sqrt{2}\,\mathbf{x}(n+1) = x(t) + \dot{x}(t), \qquad \sqrt{2}\,\mathbf{x}(n) = x(t) - \dot{x}(t),$$

we see that frequency inversion corresponds to swapping $\mathbf{x}(n) \leftrightarrow -\mathbf{x}(n)$ in the internal Cayley transform $\mathbf{\Sigma}$:

$$\begin{aligned} \begin{bmatrix} \mathbf{x}_1(n+1) \\ \mathbf{x}_2(n+1) \\ \mathbf{y}_1(n) \\ \mathbf{u}_0(n) \end{bmatrix} &= \begin{bmatrix} -1 & 0 & \mathbf{B}_1 & 0 \\ 0 & \mathbf{A}_2 & \mathbf{B}_2 & 0 \\ \mathbf{C}_1 & \mathbf{C}_2 & \mathbf{D}_1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -\mathbf{x}_1(n) \\ -\mathbf{x}_2(n) \\ \mathbf{u}_1(n) \\ \mathbf{y}_0(n) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & \mathbf{B}_1 & 0 \\ 0 & -\mathbf{A}_2 & \mathbf{B}_2 & 0 \\ -\mathbf{C}_1 & -\mathbf{C}_2 & \mathbf{D}_1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1(n) \\ \mathbf{x}_2(n) \\ \mathbf{u}_1(n) \\ \mathbf{y}_0(n) \end{bmatrix}. \end{aligned}$$

We conclude that the internal Cayley transform $\mathbf{\Sigma}_f$ of Σ_f is of the same form as the internal Cayley transform $\mathbf{\Sigma}$ of Σ (with no X_0 forming an inverted internal channel on the continuous-time side), except that $1+(\mathbf{A}_2)_f = 1-\mathbf{A}_2$ might not be invertible. After splitting off $X_0 := \mathcal{N}(1-\mathbf{A}_2)$ from X_2 as in (54), we have put $\mathbf{\Sigma}_f$ in the form of $\mathbf{\Sigma}$.

Thus the frequency-inverted system Σ_f can be written as the inverse internal Cayley transform of the system $\mathbf{\Sigma}(-\mathbf{A}, \mathbf{B}, -\mathbf{C}, \mathbf{D})$ with parameter 1, which, by Theorem 5.6, is of the same form as Σ.

Moreover, (impedance- or scattering-) passivity is invariant under frequency inversion, as is seen from the frequency-inversion invariant expression (30) in the proof of Theorem 3.3. ■

We remark that Corollary 5.9 does *not* state that the splittings of X into an inverted and a forward internal channel (on the continuous-time side) are the same for the original system and the frequency-inverted system. To see that this claim is false, note that in the original system $X_0 = \mathcal{N}(1+\mathbf{A})$, whereas for the frequency-inverted system, $X_0 = \mathcal{N}(1-\mathbf{A})$.

6 The External Cayley Transformation

The purpose of the external Cayley transform is transforming an impedance-passive (continuous- or discrete-time) system into a (continuous- or discrete-time) scattering-passive system. We treat both the continuous-time and the discrete-time case. The transformation is referred to as the "diagonal transformation" in [St1], to which the reader is referred for more details on the standard case. A similar idea is utilised in the input/output setting in [B, pp. 162–165].

The external Cayley transformation is illustrated in Figure 2, where one can also note that the transformation essentially is a feedback connection. The transformation maps positive-real transfer functions into bounded-real transfer functions, as is explained in Theorem 6.3.

In the present work, we generalise the external Cayley transformation to extended i/s/o systems as well as show that the transformation is bijective in our setting. We proceed in the same manner as with the internal Cayley transformation, i.e., by giving an abstract definition of the external Cayley transform.

Definition 6.1 *The* external Cayley transform with parameter $\beta \in \mathbb{C}_+$ *of a system Σ with input u, state x and output y is the (well-defined) system $\Sigma^\times$, whose input and output signals are given by*

$$u^\times = \frac{1}{\sqrt{2\mathrm{Re}\,\beta}}(\beta u + y) \quad \text{and} \quad y^\times = \frac{1}{\sqrt{2\mathrm{Re}\,\beta}}(\overline{\beta} u - y), \qquad \textit{respectively.} \tag{60}$$

It is easy to see that (60) can be written as

$$u = \frac{1}{\sqrt{2\mathrm{Re}\,\beta}}(u^\times + y^\times) \quad \text{and} \quad y = \frac{1}{\sqrt{2\mathrm{Re}\,\beta}}(\overline{\beta} u^\times - \beta y^\times). \tag{61}$$

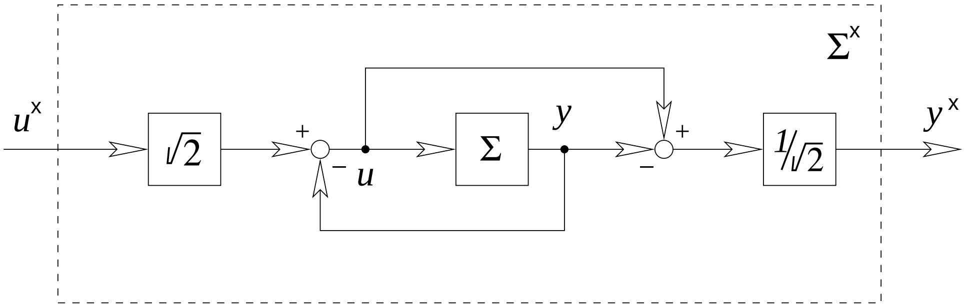

Figure 2: A schematic of the external Cayley transformation in the case $\beta = 1$. Note that the output $-y$ of Σ is fed back into the input of Σ.

From (61) it follows that $2\mathrm{Re}\,\langle u, y\rangle = \|u^\times\|^2 - \|y^\times\|^2$, which illustrates the idea of the external Cayley transform, i.e., that it often maps an impedance-passive/conservative (continuous- or discrete-time) system into a scattering-passive/conservative system. In the next section, we compute the external Cayley transform of continuous impedance-passive systems.

6.1 The Continuous-Time Case

By theorem 3.3, the general form of a continuous-time impedance-passive system is Σ:

$$\begin{bmatrix} x_0 \\ \dot{x}_1 \\ y_1 \\ u_0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & B_0 & 0 \\ 0 & A_1 & B_1 & 0 \\ B_0^* & C_1 & D_1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_0 \\ x_1 \\ u_1 \\ y_0 \end{bmatrix}.$$

If $B_0 = 0$, then Σ has only trivial inverted channels. This case is easily handled and quite uninteresting. If $B_0 \neq 0$, then split X_0 into $\big(\mathcal{R}(B_0)\big)^\perp \oplus \mathcal{R}(B_0)$ and U_1 into $\mathcal{N}(B_0) \oplus \big(\mathcal{N}(B_0)\big)^\perp$. Then $B_0 = \left[\begin{smallmatrix} 0 & 0 \\ 0 & B_{11} \end{smallmatrix}\right]$, where B_{11} is a square and invertible matrix. (The surjectivity of B_{11} is obtained from the splitting of X_0 and the injectivity is obtained from the splitting of U_1.) The corresponding splitting of the full system is (after a renumbering of the signals):

$$\begin{bmatrix} x_0 \\ x_1 \\ \dot{x}_2 \\ y_2 \\ y_1 \\ u_0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B_{11} & 0 \\ 0 & 0 & A_{22} & B_{22} & B_{21} & 0 \\ 0 & 0 & C_{22} & D_{22} & D_{21} & 0 \\ 0 & B_{11}^* & C_{12} & D_{12} & D_{11} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_0 \\ \dot{x}_1 \\ x_2 \\ u_2 \\ u_1 \\ y_0 \end{bmatrix}, \qquad B_{11} \text{ invertible.} \tag{62}$$

We have renumbered the subspaces of X and U, so that $x_i \in X_i$ and $u_i, y_i \in U_i$. In particular, $x_0(t), \dot{x}_0(t) = 0$ for all t. In the remainder of the paper we throughout denote $W_\beta = \mathrm{diag}(1, 1, 1, \sqrt{2\mathrm{Re}\,\beta}, \sqrt{2\mathrm{Re}\,\beta}, \sqrt{2\mathrm{Re}\,\beta}) \in \mathcal{L}(X_0 \oplus X_1 \oplus X_2 \oplus U_2 \oplus U_1 \oplus U_0)$. Trivially W_β is invertible.

We note that $\beta + D_{22}$ is invertible for all $\beta \in \mathbb{C}_+$, following from $D_{22} + D_{22}^* \geq 0$. Then we substitute (61) into (62), which yields that

$$W_\beta \begin{bmatrix} x_0 \\ \dot{x}_1 \\ \dot{x}_2 \\ y_2^\times \\ y_1^\times \\ y_0^\times \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{22} & B_{22} & B_{21} & 0 \\ 0 & 0 & -C_{22} & \overline{\beta} - D_{22} & -D_{21} & 0 \\ 0 & -B_{11}^* & -C_{12} & -D_{12} & \overline{\beta} - D_{11} & 0 \\ 0 & 0 & 0 & 0 & 0 & \overline{\beta} \end{bmatrix} \begin{bmatrix} \dot{x}_0 \\ \dot{x}_1 \\ x_2 \\ u_2 \\ u_1 \\ y_0 \end{bmatrix} \tag{63}$$

and

$$W_\beta \begin{bmatrix} \dot{x}_0 \\ x_1 \\ x_2 \\ u_2^\times \\ u_1^\times \\ u_0^\times \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B_{11} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & C_{22} & \beta + D_{22} & D_{21} & 0 \\ 0 & B_{11}^* & C_{12} & D_{12} & \beta + D_{11} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x}_0 \\ \dot{x}_1 \\ x_2 \\ u_2 \\ u_1 \\ y_0 \end{bmatrix}. \tag{64}$$

We denote the block matrix on the right-hand sides of (63) and (64) by $P^\times$ and $Q^\times$, respectively.

From the invertibility of B_{11} and $\beta + D_{22}$, we see that $Q^\times$ is a full-rank (square) matrix. The explicit expression for the inverse is manageable, but not very nice, and therefore we do not give give it. We have the following theorem.

Theorem 6.2 *The external Cayley transform with parameter $\beta \in \mathbb{C}_+$, of the impedance-passive continuous-time system Σ in* (62), *is the scattering passive continuous-time system $\Sigma^\times$:*

$$\begin{bmatrix} x_0(t) \\ \dot{x}_1(t) \\ \dot{x}_2(t) \\ y_2^\times(t) \\ y_1^\times(t) \\ y_0^\times(t) \end{bmatrix} = W_\beta^{-1}(P_1^\times + P_2^\times \Sigma)(Q_1^\times + Q_2^\times \Sigma)^{-1} W_\beta \begin{bmatrix} \dot{x}_0(t) \\ x_1(t) \\ x_2(t) \\ u_2^\times(t) \\ u_1^\times(t) \\ u_0^\times(t) \end{bmatrix}, \tag{65}$$

with parameter matrices $P_1^\times = \operatorname{diag}(0,1,0,\overline{\beta},\overline{\beta},-1)$, $P_2^\times = \operatorname{diag}(1,0,1,-1,-1,\overline{\beta})$, $Q_1^\times = \operatorname{diag}(1,0,1,\beta,\beta,1)$ *and* $Q_2^\times = \operatorname{diag}(0,1,0,1,1,\beta)$.

The external Cayley transform $\Sigma^\times$ can be written in the form

$$\begin{bmatrix} x_0(t) \\ \dot{x}_1(t) \\ \dot{x}_2(t) \\ y_2^\times(t) \\ y_1^\times(t) \\ y_0^\times(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_{11}^\times & A_{12}^\times & B_{12}^\times & B_{11}^\times & 0 \\ 0 & A_{21}^\times & A_{22}^\times & B_{22}^\times & 0 & 0 \\ 0 & C_{21}^\times & C_{22}^\times & D_{22}^\times & 0 & 0 \\ 0 & (B_{11}^\times)^* & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \dot{x}_0(t) \\ x_1(t) \\ x_2(t) \\ u_2^\times(t) \\ u_1^\times(t) \\ u_0^\times(t) \end{bmatrix}. \tag{66}$$

Denoting $B_{11}^{-*} = (B_{11}^{-1})^*$, we have $B_{11}^\times = \sqrt{2\mathrm{Re}\,\beta}\, B_{11}^{-*}$ and so $B_{11}^\times$ is invertible. Furthermore, the central 2×2 blocks of (66) *are the same as with the standard forward* *transform:*

$$\begin{bmatrix} A_{22}^\times & B_{22}^\times \\ C_{22}^\times & D_{22}^\times \end{bmatrix} = \begin{bmatrix} A_{22} - B_{22}(\beta + D_{22})^{-1}C_{22} & \sqrt{2\mathrm{Re}\,\beta}\, B_{22}(\beta + D_{22})^{-1} \\ -\sqrt{2\mathrm{Re}\,\beta}\,(\beta + D_{22})^{-1}C_{22} & (\overline{\beta} - D_{22})(\beta + D_{22})^{-1} \end{bmatrix}. \tag{67}$$

In particular, $D_{22}^\times = (\overline{\beta} - D_{22})(\beta + D_{22})^{-1}$, *and thus* $1 + D_{22}^\times$ *is also invertible. The remaining five blocks of* (66) *describe how the standard and extended parts connect.*

The transform $\Sigma^\times$ *is scattering conservative if and only if the original system* Σ *is impedance conservative.*

We remark that in Theorem 6.2, the first and last diagonal elements of $P_2^\times$ and $Q_2^\times$ are arbitrary. We make this particular choice only in order to illustrate a certain symmetry between the parameter matrices.

Proof of Theorem 6.2:
Most of the claims have already been dealt with. The correctness of the parameter matrices is easily verified. We omit the straightforward, but lenghty, proof of the form of the blocks in (66). After proving the passivity claims we are done.

Let $(u^\times, x, y^\times)$ be a trajectory of the transform $\Sigma^\times$. Define u and y by (61). Then, by definition of the external Cayley transformation, (u, x, y) is a trajectory of the original system Σ. Moreover, for all $t \geq 0$ we have $2\mathrm{Re}\,\langle u(t), y(t)\rangle = \|u^\times(t)\|^2 - \|y^\times(t)\|^2$. Since Σ is forward impedance passive, for all $t \geq 0$ we have

$$\|x(t)\|^2 - \|x(0)\|^2 \leq 2\mathrm{Re} \int_0^t \langle u(s), y(s)\rangle\, \mathrm{d}s = \int_0^t \|u^\times(s)\|^2 - \|y^\times(s)\|^2\, \mathrm{d}s,$$

i.e., $\Sigma^\times$ is forward scattering passive. By item 2 of Theorem 3.3, the system (66) is forward scattering passive iff it is scattering passive, because in (66), the blocks corresponding to B_0 and C_0 are both zero.

The conservative case is now trivial and the converse direction is analogous. ■

Theorem 6.3 *Let* Σ *be impedance passive and denote by* $p_i^\times$ *and* $q_i^\times$ *the* 3×3 *bottom-right corner blocks of* $P_i^\times$ *and* $Q_i^\times$*, respectively, i.e.,* $p_1^\times = \mathrm{diag}(\overline{\beta}, \overline{\beta}, -1)$*,* $p_2^\times = \mathrm{diag}(-1, -1, \overline{\beta})$*,* $q_1^\times = \mathrm{diag}(\beta, \beta, 1)$ *and* $q_2^\times = \mathrm{diag}(1, 1, \beta)$.

If $\widehat{\mathfrak{D}}$ *is the positive-real transfer function of the system* (62)*, then the transfer function* $\widehat{\mathfrak{D}}^\times$ *of* (65) *is given by*

$$\forall s \in \widehat{\mathfrak{D}}^\times: \quad \widehat{\mathfrak{D}}^\times(s) = \left(p_1^\times + p_2^\times \widehat{\mathfrak{D}}(s)\right)\left(q_1^\times + q_2^\times \widehat{\mathfrak{D}}(s)\right)^{-1}, \tag{68}$$

where $\rho(A_{22}) \cap \mathbb{C}_+ \subset \mathrm{dom}\,(\widehat{\mathfrak{D}}^\times) = \left\{s \in \rho(A_{22}) : \left(q_1^\times + q_2^\times \widehat{\mathfrak{D}}(s)\right) \text{ is invertible}\right\}$.

Moreover, $\widehat{\mathfrak{D}}^\times$ *is contractive on* $\overline{\mathbb{C}_+} \cap \mathrm{dom}\,(\widehat{\mathfrak{D}}^\times)$*. If* Σ *is impedance conservative, then* $\widehat{\mathfrak{D}}^\times$ *is unitary on the imaginary axis.*

Proof: In analogy to the derivation of the transform of the system, for the transfer function we obtain

$$\sqrt{2\mathrm{Re}\,\beta} \begin{bmatrix} \widehat{y_2} \\ \widehat{y_1} \\ \widehat{y_0} \end{bmatrix} = \left(p_1^\times + p_2^\times \widehat{\mathfrak{D}}\right) \begin{bmatrix} \widehat{u_2} \\ \widehat{u_1} \\ \widehat{y_0} \end{bmatrix}, \quad \sqrt{2\mathrm{Re}\,\beta} \begin{bmatrix} \widehat{u_2} \\ \widehat{u_1} \\ \widehat{u_0} \end{bmatrix} = \left(q_1^\times + q_2^\times \widehat{\mathfrak{D}}\right) \begin{bmatrix} \widehat{u_2} \\ \widehat{u_1} \\ \widehat{y_0} \end{bmatrix}.$$

We thus have (68) for any $s \in \operatorname{dom}(\widehat{\mathfrak{D}}^\times)$.

Theorem 5.4 says that $\widehat{\mathfrak{D}}$ is positive real on $\mathbb{C}_+$. Moreover, $q_2^\times$ is invertible and $(q_2^\times)^{-1}q_1^\times + (q_1^\times)^*(q_2^\times)^{-*} = (2\mathrm{Re}\,\beta)\operatorname{diag}(1,1,1/|\beta|^2) > 0$, so $q_1^\times + q_2^\times\widehat{\mathfrak{D}}$ is invertible, by Lemma 4.1. Therefore $q_1^\times + q_2^\times\widehat{\mathfrak{D}}(s)$ is invertible for any $s \in \rho(A_{22}) \cap \mathbb{C}_+$.

Moreover, $(q_1^\times)^*q_1^\times = (p_1^\times)^*p_1^\times$, $(q_2^\times)^*q_2^\times = (p_2^\times)^*p_2^\times$ and $(q_1^\times)^*q_2^\times - (p_1^\times)^*p_2^\times = (2\mathrm{Re}\,\beta)I_U$, with $2\mathrm{Re}\,\beta \in \mathbb{R}_+$, i.e., positive realness (skew-adjointness) of $\widehat{\mathfrak{D}}$ is equivalent to contractivity (unitarity) of $\widehat{\mathfrak{D}}^\times$, by Lemma 4.1. ■

We now show how the external Cayley transformation is inverted. Starting from the arbitrary scattering passive system

$$\begin{bmatrix} x_0(t) \\ \dot{x}_1(t) \\ y^\times(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_1^\times & B_1^\times \\ 0 & C_1^\times & D^\times \end{bmatrix} \begin{bmatrix} \dot{x}_0(t) \\ x_1(t) \\ u^\times(t) \end{bmatrix}, \tag{69}$$

split $U = \mathcal{N}(1+D^\times)^\perp \oplus \mathcal{N}(1+D^\times)$, in order to write the contractive feed-through operator in the form $D^\times = \operatorname{diag}(D_{11}^\times, -1)$, according to Lemma 3.5. Let the corresponding splittings of $B_1^\times$ and $C_1^\times$ be $B_1^\times = \begin{bmatrix} B_{11}^\times & B_{10}^\times \end{bmatrix}$ and $C_1^\times = \begin{bmatrix} C_{11}^\times \\ C_{01}^\times \end{bmatrix}$, respectively.

Taking $\lambda = -1$ in Corollary 3.6, we conclude that $C_{01}^\times = (B_{10}^\times)^*$. Further split X_1 into $\mathcal{R}(B_{10}^\times) \oplus \mathcal{R}(B_{10}^\times)^\perp$ and $\mathcal{N}(1+D^\times)$ into $\mathcal{N}(B_{10}^\times)^\perp \oplus \mathcal{N}(B_{10}^\times)$. After a renumbering of signals and spaces, $\Sigma^\times$ is in the form (66), where (the renumbered blocks) $B_{11}^\times$ and $1 + D_{22}^\times$ are square and invertible.

By combining (66) with (61), one can derive the inverse external Cayley transform utilising the same type of calculations as we have done with the preceding transformations. We arrive at the following theorem, which we give without a full proof.

Theorem 6.4 *Consider the system matrix $\Sigma^\times$ of (66). Let $P_1 = \operatorname{diag}(0,1,0,\overline{\beta},\overline{\beta},1)$, $P_2 = \operatorname{diag}(1,0,1,-\beta,-\beta,1)$, $Q_1 = \operatorname{diag}(1,0,1,1,1,\overline{\beta})$, $Q_2 = \operatorname{diag}(0,1,0,1,1,-\beta)$.*

Then $Q_1 + Q_2\Sigma^\times$ is invertible for every $\beta \in \mathbb{C}_+$ and the inverse external Cayley transform with parameter β of $\Sigma^\times$ is given by $\Sigma = W_\beta^{-1}(P_1 + P_2\Sigma^\times)(Q_1 + Q_2\Sigma^\times)^{-1}W_\beta$. The obtained inverse transform is of the form (62), where in particular $D_{22} = (\overline{\beta} - \beta D_{22}^\times)(1 + D_{22}^\times)^{-1}$ and $B_{11} = (B_{11}^\times)^{-*}/\sqrt{2\mathrm{Re}\,\beta}$, i.e., $\beta + D_{22}$ and B_{11} are invertible.

If $\Sigma^\times$ is (scattering) conservative, then Σ is (impedance) conservative.

Denote the transfer function of $\Sigma^\times$ by $\widehat{\mathfrak{D}}^\times$ and let $p_1 = \operatorname{diag}(\overline{\beta},\overline{\beta},1)$, $p_2 = \operatorname{diag}(-\beta,-\beta,1)$, $q_1 = \operatorname{diag}(1,1,\overline{\beta})$ and $q_2 = \operatorname{diag}(1,1,-\beta)$. Then the transfer function $\widehat{\mathfrak{D}}$ of Σ is

$$\forall s \in \rho(A^\times): \qquad \widehat{\mathfrak{D}}(s) = \big(p_1 + p_2\widehat{\mathfrak{D}}(s)^\times\big)\big(q_1 + q_2\widehat{\mathfrak{D}}(s)^\times\big)^{-1}.$$

If $1 \in \rho(-D)$, e.g. if $\Sigma^\times$ is the external Cayley transform of a standard system

Σ, then no splitting is required and the standard case is recovered as

$$\begin{aligned}\Sigma &= W_\beta^{-1}\begin{bmatrix} A^\times & B^\times \\ -\beta C^\times & \overline{\beta}-\beta D^\times \end{bmatrix}\begin{bmatrix} 1 & 0 \\ C^\times & 1+D^\times \end{bmatrix}^{-1} W_\beta \\ &= \begin{bmatrix} A^\times - B^\times(1+D^\times)^{-1}C^\times & \sqrt{2\mathrm{Re}\,\beta}\, B^\times(1+D^\times)^{-1} \\ -\sqrt{2\mathrm{Re}\,\beta}\,(1+D^\times)^{-1}C^\times & (\overline{\beta}-\beta D^\times)(1+D^\times)^{-1} \end{bmatrix}.\end{aligned} \tag{70}$$

We remark that the external Cayley transformation maps our class of continoustime extended impedance-passive systems bijectively onto the class of continuoustime extended scattering-passive systems.

6.2 The Discrete-Time Case

We now briefly investigate the discrete-time impedance-passive system

$$\begin{bmatrix} \mathbf{x}(n+1) \\ \mathbf{y}_1(n) \\ \mathbf{u}_0(n) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B}_1 & 0 \\ \mathbf{C}_1 & \mathbf{D}_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \mathbf{x}(n) \\ \mathbf{u}_1(n) \\ \mathbf{y}_0(n) \end{bmatrix},$$

where, by Theorem 3.3, necessarily $\mathbf{D}_1 + \mathbf{D}_1^* \geq \mathbf{B}_1^*\mathbf{B}_1 \geq 0$, and therefore $\beta + \mathbf{D}_1$ is invertible for every $\beta \in \mathbb{C}_+$, by Corollary 4.2. In the discrete-time case, the weight matrix W_β is $W_\beta = \mathrm{diag}(1, \sqrt{2\mathrm{Re}\,\beta}, \sqrt{2\mathrm{Re}\,\beta})$.

The computations leading up to Theorem 6.2 can equally well be performed for discrete-time systems and we deduce that the external Cayley transform of $\mathbf{\Sigma}$ is given by $\begin{bmatrix} \mathbf{x}(n+1) \\ \mathbf{y}_1^\times(n) \\ \mathbf{y}_0^\times(n) \end{bmatrix} = \mathbf{\Sigma}^\times \begin{bmatrix} \mathbf{x}(n) \\ \mathbf{u}_1^\times(n) \\ \mathbf{u}_0^\times(n) \end{bmatrix}$, where

$$\begin{aligned}\mathbf{\Sigma}^\times &= W_\beta^{-1}(\mathbf{P}_1^\times + \mathbf{P}_2^\times\mathbf{\Sigma})(\mathbf{Q}_1^\times + \mathbf{Q}_2^\times\mathbf{\Sigma})^{-1}W_\beta \\ &= \begin{bmatrix} \mathbf{A} - \mathbf{B}_1(\beta+\mathbf{D}_1)^{-1}\mathbf{C}_1 & \sqrt{2\mathrm{Re}\,\beta}\,\mathbf{B}_1(\beta+\mathbf{D}_1)^{-1} & 0 \\ -\sqrt{2\mathrm{Re}\,\beta}\,(\beta+\mathbf{D}_1)^{-1}\mathbf{C}_1 & (\overline{\beta}-\mathbf{D}_1)(\beta+\mathbf{D}_1)^{-1} & 0 \\ 0 & 0 & -1 \end{bmatrix},\end{aligned}$$

with $1 + (\overline{\beta} - \mathbf{D}_1)(\beta + \mathbf{D}_1)^{-1}$ invertible. In this case the parameter matrices are $\mathbf{P}_1^\times = \mathrm{diag}(0, \overline{\beta}, -1)$, $\mathbf{P}_2^\times = \mathrm{diag}(1, -1, \overline{\beta})$, $\mathbf{Q}_1^\times = \mathrm{diag}(1, \beta, 1)$ and $\mathbf{Q}_2^\times = \mathrm{diag}(0, 1, \beta)$. If $\mathbf{\Sigma}$ is impedance passive (conservative) then $\mathbf{\Sigma}^\times$ is scattering passive (conservative).

If the transfer function of $\mathbf{\Sigma}$ is $\widehat{\mathbb{D}}(z) = \begin{bmatrix} \widehat{\mathbb{D}}_1(z) & 0 \\ 0 & 0 \end{bmatrix}$, then the transfer function of $\mathbf{\Sigma}^\times$ is $\widehat{\mathbb{D}}^\times(z) = \begin{bmatrix} (\overline{\beta}-\widehat{\mathbb{D}}_1(z))(\beta+\widehat{\mathbb{D}}_1(z))^{-1} & 0 \\ 0 & -1 \end{bmatrix}$. Moreover, $\widehat{\mathbb{D}}$ is positive-real and $\widehat{\mathbb{D}}^\times$ is contractive on $\mathbb{D}_+$. If, in addition, the systems are conservative, then $\widehat{\mathbb{D}}$ is skew-adjoint and $\widehat{\mathbb{D}}^\times$ unitary on the complex unit circle $\mathbb{D}$.

In inverting the transformation we start from a standard discrete-time scattering-passive system $\mathbf{\Sigma}^\times$:

$$\begin{bmatrix} \mathbf{x}(n+1) \\ \mathbf{y}^\times(n) \end{bmatrix} = \begin{bmatrix} \mathbf{A}^\times & \mathbf{B}^\times \\ \mathbf{C}^\times & \mathbf{D}^\times \end{bmatrix}\begin{bmatrix} \mathbf{x}(n) \\ \mathbf{u}^\times(n) \end{bmatrix}.$$

Here $\mathbf{\Sigma}^\times$ is a contraction and, by factoring out a maximal eigenvalue -1 of $\mathbf{D}^\times$, we bring $\mathbf{\Sigma}^\times$ to the form

$$\begin{bmatrix} \mathbf{x}(n+1) \\ \mathbf{y}_1^\times(n) \\ \mathbf{y}_0^\times(n) \end{bmatrix} = \begin{bmatrix} \mathbf{A}^\times & \mathbf{B}_1^\times & 0 \\ \mathbf{C}_1^\times & \mathbf{D}_1^\times & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \mathbf{x}(n) \\ \mathbf{u}_1^\times(n) \\ \mathbf{u}_0^\times(n) \end{bmatrix}, \quad -1 \in \rho(\mathbf{D}_1^\times).$$

In analogy to Theorem 6.4, the inverse external Cayley transform of $\mathbf{\Sigma}^\times$ is the impedance-passive discrete-time system $\begin{bmatrix} \mathbf{x}(n+1) \\ \mathbf{y}_1(n) \\ \mathbf{u}_0(n) \end{bmatrix} = \mathbf{\Sigma} \begin{bmatrix} \mathbf{x}(n) \\ \mathbf{u}_1(n) \\ \mathbf{y}_0(n) \end{bmatrix}$, whose system matrix is

$$\mathbf{\Sigma} = \begin{bmatrix} \mathbf{A}^\times - \mathbf{B}_1^\times(1+\mathbf{D}_1^\times)^{-1}\mathbf{C}_1^\times & \sqrt{2\mathrm{Re}\,\beta}\,\mathbf{B}_1^\times(1+\mathbf{D}_1^\times) & 0 \\ -\sqrt{2\mathrm{Re}\,\beta}\,(1+\mathbf{D}_1^\times)^{-1}\mathbf{C}_1^\times & (\overline{\beta}-\beta\mathbf{D}_1^\times)(1+\mathbf{D}_1^\times)^{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

In this case we have $\mathbf{P}_1 = \mathrm{diag}(0, \overline{\beta}, 1)$, $\mathbf{P}_2 = \mathrm{diag}(1, -\beta, 1)$, $\mathbf{Q}_1 = \mathrm{diag}(1, 1, \overline{\beta})$ and $\mathbf{Q}_2 = \mathrm{diag}(0, 1, -\beta)$. The inverse transform $\mathbf{\Sigma}$ is impedance conservative if $\mathbf{\Sigma}^\times$ is scattering conservative.

Letting $\widehat{\mathbb{D}}^\times(z) = \begin{bmatrix} \widehat{\mathbb{D}}_1^\times(z) & 0 \\ 0 & -1 \end{bmatrix}$ be the transfer function of $\mathbf{\Sigma}^\times$, we obtain that $\widehat{\mathbb{D}}(z) = \begin{bmatrix} (\overline{\beta}-\beta\widehat{\mathbb{D}}_1^\times(z))(1+\widehat{\mathbb{D}}_1^\times(z))^{-1} & 0 \\ 0 & 0 \end{bmatrix}$ is the transfer function of $\mathbf{\Sigma}$.

Remark 6.5 Also here the external Cayley transformation is bijective. Thereby the proof of Theorem 3.7 is complete, since we now know that any scattering-passive system is the external Cayley transform of some impedance-passive system.

In fact, we extended the class of discrete-time standard systems in the simplest possible way that makes the discrete-time external Cayley transform bijective. Moreover, (44) is the simplest extension of the class of continuous-time impedance-passive systems that makes all Cayley transformations bijective. This is a very nice result, taking into account that we originally introduced the system (44) for its realisation capabilities.

We illustrate the bijectivity result with a corollary on (partial) flow inversion.

Definition 6.6 *Consider the extended (continuous- or discrete-time) system Σ, in which we have pre-split the forward and inverse external channels:*

$$\begin{bmatrix} x_0 \\ \dot{x}_1 \\ y_3 \\ y_2 \\ u_1 \\ u_0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & B_{03} & B_{02} & 0 & 0 \\ 0 & A_1 & B_{13} & B_{12} & 0 & 0 \\ C_{30} & C_{31} & D_{33} & D_{32} & 0 & 0 \\ C_{20} & C_{21} & D_{23} & D_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_0 \\ x_1 \\ u_3 \\ u_2 \\ y_1 \\ y_0 \end{bmatrix}.$$

Let $u_i, y_i \in U_i$. The partial flow inverse $\Sigma_\leftarrow$ *of Σ is the system that we obtain by interpreting y_1, u_2 as outputs and y_1, u_2 as inputs, i.e. by reversing the direction of the flow through Σ on U_1 and U_2, whenever this makes sense.*

If $\dim U_0 = \dim U_3 = 0$*, then we call $\Sigma_\leftarrow$ the* (full) flow inverse *of* Σ.

The following corollary is a dual result of Corollary 5.9.

Corollary 6.7 *Any partial flow inverse of an extended impedance-passive (continuous- or discrete-time) system is well defined, and the (partial) flow inverse is of the same kind as the original system.*

The particularly interesting class of extended continuous-time impedance-passive systems is invariant under both partial frequency inversion and partial flow inversion. On this class of systems, both partial frequency inversion and flow inversion always yield meaningful inverted systems.

Again we remark that the "canonical" splittings of the input/output space U, into an inverted and a forward external channel, of the original system and the (partially) flow-inverted system may differ.

We give the following example, which indicates that allowing an inverted internal channel can be useful also when originally working with standard systems.

Example 6.8 One can easily show that a standard system

$$\begin{bmatrix} \dot{x} \\ y \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}$$

can be flow inverted in the traditional sense, i.e., be written on the form

$$\begin{bmatrix} \dot{x} \\ u \end{bmatrix} = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

if and only if D is invertible. (The proof is a simpler analogue of this example.)

In this example we consider the standard-type impedance-passive system

$$\begin{bmatrix} \dot{x} \\ y \end{bmatrix} = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}, \tag{71}$$

that is obviously not flow-invertible in the traditional sense. Lemma 3.4 applied to the impedance-passivity condition $\left[\begin{smallmatrix} A & B \\ -C & 0 \end{smallmatrix}\right] + \left[\begin{smallmatrix} A & B \\ -C & 0 \end{smallmatrix}\right]^* \leq 0$ implies that $C = B^*$. Furthermore, for ease of formulation, we make the very non-restrictive assumption that U has lowest possible dimension, i.e., that B is injective making C surjective.

We split $X := X_0 \oplus X_1$, where $X_0 = \mathcal{N}(C)$ and $X_1 = X_0^\perp$. Then (71) becomes

$$\begin{bmatrix} \dot{x}_0 \\ \dot{x}_1 \\ y \end{bmatrix} = \begin{bmatrix} A_{00} & A_{01} & C_0^* \\ A_{10} & A_{11} & 0 \\ C_0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ u \end{bmatrix}, \tag{72}$$

with C_0 invertible. Noting that

$$\begin{bmatrix} x_0 \\ \dot{x}_1 \\ u \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ A_{10} & A_{11} & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ u \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \dot{x}_0 \\ x_1 \\ y \end{bmatrix} = \begin{bmatrix} A_{00} & A_{01} & C_0^* \\ 0 & I & 0 \\ C_0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ u \end{bmatrix},$$

we obtain

$$\begin{bmatrix} x_0 \\ \dot{x}_1 \\ u \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ A_{10} & A_{11} & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} 0 & 0 & C_0^{-1} \\ 0 & I & 0 \\ C_0^{-*} & -C_0^{-*}A_{01} & -C_0^{-*}A_{00}C_0^{-1} \end{bmatrix} \begin{bmatrix} \dot{x}_0 \\ x_1 \\ y \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & C_0^{-1} \\ 0 & A_{11} & A_{10}C_0^{-1} \\ C_0^{-*} & -C_0^{-*}A_{01} & -C_0^{-*}A_{00}C_0^{-1} \end{bmatrix} \begin{bmatrix} \dot{x}_0 \\ x_1 \\ y \end{bmatrix},$$

which is a well-defined impedance-passive extended continuous-time system.

7 Realisation of rational relations

In Section 2, we claimed that any positive-real right-coprime rational relation on $\Omega = \overline{\mathbb{C}_+} \cup \{\infty\}$ over $\left[\begin{smallmatrix} U \\ U \end{smallmatrix}\right]$ can be realised by a continuous-time impedance-passive extended system. We now conclude the paper by proving this claim.

Lemma 7.1 *Let R be a right-coprime positive-real rational relation on Ω over $\left[\begin{smallmatrix} U \\ U \end{smallmatrix}\right]$. Then R has a unique right-coprime factorisation (P_1, Q_1), such that $P_1(s)+Q_1(s) = 1$ for all $s \in \Omega$. Moreover, $P_1(s)$ and $Q_1(s)$ are contractions on U for all $s \in \Omega$.*

Proof: We start by proving existence of P_1 and Q_1. For any $s \in \Omega$, $P(s) + Q(s)$ is injective, and thus invertible, which we now show. We have $\mathrm{Re}\,\langle y, u\rangle \geq 0$ for all $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in R(s)$ by positive-realness, and thus

$$\left\|\big(P(s) + Q(s)\big)v\right\|^2 = \|P(s)v\|^2 + 2\mathrm{Re}\,\langle P(s)v, Q(s)v\rangle + \|Q(s)v\|^2$$
$$\geq \|P(s)v\|^2 + \|Q(s)v\|^2 = \left\| \begin{bmatrix} P(s) \\ Q(s) \end{bmatrix} v \right\|^2.$$

The right-hand side is zero only if $v \in \mathcal{N}\left(\left[\begin{smallmatrix} P(s) \\ Q(s) \end{smallmatrix}\right]\right) = \{0\}$, by right coprimeness. We have now established that the (square) matrix $P(s) + Q(s)$ maps U one-to-one onto itself for all $s \in \Omega$. Thus, we have

$$R(s) = \begin{bmatrix} P(s) \\ Q(s) \end{bmatrix} U = \begin{bmatrix} P(s) \\ Q(s) \end{bmatrix} \big(P(s) + Q(s)\big)^{-1} U = \begin{bmatrix} P_1(s) \\ Q_1(s) \end{bmatrix} U,$$

with $P_1(s) = P(s)\big(P(s) + Q(s)\big)^{-1}$ and $Q_1(s) = Q(s)\big(P(s) + Q(s)\big)^{-1}$. Obviously, $P_1(s) + Q_1(s) = 1$ for all $s \in \Omega$.

Now over to uniqueness. Let both (P_1, Q_1) and (P_1', Q_1') be right-coprime factorisations of R satisfying $P_1(s) + Q_1(s) = 1$ and $P_1'(s) + Q_1'(s) = 1$ for all $s \in \Omega$. Fix $s \in \Omega$ and note that $\left[\begin{smallmatrix} P_1(s) \\ Q_1(s) \end{smallmatrix}\right]$ is injective by coprimeness, thus having a left inverse T_s. One easily shows that $P_s := \left[\begin{smallmatrix} P_1(s) \\ Q_1(s) \end{smallmatrix}\right] T_s$ is a projection onto $\mathcal{R}\left(\left[\begin{smallmatrix} P_1(s) \\ Q_1(s) \end{smallmatrix}\right]\right)$ and, moreover, that P_s acts as an identity on $\mathcal{R}\left(\left[\begin{smallmatrix} P_1(s) \\ Q_1(s) \end{smallmatrix}\right]\right) = R(s) = \mathcal{R}\left(\left[\begin{smallmatrix} P_1'(s) \\ Q_1'(s) \end{smallmatrix}\right]\right)$. Then

$$\begin{bmatrix} P_1'(s) \\ Q_1'(s) \end{bmatrix} = P_s \begin{bmatrix} P_1'(s) \\ Q_1'(s) \end{bmatrix} = \begin{bmatrix} P_1(s) \\ Q_1(s) \end{bmatrix} T_s \begin{bmatrix} P_1'(s) \\ Q_1'(s) \end{bmatrix} =: \begin{bmatrix} P_1(s) \\ Q_1(s) \end{bmatrix} V_s$$

and, moreover,

$$1 = P_1'(s) + Q_1'(s) = P_1(s)V_s + Q_1(s)V_s = V_s,$$

implying that $V_s = 1$, i.e. that $P_1 = P_1'$ and $Q_1 = Q_1'$.

It still remains to show that $P_1(s)$ and $Q_1(s)$ are contractions. Due to positive realness of R, we have $\mathrm{Re}\,\langle P_1(s)v, Q_1(s)v\rangle \geq 0$, and therefore, that

$$\begin{aligned}\|v\|^2 &= \langle \big(P_1(s)+Q_1(s)\big)v, \big(P_1(s)+Q_1(s)\big)v\rangle \\ &= \|P_1(s)v\|^2 + 2\mathrm{Re}\,\langle P_1(s)v, Q_1(s)v\rangle + \|Q_1(s)v\|^2 \\ &\geq \|P_1(s)v\|^2 + \|Q_1(s)v\|^2. \quad \blacksquare\end{aligned}$$

Let Σ be a continuous-time impedance-passive extended i/s/o system with transfer function $\widehat{\mathfrak{D}}(s) = \left[\begin{smallmatrix}\widehat{\mathfrak{D}}_1(s) & 0\\ 0 & 0\end{smallmatrix}\right] \in \mathcal{L}\left(\left[\begin{smallmatrix}U_1\\ U_0\end{smallmatrix}\right]\right)$, cf. (8). By Theorem 3.7, $\mathbb{C}_+ \subset \mathrm{dom}\,(\widehat{\mathfrak{D}})$ and we can define a (usually noncoprime) rational relation on $\mathbb{C}_+$ (not Ω) over $\left[\begin{smallmatrix}U\\ U\end{smallmatrix}\right]$ by

$$R_+(s) = \left\{ \begin{bmatrix} y_1\\ y_0\\ u_1\\ u_0\end{bmatrix} \in \begin{bmatrix} U_1\\ U_0\\ U_1\\ U_0\end{bmatrix} \mid \begin{bmatrix} y_1\\ u_0\end{bmatrix} = \widehat{\mathfrak{D}}(s)\begin{bmatrix} u_1\\ y_0\end{bmatrix}\right\}, \quad s \in \mathbb{C}_+. \tag{73}$$

The final theorem of the paper characterises the class of relations that we can realise by continuous-time impedance-passive extended systems.

Theorem 7.2 *We have the following two results.*

1. *If Σ is a continuous-time impedance-passive extended i/s/o system with transfer function $\widehat{\mathfrak{D}}$, then there exists a unique positive-real right-coprime rational relation R on Ω over $\left[\begin{smallmatrix}U\\ U\end{smallmatrix}\right]$, such that R_+, given in (73), is the restriction $R_+ = R|_{\mathbb{C}_+}$ of R to $\mathbb{C}_+$.*

   For $s \in \mathbb{C}_+$, define

$$\begin{aligned} P_+(s) &:= \begin{bmatrix} \widehat{\mathfrak{D}}_1(s)\big(1+\widehat{\mathfrak{D}}_1(s)\big)^{-1} & 0\\ 0 & 1\end{bmatrix} \quad \text{and} \\ Q_+(s) &:= \begin{bmatrix} \big(1+\widehat{\mathfrak{D}}_1(s)\big)^{-1} & 0\\ 0 & 0\end{bmatrix}. \end{aligned} \tag{74}$$

   Then P_+ and Q_+ are contractions on $\mathbb{C}_+$. The unique right-coprime factorisation (P_1, Q_1) of R with $P_1(s)+Q_1(s) = 1$ is obtained by taking $P_1 := P_+|_\Omega$ and $Q_1 := Q_+|_\Omega$, i.e. by continuously extending P_+ and Q_+ to the imaginary axis $i\mathbb{R}$ and the point ∞.

2. *Conversely, if R is a right-coprime positive-real rational relation on Ω over $\left[\begin{smallmatrix}U\\ U\end{smallmatrix}\right]$, then there exists a subspace $U_1 \subset U$, $U_0 = U_1^\perp \cap U$ and an extended continuous-time impedance-passive system with transfer function $\widehat{\mathfrak{D}}$, such that $R_+ = R|_{\mathbb{C}_+}$.*

Proof:

1. Recalling that for $s \in \mathbb{C}_+$, $1+\widehat{\mathfrak{D}}_1(s)$ maps U_1 one-to-one onto itself, whenever the system Σ it originates from is impedance passive (Theorem 3.7 and Corollary 4.2), we can write (73) as (for $s \in \mathbb{C}_+$):

$$R_+(s) = \begin{bmatrix} \widehat{\mathfrak{D}}_1(s) & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \big(1+\widehat{\mathfrak{D}}_1(s)\big)^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} U_1 \\ U_0 \end{bmatrix} = \begin{bmatrix} P_+(s) \\ Q_+(s) \end{bmatrix} \begin{bmatrix} U_1 \\ U_0 \end{bmatrix}.$$

One very easily checks that $P_+(s)+Q_+(s)=1$ for all $s \in \mathbb{C}_+$.

Necessarily $R_+(s)$ is positive real, because for any $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in R_+(s)$ we have

$$\mathrm{Re}\left\langle \begin{bmatrix} u_1 \\ u_0 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_0 \end{bmatrix} \right\rangle = \mathrm{Re}\left\langle u_1, \widehat{\mathfrak{D}}_1(s) u_1 \right\rangle \geq 0.$$

Contractivity of $P_+(s)$ and $Q_+(s)$ is proved very similarly to the proof of Lemma 7.1. The continuous extensions P_1 and Q_1 of P_+ and Q_+ to all of Ω are (contractive and) unique, since $\mathbb{C}_+$ is dense in Ω.

Positive realness of R_+ implies positive realness of R. Moreover, R is right coprime, because for any $s_0 \in \Omega$, we have $P_1(s_0)+Q_1(s_0) = \lim_{\mathbb{C}_+ \ni s \to s_0} \big(P_+(s) + Q_+(s)\big) = 1$, which in turn implies that $P_1(s)v = Q_1(s)v = 0$ only if $v = P_1(s)v + Q_1(s)v = 0$.

2. Let $R(s)$ be the right-coprime positive-real rational relation

$$R(s) = \mathcal{R}\left(\begin{bmatrix} P_1(s) \\ Q_1(s) \end{bmatrix}\right), \quad P_1(s)+Q_1(s) = 1, \quad s \in \Omega.$$

For all $s \in \Omega$, perform an (invertible) external Cayley transformation of $R(s)$ with parameter $\beta = 1$, as defined in Definition 6.1:

$$R^\times(s) := \left\{ \begin{bmatrix} y^\times \\ u^\times \end{bmatrix} \in \begin{bmatrix} U \\ U \end{bmatrix} \mid \exists \begin{bmatrix} y \\ u \end{bmatrix} \in R(s): \quad y^\times = \frac{u-y}{\sqrt{2}}, u^\times = \frac{u+y}{\sqrt{2}} \right\}.$$

We readily check that $\left[\begin{smallmatrix} y^\times \\ u^\times \end{smallmatrix}\right] \in R(s)^\times$ if and only if there is some $v \in U$, such that $y^\times = \big(Q_1(s) - P_1(s)\big)v$ and $u^\times = \big(Q_1(s)+P_1(s)\big)v = v$. Thus, letting $\widehat{\mathfrak{D}^\times}(s) := Q_1(s) - P_1(s)$, it follows that $\left[\begin{smallmatrix} y^\times \\ u^\times \end{smallmatrix}\right] \in R(s)^\times$ if and only if $u^\times \in U$ and $y^\times = \widehat{\mathfrak{D}^\times}(s) u^\times$.

Moreover, R being positive real implies that

$$\|u^\times\|^2 - \|\widehat{\mathfrak{D}^\times}(s)u^\times\|^2 = \|u^\times\|^2 - \|y^\times\|^2 = 2\mathrm{Re}\,\langle u, y\rangle \geq 0,$$

i.e., that $\widehat{\mathfrak{D}^\times}(s)$ is a contraction (depending rationally on $s \in \Omega$).

Let $\Sigma^\times = \left[\begin{smallmatrix} A^\times & B^\times \\ C^\times & D^\times \end{smallmatrix}\right]$ be a continuous-time scattering-passive realisation of $\widehat{\mathfrak{D}^\times}$. (By combining Theorems 1.2 and 5.2 of [AS] one obtains that such a realisation, denoted by Σ_Q in that paper, always exists. Existence of a finite-dimensional realisation is guaranteed by the rationality of $\widehat{\mathfrak{D}^\times}$.)

By Remark 6.5, the inverse external Cayley transform Σ, with parameter $\beta = 1$, of the scattering-passive system $\Sigma^\times$ is well-defined. Futhermore, according to (8), the positive-real transfer function of Σ is of the type $\widehat{\mathfrak{D}}(s) = \left[\begin{smallmatrix} \widehat{\mathfrak{D}}_1(s) & 0 \\ 0 & 0 \end{smallmatrix}\right] \in \mathcal{L}\left(\left[\begin{smallmatrix} U_1 \\ U_0 \end{smallmatrix}\right]\right)$, with $\mathbb{C}_+ \subset \operatorname{dom}(\widehat{\mathfrak{D}}_1)$, for some $U_1, U_0 = U_1^\perp$. This means that the inputs and outputs of Σ are related by

$$\begin{bmatrix} \widehat{y_1}(s) \\ \widehat{u_0}(s) \end{bmatrix} = \begin{bmatrix} \widehat{\mathfrak{D}}_1(s) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \widehat{u_1}(s) \\ \widehat{y_0}(s) \end{bmatrix}, \quad \widehat{u_1}(s) \in U_1, \widehat{y_0}(s) \in U_0, s \in \rho\big(\widehat{\mathfrak{D}}_1\big)$$

in the frequency domain. (Possibly $\widehat{\mathfrak{D}}_1$ has poles in $i\mathbb{R} \cup \{\infty\}$.)

Due to the invertibility of the external Cayley transformation, we now in particular have that for $s \in \mathbb{C}_+$:

$$\begin{bmatrix} y_1 \\ y_0 \\ u_1 \\ u_0 \end{bmatrix} \in R(s) \iff \begin{bmatrix} y_1^\times \\ y_0^\times \\ u_1^\times \\ u_0^\times \end{bmatrix} \in R^\times(s) \iff \begin{bmatrix} y_1^\times \\ y_0^\times \end{bmatrix} = \widehat{\mathfrak{D}^\times}(s) \begin{bmatrix} u_1^\times \\ u_0^\times \end{bmatrix}$$
$$\iff \begin{bmatrix} y_1 \\ u_0 \end{bmatrix} = \begin{bmatrix} \widehat{\mathfrak{D}}_1(s) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ y_0 \end{bmatrix},$$

implying that $R|_{\mathbb{C}_+} = R_+(s)$, as given by (73). ∎

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References

[AG] D.Z. Arov and I.P. Gavrilyuk, A method for solving initial value problems for Linear differential equations in Hilbert space based on the Cayley transform, *Numer. funct. anal. and optimiz.* 14 (5&6) (1993), 459–473.

[AS] D.Z Arov and O.J. Staffans, The infinite-dimensional continuous time Kalman-Yakubovich-Popov inequality, to appear in *Operator Theory: Advances and Applications*, obtainable from `http://www.abo.fi/~staffans`.

[B] V. Belevitch, *Classical Network Theory*, Holden-Day, San Francisco, 1968.

[FP] G.F. Franklin and J.D. Powell, *Digital Control of Dynamic Systems*, Addison Wesley, 1980.

[HM] V. Havu and J. Malinen, Approximation of the Laplace transform by the Cayley transform, *Helsinki University of Technology, institute of Mathematics, Research Reports* A480 (2004).

[Sc] A.J. van der Schaft, *L_2-gain and Passivity Techniques in Nonlinear Control*, Springer-Verlag, London, 2000.

[St1] O.J. Staffans, Passive and conservative infinite-dimensional impedance and scattering systems from a personal point of view, *Mathematical Systems Theory in Biology, Communication, Computation and Finance, IMA Volumes in Mathematics and its Applications* 134, Springer-Verlag, New York, 2002, 375–413.

[W] M.R. Wohlers, *Lumped and Distributed Passive Networks*, Academic Press, New York, 1969.

[Å] K. Åström, T. Hägglund, PID *Controllers: Theory, Design, and Tuning*, Instrument Society of America, 1995.