Doing High-School Mathematics Carefully

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Logic in mathematics

- Mathematical proofs form one of the cornerstones of formal reasoning.
- However, proofs are often considered difficult and consequently the modern Finnish High School mention proofs only in connection with geometry.
- Would need more training in rigorous reasoning in school.
- Logical notation is used very little, and it is not taught in a systematic way.
- The notion of proof in school mathematics is informal and not uniform.
- Where logic is taught, it is seen as a separate object of study, rather than a tool to be used when solving mathematical problems.

Solving problems

- Mathematics is mostly problem solving.
- Students are taught new concepts and then apply them by solving problems (exercises).
- As in proofs, this problem solving is done informally.
- A solution typically involves a number of interrelated formulas or expressions, but the relationships are not made explicit.
- This makes it hard for students to know when a problem has been acceptably solved, and it also makes it difficult to look at solutions afterward and discuss them.
- A more uniform format is sometimes used in some areas, like equation solving and rewriting algebraic expressions

Logic in Computer Science

Within the Computer Science community, a *calculational* paradigm for manipulating mathematical expressions has emerged (originated by E.W. Dijkstra, Wim Feijen, N.van Gaesteren). Textbook by David Gries and Fred Schneider.

The following is an example of this, deriving a distribution property for set operators:

$$
\begin{array}{cl}
 & v \in A \cap (B \cup C) \\
\equiv & \{\text{set comprehension (intersection)}\} \\
 & v \in A \wedge v \in B \cup C \\
\equiv & \{\text{set comprehension (union)}\} \\
 & v \in A \wedge (v \in B \vee v \in C) \\
\equiv & \{\text{distributivity of logical operators}\} \\
 & (v \in A \wedge v \in B) \vee (v \in A \wedge v \in C) \\
\equiv & \{\text{set comprehension (intersection, union)}\} \\
 & v \in (A \cap B) \cup (A \cap C)
\end{array}
$$

Calculational proofs

- The initial expression is transformed step by step.
- Each new version of the expression is written on a new line.
- Between the two lines we write a symbol denoting the relationship between the expressions (equality in this case) and a justification for the validity of the step.
- The level of detail in the steps can be varied; in this example the last step is similar to the combination of the first two steps.

Advantages

- The calculational proof forces the student to be explicit about what strategies and rules are used, and in what order.
- Solutions to problems can be inspected and discussed afterwards, and they are easy to check.
- It is possible to refine an existing solution, making it more accurate and more detailed.
- The method supports self-study.
- It is possible to have computer-support for constructing, checking and browsing calculational proofs.

Successful use of the calculational method requires the basic notions of propositional logic and predicate calculus. The amount of necessary logical notation is small, and it can be shown to be very useful in practice, so the extra effort spent on learning logic is a good investment for the student.

Structured derivations

We will describe here *structured derivations*, an extension of the calculation proof format (mainly by way of examples)

Show how structured derivations can be used in High-School Mathematics, to derive solutions to typical problems.

Equation solving

Traditional way of solving a linear equation: Each new "version" of the equation on a new line, indicate the "operation" used to get from one version to the next

$$
\begin{array}{ll}
2x + 3 = 5 & \mid\ -3 \\
2x = 2 & \mid\ /2 \\
x = 1 &
\end{array}
$$

Finally the student typically checks the solution, to make sure that it is not a "false solution":

- If the last line is of the form $0 = 0$, then the student knows that the answer is that "the equation has infinitely many solutions"
- If the last line is of the form $0 = 1$, then the student knows that the answer is that "the equation has no solutions".

Equation solving calculational style

The same equation, solved in the same way, but with the solution written in the calculational style:

$$
\begin{array}{ll}
 & 2x + 3 = 5 \\
\equiv & \{\text{subtract 3 from both sides}\} \\
 & 2x = 2 \\
\equiv & \{\text{divide both sides by 2}\} \\
 & x = 1
\end{array}
$$

- The logical connection (equivalence) between the formulas is made explicit
- Next to the equivalence symbol $\equiv$ is a justification for the transformation step
- By transitivity of equivalence, this *derivation* shows that the original equation is equivalent to $x = 1$, i.e., $2x + 3 = 5$ is true exactly when $x = 1$ is true.
- The fact that $x = 1$ is such a simple expression (it gives us immediate and full information about x) makes us accept it as an “answer”.

Derivations with truth and falsity

Next we introduce T and F (logical truth and falsity). Consider the following two equation-solving derivations:

$$
\begin{array}{ll}
 & 2x + 3 = 2x \\
\equiv & \{\text{subtract 3 from both sides}\} \\
 & 3 = 0 \\
\equiv & \{\text{unequal numbers}\} \\
 & \mathsf{F}
\end{array}
\qquad
\begin{array}{ll}
 & 2x = 2(x + 1) - 2 \\
\equiv & \{\text{simplify right-hand side}\} \\
 & 2x = 2x \\
\equiv & \{\text{equality is reflexive}\} \\
 & \mathsf{T}
\end{array}
$$

$2x + 3 = 2x \equiv \mathsf{F}$ shows that the equation never holds: regardless of what value x has, the equation is false, there are no solutions.

Similarly, we infer that the second equation always holds: any real number is a solution to the equation $2x = 2(x + 1) - 2$.

Equation solving, a simple pattern

Equation solving now follows a uniform pattern:

- We manipulate the original equation using equivalence-preserving transformations,
- We try to reach an expression that is as simple as possible (according to some accepted criteria)
- For equations, expressions of the form T, F, and $x = 5$ are simple enough for the derivation to be considered completed.

Derivations with conjunction and disjunction

Logical connectives give us the tool we need to keep a derivation togetherin situations where one has traditionally split the solution into seemingly disconnected parts. Consider solving the equation $x(x-2) = 3(x-2)$:

$$
\begin{array}{ll}
 & x(x-2) = 3(x-2) \\
\equiv & \{\text{subtract } 3(x-2) \text{ from both sides}\} \\
 & x(x-2) - 3(x-2) = 0 \\
\equiv & \{\text{distributivity}\} \\
 & (x-3)(x-2) = 0 \\
\equiv & \{\text{zero-product rule: } ab = 0 \ \equiv \ a = 0 \lor b = 0\} \\
 & x - 3 = 0 \lor x - 2 = 0 \\
\equiv & \{\text{add 3 to both sides in left disjunct}\} \\
 & x = 3 \lor x - 2 = 0 \\
\equiv & \{\text{add 2 to both sides in right disjunct}\} \\
 & x = 3 \lor x = 2
\end{array}
$$

Discussion

- The original equation is found to be equivalent to the logical expression $x = 3 \vee x = 2$.
- This can now be interpreted to say that "the equation has two solutions".
- If the equation solving effort is divided up into the two separate equations $x - 3 = 0$ and $x - 2 = 0$, then students are often confused about the connection between the two separate solutions $x = 3$ and $x = 2$.
- This problem becomes even more evident when the equations involve two or more unknowns and some of the "subequations" do not have a unique solution.
- The use of explicit logical connectives also makes it easier to avoid the confusing $\pm$ notation; prefer $x = 1 \vee x = 3$ to $x = 2 \pm 1$.
- Note that the explicit connective $\vee$ also makes it easy to write down rules like the zero-product rule. Students have to learn to read and manipulate connectives, but the effort pays off quickly.

Second example

As a second example, consider the equation $(x-1)(x^2+1)=0$:

$$
\begin{array}{ll}
 & (x-1)(x^2+1)=0 \\
\equiv & \{\text{zero-product rule: } ab=0 \ \equiv \ a=0 \vee b=0\} \\
 & x-1=0 \vee x^2+1=0 \\
\equiv & \{\text{add 1 to both sides in left disjunct}\} \\
 & x=1 \vee x^2+1=0 \\
\equiv & \{\text{add -1 to both sides in right disjunct}\} \\
 & x=1 \vee x^2=-1 \\
\equiv & \{\text{square is never negative}\} \\
 & x=3 \vee \mathsf{F} \\
\equiv & \{\text{rules for connectives}\} \\
 & x=3
\end{array}
$$

Comments

Here the connection between the two subequations is explicit, and the last step of the derivation uses simple rules for calculation with logical formulas

With a little practice, logical rules like $p \vee \mathsf{F} \equiv p$, $p \wedge \mathsf{F} \equiv \mathsf{F}$, etc should be as self-evident as arithmetical rules like $x + 0 = x$ or $0 \cdot x = 0$).

Thus, the whole solution process is kept together as a single derivation where the problem expression is gradually transformed into a solution.

Level of detail

We have chosen a certain level of detail for the derivations here.

Typically, the justifications in a very detailed derivation refer to explicit rules (like the zero-product rule), while in a less detailed derivation the justifications refer to more general strategies (like "solve the equation").

In the case when a less detailed step is used, it should always be possible to think of a more detailed derivation being hidden.

Subderivations

- A rule that is used in a proof step can be established by a separate derivation.
- Alternatively, we can establish the rule in-line with a *subderivation*, indented one level to the right and written immediately below a comment that justifies using the result of the subderivation.
- We indicate the start of the subderivation by a label (this label can be a "bullet" or, if there is more than one subderivation, a number or some text).
- The end is indicated by a small dot to the left of the term immediately following the end of the subderivation.
- If the rule is so general that it may be useful in other contexts, or the proof of the rule is long, then an explicit lemma is usually a good idea.
- However, if the rule is specific to the derivation at hand and its proof is short, then a subderivation is preferred.

Focusing

A special kind of subderivation is one that we start by *focusing* on a subterm.

- Assume that in a derivation, we have reached a term $t[t_1]$ where t_1 occurs as a subterm.
- A typical step in a linear derivation can replace the subterm t_1 with an equivalent term t_2, leaving the rest of t unchanged.
- If the transformation from t_1 to t_2 involves a long sequence of steps, we have to write down the unchanged part of the original term repeatedly.
- To avoid this, we can start a subderivation by *focusing* on the subterm t_1.
- If the subderivation proves $t_1 \equiv t_2$, then it can be used to justify replacing t_1 by t_2 in t.

This argument can be formalised as an *inference rule* of the form

$$\frac{t_1 \equiv t_2}{t[t_1] \equiv t[t_2]}$$

Focusing on a subexpression

Consider the following typical exercise: *For what values of x is the expression* $\dfrac{x-1}{x^2-1}$ *defined?*

We take the definedness statement as our starting point and try to manipulate it until we reach an expression that characterises a set of values for x in a simple way:

Solution

$$
\begin{array}{ll}
 & (x-1)/(x^2-1) \text{ is defined} \\
\equiv & \{\text{definedness of rational expressions}\} \\
 & x^2 - 1 \neq 0 \\
\equiv & \{\text{switch to logic notation}\} \\
 & \neg(x^2 - 1 = 0) \\
\equiv & \{\text{solve } x^2 - 1 = 0\} \\
 & \begin{array}{ll}
 \bullet & x^2 - 1 = 0 \\
 \equiv & \{\text{factorisation rule}\} \\
 & (x+1)(x-1) = 0 \\
 \equiv & \{\text{rule for zero product}\} \\
 & x = -1 \vee x = 1
 \end{array} \\
\bullet & \neg(x = -1 \vee x = 1) \\
\equiv & \{\text{de Morgan rules}\} \\
 & \neg x = -1 \wedge \neg x = 1 \\
\equiv & \{\text{change notation}\} \\
 & x \neq -1 \wedge x \neq 1
\end{array}
$$

Comments

Thus, the expression $\dfrac{x-1}{x^2-1}$ is defined for all values of x except -1 and 1.

In this derivation, focusing on the subterm $x^2 - 1 = 0$ allowed us to do equation solving in a subderivation.

In Mathematics textbooks, a graph of some kind is often used to justify the final solution to problems involving subsets of the real line. Such a graph can illustrate a situation, it can help finding a proof strategy, and it can be used to check that the solution is feasible, but it does not really prove anything.

Also note that by using logical connectives and the de Morgan rules the role of the two “solutions” -1 and 1 is made explicit.

Surpressing subderivation

A subderivation is a level of detail that can be surpressed if we want to present the solution in a less detailed form. The result of surpressing the subderivation is the following:

$$
\begin{array}{ll}
& \dfrac{x-1}{x^2-1} \text{ is defined} \\
\equiv & \{\text{definedness of rational expressions}\} \\
& x^2 - 1 \neq 0 \\
\equiv & \{\text{switch to logic notation}\} \\
& \neg(x^2 - 1 = 0) \\
\equiv & \{\text{solve } x^2 - 1 = 0\} \\
& \neg(x = -1 \vee x = 1) \\
\equiv & \{\text{de Morgan rules}\} \\
& \neg x = -1 \wedge \neg x = 1 \\
\equiv & \{\text{change notation}\} \\
& x \neq -1 \wedge x \neq 1
\end{array}
$$

Browsing structured derivations on the net

We have developed presentation software that allows this kind of browsing of mathematical derivations on the web.

Can collapse and expand subderivations (outlining)

Large collection of examples (solutions to the Finnish Matriculation Exam 1995-98, Spring and Fall).

Web allows distribution of solutions, from student to teacher as well as from teacher to student.

Allows student to go throught solutions at own pace.

Web system uses Unicode fonts for math symbols.

Would need better math support, like MathML (an XML language).

Focusing and local assumptions

When we focus on either operand of a conjunction or disjunction, or on the consequent of an implication, the subderivation can make use of the context that the subterm occurs in.

Thus, the subderivation may have additional local assumptions compared with the enclosing derivation.

1. When focusing on a in $a \wedge b$, we may add b as a local assumption.
2. When focusing on a in $a \vee b$, we may add $\neg b$ as a local assumption.
3. When focusing on b in $a \Rightarrow b$, we may add a as a local assumption.

When a subderivation involves focusing that adds assumptions, we start the subderivation with a line that states the *added assumptions inside square brackets*.

Window inference

These rules can be formally explained as inference rules, using *sequents*. For example, the first rule can be written as follows:

$$\frac{\Phi, t \vdash t_1 \equiv t_2}{\Phi \vdash t \wedge t_1 \ \equiv \ t \wedge t_2}$$

where t is the conjunct that becomes a local assumption and Φ is the (possibly empty) set of assumptions that are inherited from the outer derivation to the subderivation.

Applying such rules amounts to *contextual rewriting* and the method of focusing described lies at the heart of the *window inference* reasoning paradigm .

Example

As an example, we solve an equation involving two unknowns.

$$
\begin{array}{ll}
 & |x-1| + |2x-y| = 0 \\
\equiv & \{\text{property of absolute values}\} \\
 & x-1=0 \wedge 2x-y=0 \\
\equiv & \{\text{add 1 to both sides of left equation}\} \\
 & x=1 \wedge 2x-y=0 \\
\equiv & \{\text{simplify right conjunct}\} \\
 & \quad \begin{array}{ll}
 \bullet & [x=1] \\
  & 2x-y=0 \\
 \equiv & \{\text{substitute using assumption}\} \\
  & 2-y=0 \\
 \equiv & \{\text{solve}\} \\
  & y=2
 \end{array} \\
\bullet & x=1 \wedge y=2
\end{array}
$$

Comments

The solution, here written with logical conjunction ($x = 1 \wedge y = 2$) could also be written as

$$\begin{cases} x = 1 \\ y = 2 \end{cases}$$

This notation has an implicit conjunction; but it better to be more explicit about the connection between the two parts of the solution: $x = 1$ and $y = 2$.

This is important, since a similar notation is sometimes used to show the two solutions of a second-degree equation, e.g.,

$$\begin{cases} x_1 = 1 \\ x_2 = 2 \end{cases}$$

but in this case, the implicit connective involved is disjunction rather than conjunction.

An induction proof

A simple induction proof illustrates how multiple subderivations arise. The induction requires one subderivation for the base case and one for the step case. As our example, we prove the formula

$$1 + 2 + \cdots + n \quad = \quad \frac{n(n+1)}{2}$$

The derivation

The derivation is as follows:

$$
\begin{array}{ll}
 & 1+2+\cdots+n \\
= & \{\text{induction}\} \\
 & \quad \begin{array}{ll}
 \textbf{(base)} & (1+2+\cdots+n)[n:=0] \\
 = & \{\text{empty sum is } 0\} \\
  & 0 \\
 = & \{\text{arithmetic}\} \\
  & \dfrac{0(0+1)}{2} \\
 = & \{\text{expressed using substitution}\} \\
  & \dfrac{n(n+1)}{2}[n:=0]
 \end{array}
\end{array}
$$

$$
\begin{array}{ll}
\textbf{(step)} & [1+2+\cdots+n = \dfrac{n(n+1)}{2}] \\
& (1+2+\cdots+n)[n := n+1] \\
= & \{\text{carrying out substitution}\} \\
& 1+2+\cdots+n+n+1 \\
= & \{\text{induction assumption}\} \\
& \dfrac{n(n+1)}{2}+n+1 \\
= & \{\text{rules for fractions}\} \\
& \dfrac{n^2+n+2n+2}{2} \\
= & \{\text{simplify the numerator}\} \\
& \dfrac{n^2+3n+2}{2} \\
= & \{\text{factorisation}\} \\
& \dfrac{(n+1)(n+2)}{2} \\
= & \{\text{expressed using substitution}\} \\
& \dfrac{n(n+1)}{2}[n := n+1]
\end{array}
$$

- $$\frac{n(n+1)}{2}$$

Comments

We give explicit names (rather than "bullets") to the two subderivations, making their roles explicit.

We also use explicit substitution operations to indicate what exactly has to be proved.

If we hide the subderivations, then there remains only an indication of the strategy used, but no details of its actual workings:

$$\begin{array}{ll} & 1 + 2 + \cdots + n \\ = & \{\text{induction}\} \\ & \dfrac{n(n+1)}{2} \end{array}$$

Although this would hardly be acceptable as a solution to a classroom exercise, there are other situations when the intended reader can be assumed to accept this lack of detail.

A minimisation problem

We determine how the value of x should be chosen in order for the expression $x^2 - x - 6$ to attain its smallest value. This example illustrates the use of text as part of the description that is manipulated.

$$
\begin{array}{ll}
 & x^2 - x - 6 \text{ attains its smallest value} \\
\equiv & \{\text{transform the quadratic expression}\} \\
 & \quad \bullet \quad x^2 - x - 6 \\
 & \quad = \quad \{\text{complete the square}\} \\
 & \qquad\quad x^2 - x + \dfrac{1}{4} - 6 - \dfrac{1}{4} \\
 & \quad = \quad \{\text{arithmetic}\} \\
 & \qquad\quad (x - \dfrac{1}{2})^2 - \dfrac{25}{4} \\
\bullet & (x - \dfrac{1}{2})^2 - \dfrac{25}{4} \text{ attains its smallest value} \\
\equiv & \{\text{constant term does not affect location of minimum}\} \\
 & (x - \dfrac{1}{2})^2 \text{ attains its smallest value}
\end{array}
$$

$$\equiv \quad \{\text{smallest value of a square is zero}\}$$

$$x - \frac{1}{2} = 0$$

$$\equiv \quad \{\text{add } \frac{1}{2} \text{ to both sides}\}$$

$$x = \frac{1}{2}$$

Comments

- Note that on the outermost level we work with logical equivalence.
- In the subderivation, however, we transform a real-valued expression under equality.
- We make a number of implicit assumptions about the extreme values of quadratic expressions; without them, we would have to make a much more elaborate derivation starting from an expression with quantifiers (the expression $(\forall y \bullet y^2 - y - 6 \leq x^2 - x - 6)$.

More advanced derivations

- So far all our derivations have preserved equivalence relations (logical equivalence or arithmetic equality).
- The same method can be used for preserving any *transitive* relation.
- Typical examples of relations that we want to preserve are (forward and backward) implication, and orderings (such as $\leq$ and $<$) on natural numbers, integers or reals, etc.

Monotonicity

This extension means that we have to consider *monotonicity properties* when focusing on subterms.

Example: assume that we want to transform an expression of the form $t_1 - t_2$ under the ordering $\leq$ on the real numbers. We can focus on t_1, show $t_1 \leq t_1'$ and conclude $t_1 - t_2 \leq t_1' - t_2$.

In this case, the same relation ($\leq$) can be used in the subderivation; we say that *the context is monotonic*.

However, if we focus on t_2 then we must reverse the direction of the ordering in the subderivation, because t_2 occurs in a *negative position* with respect to the ordering (in this case we say that the context is *antimonotonic*).

The inference rules involved are the following:

$$\frac{\vdash t_1 \leq t_2}{\vdash t_1 - t \leq t_2 - t} \qquad \frac{\vdash t_1 \geq t_2}{\vdash t - t_1 \leq t - t_2}$$

An exercise from real analysis

We now show how a typical problem from real analysis can be solved. We show that the function $f(x) = 2x$ is uniformly continuous.

In order to express continuity we need (nested) quantifiers

The rules used are essentially formalisations of well-known intuitive strategies used to handle quantifiers. Here we need general monotonicity and in addition the rule of $\exists$-introduction (proving existence by supplying a witness) is used:

$$\frac{\vdash t[t'/x]}{\vdash (\exists x \bullet t)}$$

Goal oriented proofs

The derivation illustrates *nested subderivations*. It preserves backward implication.

This is typical for *goal-oriented proof*; we start from the thing we want to prove and try to transform (reduce) it to T while preserving backward implication.

The justification of this method is that we if we have proved $t \Leftarrow \mathsf{T}$ then we can infer t.

Derivation

$$
\begin{array}{ll}
 & f \text{ uniformly continuous} \\
\equiv & \{\text{definition of uniform continuity}\} \\
 & (\forall\, \varepsilon \bullet \varepsilon > 0 \Rightarrow (\exists\, \delta \bullet \delta > 0 \wedge (\forall\, x\ y \bullet x - y < \delta \wedge x > y \Rightarrow |f(x) - f(y)| < \varepsilon))) \\
\Leftarrow & \{\text{focus on consequent, replace in monotonic context}\}
\end{array}
$$

$$
\begin{array}{ll}
\bullet & [\varepsilon > 0] \\
& (\exists\, \delta \bullet \delta > 0 \wedge (\forall\, x\ y \bullet x - y < \delta \wedge x > y \Rightarrow |f(x) - f(y)| < \varepsilon)) \\
\Leftarrow & \{\text{focus on innermost consequent, replace in monotonic context}\} \\
& \quad \begin{array}{ll}
\bullet & [\delta > 0, x - y < \delta, x > y] \\
& |f(x) - f(y)| < \varepsilon \\
\equiv & \{\text{definition of } f\} \\
& |2x - 2y| < \varepsilon \\
\equiv & \{\text{simplify using assumptions}\} \\
& x - y < \varepsilon/2 \\
\Leftarrow & \{\text{transitivity using assumptions}\} \\
& \delta \le \varepsilon/2
\end{array} \\
\bullet & (\exists\, \delta \bullet \delta > 0 \wedge (\forall\, x\ y \bullet x - y < \delta \wedge x > y \Rightarrow \delta \le \varepsilon/2)) \\
\Leftarrow & \{\exists\text{-introduction rule, witness for } \delta \text{ is } \varepsilon/2\} \\
& \varepsilon/2 > 0 \wedge (\forall\, x\ y \bullet x - y < \delta \wedge x > y \Rightarrow \varepsilon/2 \le \varepsilon/2) \\
\equiv & \{\text{simplify using assumptions}\} \\
& \mathsf{T}
\end{array}
$$

$$
\begin{array}{ll}
\bullet & (\forall\, \varepsilon \bullet \varepsilon > 0 \Rightarrow \mathsf{T}) \\
\equiv & \{\text{simplify using basic rules of predicate logic}\} \\
& \mathsf{T}
\end{array}
$$

Thus the function is uniformly continuous.

Discussion on proof

- The nested quantifiers require separate focusing steps, so that it is clear what variables occur free in the assumption of what subderivation.
- The function in the above example was very simple. However, the same scheme can be used to prove uniform continuity of more complicated functions; only the innermost subderivation becomes longer.

Conclusion

- We have shown how structured derivations can be used when solving problems of the kind that are typical for High School and first-year University Mathematics.
- The format allows us to proceed from problem formulation to solution as a single structured derivation, where the steps done in solving the problem are written out explicitly.
- This derivation can be inspected and discussed as an object in its own right.
- The level of formality used in the derivation can be varied according to the level of students and the difficulty of the problem.
- The derivation can by built up top-down (starting with the steps on the outermost level), bottom-up (starting with lemmas that provide detail) or linearly from problem solution.

Comparisons

- Structured derivations are based on the same principle as standard equation solving, where a new (equivalent) version of an equation is written immediately below the previous version.
- The use of the logical symbol shows the relationship between the two equations and the justification for the step.
- In traditional equation solving, the relationship between equations ($\equiv$, $\Rightarrow$ or $\Leftarrow$) is usually left implicit. This easily leads students to draw erroneous conclusions, in particular in situations when the equation in question has either no solutions or more than one solution.
- In cases with more than one solution, rigorous use of connectives ($\wedge$ and $\vee$) also helps avoid mistakes in the final solution steps.
- Structured derivations extend the equation solving paradigm to more general problem solving, by using logical formalism.