Multimodel $\text{H}_2/\text{H}_\infty$ Control:
Numerical Methods and Applications

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Preface

The research for this thesis was carried out during the years 1996 to 2000 under the guidance of Professor Hannu T. Toivonen. The research took place at the Process Control Laboratory of Åbo Akademi University, with the exception of an 18-month period from 1997 to 1999 during which I had the pleasure of being a visiting researcher at the Laboratory of Process Systems Engineering of Kyoto University, hosted by Professor Iori Hashimoto.

I should like to express my gratitude to Professor Toivonen for his support, his expertise and his understanding. The work presented here was possible due to his thorough knowledge of the field of robust control, as well as his ability to provide insight at moments of doubt and confusion.

Further, I want to thank Professor Hashimoto for his invitation and his kind support of my stay in Kyoto, as well as the rest of the people at the PSE Laboratory for their patience and cheerful company.

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where the granters are listed in chronological order.

The research was done as team work with Professor Toivonen, Dr. Kurt-Erik Hägghblom, Dr. Tore Gustafsson, Dipl. Ing. Jari Böling, Dipl. Ing. Kati Sandström and Dipl. Ing. Bernt...
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Finally, I wish to thank my mother, my father and my sister for being there, as well as my friends in Kyoto and Finland.

Åbo, June 2001

Rasmus Nyström
Abstract

The thesis demonstrates the use of multimodel control design methods for different types of example processes. The primary objective is to use multimodelling methods for control of two classes of processes: uncertain ones and nonlinear ones. For the former case, a distillation column is used as a design example. For the latter case, a highly nonlinear pH neutralization process is used. A secondary objective is to use parametric methods for control. Multimodel methods based on gain-scheduled $H_2$ and $H_\infty$ control are in the main focus, and other methods are studied for comparison.
List of Publications

The thesis is based on the following publications and conference papers. The labels refer to the bibliography section.


——— Accepted for publication: ———


Contributions

The thesis is mainly a sequel to the work of Johan Pensar [70]. The parametric problems are stated in more generalized forms and the gradient expressions revised and presented in a more uniform manner. Some additional auxiliary costs have been introduced by the author.

In the application to a pH control process in Part II, the author was responsible for the $H_2/H_\infty$ and gain-scheduled control design, whereas the process model and the PID controllers are the work of Kati Sandström and Tore Gustafsson [79, 26].

In the application to a distillation column in Part III, the process model is the work of Kurt-Erik Häggblom and Jari Böling [30]. The author was responsible for the designs connected with $H_2$ and $H_\infty$ control, and made preliminary sketches of the uncertainty formulation, whereas the mathematical proof was done by Häggblom and Böling. The problem of forming an uncertainty description for the column led to the papers [11, 12] in which the author is co-author. The IMC, $\mu$ and PID control designs are the work of Böling, and the robust loop shaping controller is the work of Jan Ramstedt.

In the brief example application to a diesel engine in Section 3.4, the process model was provided by Wärtsilä NSD Corp., and was given in state-space form by Aslak Nuortio [56].

In the LPV study in Part IV, the author took part in formulating the LPV model based on the original nonlinear model by Sandström [79], and provided the Riccati-based control designs. The MPC and deadbeat controllers are the work of Bernt Akesson [1] and the IMC controller in [62] is the work of Sandström.
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Part I

Introduction
Chapter 1

Introduction and overview

1.1 Overview

The theory and methodology for solving the \( H_2 \) (LQ), \( H_\infty \) and related mixed-norm and multimodel control problems have by now advanced to a point where these methods can be used with good results in practical control design. For the individual \( H_2 \) and \( H_\infty \) control problems as such, analytical closed-form solutions exist. However, numerical optimization is required if the problems are to be solved in a mixed-norm or multimodel setting. With such parametric controller design, it is not only possible to solve problems which do not have analytical solutions, an additional benefit is that the structure of the controller can be fixed to a desired form, typically a low-order state-space controller or a PI(D) controller.

It can be stated that for the \( H_2 \) and \( H_\infty \) control methods the theory of synthesis is still ahead of the theory of design. At present, \( H_2/H_\infty \)-based design involves a large amount of work before useful control is obtained. The synthesis methods can seldom be applied in a straightforward manner, but the control problem must be formulated in a way which fits into the \( H_2/H_\infty \) framework. A potentially large number of parameters, mainly weighting filters, must be tuned by alternating synthesis and simulation in order to produce satisfactory performance. The tuning work is based on trial and error, and quite often it compensates for design criteria which cannot be stated in terms of \( H_2/H_\infty \) criteria.

The \( H_2/H_\infty \) framework allows the design of robust control through the use of uncertainty estimates. However, a rigorous estimate of the uncertainty is seldom available, so the uncertainty description is often reduced to a tuning parameter, making it difficult to motivate why the method is used in the first place. By contrast, if a norm-bounded uncertainty description is available from the modeling stage \( H_2/H_\infty \) design is an excellent way of utilizing it. Further, if the uncertainty description is based on a multimodel set, an attractive and straightforward alternative is to use multimodel \( H_2/H_\infty \) control.

In its basic form, \( H_2/H_\infty \) design is a linear design methodology which, if the system to be controlled is strongly nonlinear, has to be extended to account for the nonlinearity in
some suitable manner. One approach is to treat the nonlinearity as uncertainty, and to use $H_2/H_\infty$ control or multimodel $H_2/H_\infty$ control. This approach yields linear robust control which, depending on the circumstances, might be too conservative.

An alternative way to handle a nonlinear system by $H_2/H_\infty$ control is to extend the control to be nonlinear. Nonlinear control can be achieved through simple interpolating gain scheduling based on locally optimal controllers. For such a gain-scheduled controller, however, not much can be said about performance and robustness, except through simulation and testing.

The concept of gain scheduling can be extended through the introduction of linear parameter-varying (LPV) systems and by using associated $H_2/H_\infty$ synthesis techniques. The theory of such LPV design is relatively recent.

1.1.1 Thesis outline

In Part I the synthesis technique for parametric $H_2/H_\infty$ design is recapitulated. This part contains material which is partly unpublished, such as the revised and extended gradient expressions used in the optimizations. Some design issues are also discussed in this part, particularly the derivation of an uncertainty description from a multimodel set. Part I also contains a brief description of an example process to which parametric, $H_2$-based PID control has been applied.

Part II contains a case study, where $H_2/H_\infty$-based multimodel and gain-scheduling control are used to control a simulated pH neutralization system. Linear multimodel control and some gain scheduling schemes are tested with varying success. The part consists of the two reprinted publications [60] and [58], and can be seen as an introduction to Part IV.

In Part III, an ill-conditioned, high-purity distillation column is used as an example process to which linear multimodel $H_2$ control, $H_2/H_\infty$ control and robust $H_2$ control are applied. The methods are compared to other well-known control schemes. It is experimentally shown that a very good control result can be obtained by parametric low-order controller design and, especially, multimodel design. The part consists of a reprint of [63], and a slightly modified combination of the papers [61, 64].

Part IV consists of the paper [65], where the pH control process is revisited and formulated as an LPV system. A number of $H_2$-based gain-scheduling methods are designed by gradually increasing the complexity or accuracy of the nonlinear design, and the control results obtained by the different methods are compared.

Some helpful information about the derivation of the gradients in Part I can be found in the appendices A and B. Discussion on the background of the different methods and references to the literature can be found in the respective papers in parts II through IV.
Chapter 2

Fundamental concepts of parametric $H_2/H_\infty$ control

This chapter contains background material and material (the generalized and improved gradient expressions) which, due to space considerations, has not been previously published, although extensively used in the papers [58]–[64].

2.1 A plant with feedback

2.1.1 Nomenclature and definitions

![Diagram](image_url)

Figure 2.1: A plant with feedback.

Figure 2.1 depicts a system of the form studied in this work. The plant, denoted $P$, is a description of the process which is to be controlled. The model is equipped with input and output signals, each of which may be a vector of several scalar signals.

The signal denoted $y$ is the measured output. Its value is fed to a controller, $K$, which manipulates the control input $u$. The objective is to design controllers which fulfill some design criterion, stated by using the other signals in Figure 2.1. Generally, the signals $w_0$ and $w_1$ symbolize different forms of disturbances entering the plant $P$, and
the effect these disturbances have on the system is expressed in terms of the signals $z_0$ and $z_1$. The latter signals may consist of measured outputs, states of the plant, or suitable combinations thereof.

In this work, $z_0$ usually denotes a signal connected with control specifications based on the $H_2$ norm, and $z_1$ denotes a signal connected with the $H_\infty$ norm. These norms will be described in more detail in the sequel.

### 2.1.2 Obtaining the plant description

The description of the plant $P$ is formulated in the control design phase. A model describing the input–output behaviour of the process is a prerequisite. This input–output model is augmented so that the control objectives are expressed in some suitable manner. Several types of disturbances (e.g., steps, ramps, pulses, oscillations or random noise) can be modeled to enter the system via the inputs $w_0$ and $w_1$ to affect suitable states or signals of the process. The outputs $z_0$ and $z_1$ have to be constructed to correspond to the design objectives. During the construction of these signals, the importance of one disturbance has to be weighted against that of another. Correspondingly, the controlled signals have to be weighted against each other. Additional filters may have to be introduced in order to weight particular frequency ranges against others.

In this work, the plant $P$ is usually described by state-space expressions of the form

$$\begin{align*}
\sigma x &= Ax + B_0 w_0 + B_1 w_1 + B_2 u \\
z_0 &= C_0 x + D_{02} u \\
z_1 &= C_1 x + D_{12} u \\
y &= C_2 x + D_{20} w_0 + D_{21} w_1 \\
u &= Ky
\end{align*}$$

(2.1)

where the operator $\sigma$ refers to a derivative in the case of continuous-time modelling, and to the forward shift operator $q$ in the case of discrete-time modelling. A more compact notation for the plant $P$ in (2.1) is given by

$$P = \begin{bmatrix}
A & B_0 & B_1 & B_2 \\
C_0 & 0 & 0 & D_{02} \\
C_1 & 0 & 0 & D_{12} \\
C_2 & D_{20} & D_{21} & 0
\end{bmatrix}$$

(2.2)

The matrices $D_{00}$, $D_{01}$, etc. are dropped from the above description for reasons of convenience, and since they are rarely needed. Exceptions to this can be handled through suitable augmentations and/or approximations.

Here, the controller dynamics are supposed to be contained in the plant model $P$. This can be realized through proper augmentation of the model.

### 2.1.3 The closed-loop system description

Particularly in parametric control design we are interested in working with expressions of closed-loop systems, i.e. systems where the feedback $K$ has been incorporated in the
plant description. Thus, we define $T_{z,w}$ as the closed-loop transfer function from the signal $w$, to the signal $z$.

For instance, for the plant in Figure 2.1, described by the state-space equations (2.1), the closed-loop system related to $H_2$ design, i.e., the transfer function from $w_0$ to $z_0$ is denoted $T_{z_0w_0}$ and is expressed in the compact form

$$T_{z_0w_0} = \begin{bmatrix} A + B_2 K C_2 \\ C_0 + D_{05} K C_2 \end{bmatrix} \begin{bmatrix} B_0 + B_2 K D_{20} \\ D_{05} K D_{20} \end{bmatrix}$$  \quad (2.3)

(Notice that in this case we have to make the assumption $D_{05} K D_{20} = 0$ for the $H_2$ norm to be defined.)

## 2.2 The $H_2$ norm and $H_2$ design

### 2.2.1 The $H_2$ norm

A vast number of control design problems can be conveniently expressed in terms of an $H_2$ or LQ control problem, where the $H_2$ norm of the system is minimized. The definition of the $H_2$ norm of a system $T_{z_0w_0}$ (cf. Section 2.1) in continuous time is

$$||T_{z_0w_0}||_2 := \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} tr \{ T_{z_0w_0}^*(j\omega) T_{z_0w_0}(j\omega) \} \, d\omega \right]^{\frac{1}{2}}$$  \quad (2.4)

where $T^*$ denotes the complex conjugate transpose of $T$. Equivalently, the $H_2$ norm can be expressed using the singular values $\sigma_i, i = 1, \ldots, n$,

$$||T_{z_0w_0}||_2 := \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{i=1}^{n} \sigma_i^2(T_{z_0w_0}(j\omega)) \, d\omega \right]^{\frac{1}{2}}$$  \quad (2.5)

In discrete time, the integral in expressions (2.4) and (2.5) is taken over $[0, 2\pi]$ instead of $[\infty, \infty]$.

The $H_2$ norm gives an expression of the covariance of the output $z_0$ of the system $T_{z_0w_0}$ when it is subjected to a white-noise disturbance through $w_0$. Alternatively, it expresses the $l_2$ norm of the output of $T_{z_0w_0}$ when subjected to a pulse disturbance or an initial-value disturbance.

By handling and/or creating suitable signals in the system, the $H_2$ control problem can be used to describe noise rejection problems, tracking problems, feed-forward control, etc, for all typical types of disturbances or tracking trajectories.
2.2.2 The $H_2$ control problem

For the plant in Figure 2.1 described by the state-space equations (2.1), we can define the $H_2$ control problem as below.

**Problem 2.1 (The $H_2$ control problem)** Find a stabilizing controller from $y$ to $u$ which minimizes the $H_2$ norm $\|T_{2wu}\|_2$.

An analytical solution of the $H_2$ control problem can be obtained through a double-Riccati equation. As an alternative, it can be parametrically solved through numerical minimization of the $H_2$ norm, using the feedback $K$ as an optimization variable. The parametric approach is particularly useful in the following cases:

1. the $H_2$ control problem is a subproblem of a more complex problem, such as a multimodel control problem or a multiobjective control problem (whereby an analytical solution rarely exists), or

2. a low-order or structurally fixed controller is called for. (The analytical solution yields an unstructured controller of the same order as the plant.)

2.2.3 The $H_2$ cost

In order to solve the $H_2$ control problem parametrically, we need the following result.

**Result 2.1** Let the closed-loop stable transfer function $T_{2wu}$ be described by

$$\begin{bmatrix} A_f & B_f \\ C_f & D_f \end{bmatrix},$$

(cf. (2.3)). Then the $H_2$ cost is given by

$$J_2(T_{2wu}) := J_2(P, K) := \|T_{2wu}\|_2^2 = \text{tr}[B_f^T Q B_f]$$

(2.7)

where, in continuous time, $Q$ is the positive semidefinite solution of the Lyapunov equation

$$A_f^T Q + Q A_f + C_f^T C_f = 0$$

(2.8)

and, in discrete time, of the Lyapunov equation

$$Q = A_f^T Q A_f + C_f^T C_f.$$  

(2.9)

This result can be obtained through integration or from standard literature on LQ control.
2.3 The $H_\infty$ norm and $H_\infty$ design

2.3.1 The $H_\infty$ norm

The $H_\infty$ norm of the stable transfer function $T_{z_1,w_1}$ is defined as

$$\|T_{z_1,w_1}\|_\infty := \sup_{\omega} \tilde{\sigma}[T_{z_1,w_1}(j\omega)]$$  \hspace{1cm} (2.10)

where $\tilde{\sigma}$ denotes the maximum singular value.

The $H_\infty$ norm is the maximum gain of the system $T_{z_1,w_1}$, as induced by signals in $L_2$, and is therefore often referred to as the $L_2$-induced norm. The $H_\infty$ norm is particularly useful for control problems related to robustness.

The following problem is of interest when the minimum achievable $H_\infty$ norm is sought for, such as in loop shaping.

**Problem 2.2 (The $H_\infty$ norm minimization problem)** Consider the control system in Figure 2.1. Let $T_{z_1,w_1}$ denote the closed-loop transfer function from $w_1$ to $z_1$. Find a stabilizing controller $K$ such that the $H_\infty$ norm (2.10) is minimized.

However, in parametric design it is often of greater interest to ensure that an $H_\infty$ norm bound is satisfied, i.e. to solve the problem

**Problem 2.3 (The $H_\infty$ norm existence problem)** Given the system $T_{z_1,w_1}$, find a stabilizing controller $K$ such that the $H_\infty$ norm bound

$$\|T_{z_1,w_1}\|_\infty \leq \gamma$$  \hspace{1cm} (2.11)

is satisfied.

or, particularly, the problem

**Problem 2.4 (The $H_\infty$ norm satisfaction problem)** Given the system $T_{z_1,w_1}$ and the controller $K$, find out if the bound (2.11) holds.

For solving any of the above problems by parametric design, the following result is of great utility.

**Result 2.2** The stable system

$$T_{z_1,w_1} := \begin{bmatrix} A_f & B_f \\ C_f & D_f \end{bmatrix}$$  \hspace{1cm} (2.12)

satisfies the $H_\infty$ norm bound (2.11) if and only if
1. In continuous time: the stationary Riccati equation
\[
A_f^T S + S A_f + (B_f^T S + D_f^T C_f)^T (\gamma^2 I - D_f^T D_f)^{-1} \cdot (B_f^T S + D_f^T C_f) + C_f^T C_f = 0
\]  
(2.13)

has a symmetric positive definite solution \( S \), such that the matrix
\[
A_S := A_f + B_f (\gamma^2 I - D_f^T D_f)^{-1} (B_f^T S + D_f^T C_f)
\]
(2.14)
is stable, where \( \gamma^2 I - D_f^T D_f > 0 \).

2. In discrete time: the stationary Riccati equation
\[
S = A_f^T S A_f + (B_f^T S A_f + D_f^T C_f)^T (\gamma^2 I - D_f^T D_f - B_f^T S B_f)^{-1} \cdot (B_f^T S A_f + D_f^T C_f) + C_f^T C_f
\]
(2.15)

has a symmetric positive definite solution \( S \), such that the matrix
\[
A_S := A_f + B_f (\gamma^2 I - D_f^T D_f - B_f^T S B_f)^{-1} (B_f^T S A_f + D_f^T C_f)
\]
(2.16)
is stable, where \( \gamma^2 I - D_f^T D_f - B_f^T S B_f > 0 \).

The proof of the result can be found in standard \( H_\infty \) control literature.

### 2.3.2 An uncertain plant

![Uncertain plant diagram](image)

Figure 2.2: Uncertain plant.

Figure 2.2 depicts a generalized form of an uncertain plant which corresponds to a large class of control problems. The plant model and control objectives are contained in the block denoted \( P \), as stated in section 2.1.

The block denoted \( \Delta \) corresponds to uncertain dynamics, which is fed back through the signals \( z_1 \) and \( w_1 \). This setup can be used to describe common types of uncertainty structures, such as additive and multiplicative plant uncertainty. Here, the uncertainty \( \Delta \) is assumed to be norm-bounded, i.e., its \( H_\infty \) norm satisfies
\[
\| \Delta \|_\infty \leq \gamma^{-1}.
\]
(2.17)
In order to have more specific, possibly frequency-dependent bounds on the uncertainty, the plant \( P \) is augmented in some suitable manner. Augmenting for a block-structured uncertainty is also possible, but \( \Delta \) is usually assumed to be unstructured in this work.

The signals \( w_0 \) and \( z_0 \) pertain to the \( (H_2) \) performance of the system. In the setup in Figure 2.2, the controller is usually designed to minimize the effect that the disturbance \( w_0 \) has on the signal \( z_0 \), whilst taking into account the possible plant variations caused by the uncertainty \( \Delta \).

### 2.3.3 Robust stability

If the plant in Figure 2.2 is stable for any \( \Delta : \|\Delta\|_\infty \leq \gamma^{-1} \) then it is said to be robustly stable with respect to this uncertainty class. The fundamental tool for testing robust stability is the following theorem.

**Theorem 2.1 (The small-gain theorem)** The closed-loop system in Figure 2.2 is stable for all norm-bounded uncertainties \( \Delta \) such that \( \|\Delta\|_\infty \leq \gamma^{-1} \) if and only if the closed-loop transfer function \( T_{z_1w_1} \) from \( w_1 \) to \( z_1 \) of the nominal system (\( \Delta = 0 \)) is stable and satisfies the norm bound

\[
\|T_{z_1w_1}\|_\infty \leq \gamma
\]

(2.18)

A proof can be found for instance in [100].

### 2.4 Mixed \( H_2/H_\infty \) design

Whereas an \( H_\infty \) measure is often connected with robustness, the \( H_2 \) criterion is most useful to express performance. In order to have control specifications of both types in one design, mixed \( H_2/H_\infty \) has been introduced. Other types of mixed-norm problems than combined performance and robustness could also be motivated, for instance combining a specification based on \( H_\infty \) loop shaping with a criterion for LQ optimality.

With respect to the plant in Figure 2.1, the following type of problem can be stated.

**Problem 2.5 (The mixed \( H_2/H_\infty \) problem I)** Find a stabilizing linear controller \( K \) for the plant \( P \) which minimizes a combination of the \( H_2 \) norm \( \|T_{z_0w_0}\|_2 \) and the \( H_\infty \) norm \( \|T_{z_1w_1}\|_\infty \).

Consider now the uncertain plant in Figure 2.2. As stated in section 2.3.2, robustness against norm-bounded uncertainties can be tested using the small-gain theorem. This approach, however, does not take performance into account in other ways than by preserving robust stability within the uncertain model region. A relatively straightforward approach is then to add a performance measure in terms of the \( H_2 \) norm with respect to the nominal model. This leads to the following problem statement.
Problem 2.6 (The mixed $H_2/H_\infty$ problem II) Find a stabilizing linear controller $K$ for the plant $P$ which minimizes the $H_2$ norm $\|T_{z,w}\|_2$ subject to the $H_\infty$ constraint $\|T_{z,w}\|_\infty < \gamma$.

Numerical methods for the computation of a fixed-order controller which solves the mixed $H_2/H_\infty$ problems have been described in [34] and [69], among others.

A potential drawback of the design approach of Problem 2.6 is that it ensures good $H_2$ performance for the nominal plant model only. The method provides robust stability with respect to the plant uncertainty, but there is no guarantee for robust performance. One approach to achieve robust performance in addition to robust stability is to generalize the procedure to the multimodel setting [69]. Another approach is to use robust $H_2$ design, which is explained in more detail in Section 2.6.

Numerical methods to solve problems 2.5 and 2.6 are given in Chapter 3.

2.5 Multimodel control

In many cases the process description is not restricted to a single linear model $P$, but contains a set of alternative or complementary models $P_n$, $n = 1, \ldots, N$, describing the operating point or the operating range in question. The model set may for instance be a result of several identification experiments, and can then provide a description of the plant uncertainty, time variations or nonlinearity.

In order to provide robustness with respect to these factors, it is therefore of interest to calculate controllers which are optimal in some sense with respect to all models in the set. Parametric design enables the formulation of a vast amount of multimodel control problems. In the sequel a few, which will subsequently be used in the applications sections, are given as examples.

2.5.1 Multimodel $H_2$ control

In the multimodel $H_2$ (LQ) problem [48, 50, 69], a controller is computed which stabilizes the discrete models $P_n$, $n = 1, \ldots, N$ in a given model set, and makes the quadratic cost small for all the individual models simultaneously. The problem can be formulated by introducing a weighted sum of $H_2$ costs evaluated for the individual models,

$$J_{MM}(P_n, K) := \sum_{n=1}^{N} c_n J_2(P_n, K)$$

Here $c_n$ are nonnegative weights which reflect the importance of the models in the quadratic cost. The multimodel $H_2$ problem can be stated as follows.

Problem 2.7 (Multimodel $H_2$ problem) Find a linear controller $K$ which stabilizes the models $P_n$, $n = 0, \ldots, N$, such that the cost (2.19) is minimized.
2.6. ROBUST $H_2$ CONTROL

The multimodel $H_2$ problem has no closed-form solution, and it must therefore be solved by numerical optimization. Numerical tools for this are given in Chapter 3.

In the approach of Problem 2.7, robustness is achieved by requiring closed-loop stability and acceptable performance for the discrete model set. This provides robust performance as well as stability, but provides robustness in the convex hull of the model set only by assumption. The multimodel controller design can, however, be generalized to the case when the individual models are equipped with norm-bounded uncertainties as well. A simple approach is given below.

2.5.2 Multimodel $H_2/H_\infty$ control

Problem 2.8 (Multimodel $H_2$ problem with robust stability) Find a linear controller $K$ which stabilizes the models $P_n$, $n = 0, \ldots, N$, such that the cost (2.19) is minimized, subject to the constraints

$$\|T_{z,w_1} (P_n,K)\|_\infty \leq \gamma, \text{ all } n.$$  (2.20)

2.6 Robust $H_2$ control

An approach to combine $H_\infty$ design with an $H_2$ design criterion is to use mixed $H_2/H_\infty$ design, where the nominal $H_2$ performance is minimized under the requirement that robust stability be maintained within the uncertain region, see Section 2.4. This can be done by minimizing a nominal $H_2$ performance criterion in conjunction with an $H_\infty$ auxiliary cost. However, it is often of more interest to treat measures of the $H_2$ norm with respect to a norm-bounded uncertainty description, i.e. to consider the $H_2$ norm of a system not only for one nominal model, but within the whole uncertain region.

In this section a worst-case control problem which achieves robust $H_2$ performance with respect to norm-bounded uncertainties is described. The problem is reformulated as a minimization of an upper bound on the worst-case $H_2$ cost within the uncertainty region. More thorough treatment and extensive literature references can be found in [85, 88, 89, 90].

Consider the linear system

$$z_R = \begin{bmatrix} T_{z,w_0} & T_{z,w_1} \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \end{bmatrix}, \quad w_1 = Tw_0$$  (2.21)

where $T$ is a strictly causal operator in $H_\infty$. Further, define the auxiliary cost

$$\mathcal{L} := \sup_T \left\{ \sum_{i=1}^{m} \|z_i\|_2^2 - \gamma^2 \|w_1\|_2^2 \mid w_0 = e_i \delta_0 \right\} : \quad w_1 = Tw_0$$  (2.22)

where $\delta_0$ is the unit impulse.
Given this setup, the cost $\mathcal{L}$ can be evaluated by solving Riccati equations similar to (2.13) in the continuous case and (2.15) in the discrete-time case. Further details of the evaluation of the cost will be given in sections 3.1.6 and 3.1.7.

The assumption that $T$ is causal is made because this uncertainty structure is compliant with a large class of uncertainty descriptions and results in a less conservative design, as well as a relatively simple expression for the auxiliary cost in (2.22).

Consider now the uncertain plant of Figure 2.2. Let $T_{z_0 w_0}(P, K, \Delta)$ stand for the closed-loop transfer function from $u_0$ to $z_0$ and $T_{z_1 w_0}(P, K)$ that from $w_0$ to $z_1$ for the nominal system ($\Delta = 0$). We introduce the worst-case $H_2$ cost with respect to the norm-bounded uncertainty block $\Delta$,

$$I(P, K) := \sup_{\Delta} \left\{ \|T_{z_0 w_0}(P, K, \Delta)\|_2^2 : \|\Delta\|_{\infty} \leq \gamma^{-1}, \Delta T_{z_1 w_0}(P, K) \text{ strictly causal} \right\}.$$  

The evaluation of robust $H_2$ performance defined by (2.23) is very hard. Therefore it is well-motivated to study more tractable robust performance measures. In particular, a useful upper bound on the cost (2.23) can be obtained as follows. Notice that a signal representation of the worst-case $H_2$ cost (2.23) is

$$I(P, K) := \sup_{\Delta} \left\{ \sum_{i=1}^m \|z_0\|_2^2 : w_0 = e_i \delta_0 \right\} : \|\Delta\|_{\infty} \leq \gamma^{-1}, \Delta T_{z_1 w_0}(P, K) \text{ strictly causal} \right\}$$

where $\delta_0$ is the unit impulse. The fact that $\Delta$ is norm-bounded implies the inequality

$$\sum_{i=1}^m \|z_i\|_2^2 - \gamma^2 \|w_1\|_2^2 : w_0 = e_i \delta_0 \geq 0.$$  

An upper bound on the worst-case $H_2$ cost (2.24) is therefore obtained by solving the constrained maximization problem

$$\sup_{T} \left\{ \sum_{i=1}^m \|z_0\|_2^2 : w_0 = e_i \delta_0 \right\} : w_1 = Tw_0, \ T \text{ strictly causal} \right\}$$

subject to the inequality (2.25). This constrained maximization problem can be handled by introducing the Lagrange-type function associated with the cost in (2.26) and the constraint (2.25),

$$\mathcal{L}(P, K, d) := \sup_{T} \left\{ \sum_{i=1}^m \|z_0\|_2^2 + d^2 \|z_1\|_2^2 - \gamma^2 \|w_1\|_2^2 : w_0 = e_i \delta_0 \right\} : w_1 = Tw_0, \ T \text{ strictly causal} \right\}.$$
Here \( d \) is a real-valued scalar parameter which corresponds to a Lagrangian multiplier associated with the constraint (2.25). It can be shown \([85]\) that if the nominal closed-loop system has \( H_\infty \) norm from \( w_1 \) to \( z_1 \) less than one, i.e., if the criterion for robust stability holds, then there exists \( d > 0 \) such that \( \mathcal{L}(P, K, d) \) is bounded. Moreover, the worst-case cost (2.27) is an upper bound on the robust \( H_2 \) performance measure (2.23),

\[
I(P, K) \leq \mathcal{L}(P, K, d)
\]  

(2.28)

The cost (2.27) can be written as

\[
\mathcal{L}(P, K, d) = \sup_{T} \left\{ \sum_{i=1}^{m} \left[ \| z_0 - \gamma^2 \| \left\| dw_1 \right\| \right] \left\| \frac{z_0}{d} \right\| - w_0 = e_i \delta_0 \right\} : \\
w_1 = Tw_0, \ T \text{ strictly causal}
\]

(2.29)

Finally, this cost is of the same form as the one defined in (2.22), and we are ready to introduce the robust \( H_2 \) performance problem as follows.

**Problem 2.9 (Robust \( H_2 \) problem)** Find a stabilizing linear controller \( K \) for the plant \( P \) and a weighting parameter \( d > 0 \) such that the worst-case cost \( \mathcal{L}(P, K, d) \) is minimized.

The worst-case cost (2.27) is an upper bound on robust performance with respect to the set of nonlinear and time-varying perturbations \( \Delta \) \([85]\). When a linear time-invariant uncertainty is assumed, a less conservative upper bound can be obtained by replacing the real-valued weighting parameter \( d \) by a frequency-dependent scaling filter \( d(\delta) \in H_\infty \) \([99]\). This results in a significantly more complex optimization problem which often displays convergence problems \([88, 27]\). In addition, numerical studies have implied that the benefit from using a dynamic scaling filter instead of a real-valued weighting parameter tends to be quite small \([88]\). Nevertheless, such a structure of the weighting filter is assumed as the final expression for the cost is given in sections 3.1.6 and 3.1.7. In order to handle higher signal dimensions than one, we introduce the diagonal weighting filter

\[
D(\delta) := d(\delta)I
\]

(2.30)

and the signals

\[
w_\Delta := Dw_1, \ z_\Delta := Dz_1.
\]

(2.31)

We then augment the plant as in Figure 2.3.

It is easy to spot the similarity between the plant of Figure 2.3 and that which is used to solve the \( \mu \) synthesis problem by \( D-K \) iteration. In fact, an \( r \)-block-structured uncertainty description can readily be incorporated in the setup at hand, by setting \( D = \text{diag} (d_1I_1, \ldots, d_rI_r) \), corresponding to the partitioning of the uncertainty \([88]\).
Figure 2.3: Plant used for robust LQ synthesis.
Chapter 3

Parametric controller design

The concept of parametric controller design has been touched upon in the previous sections, where different control problems were introduced. In parametric controller design, numerical optimization is performed on a controller $K$, so that a control objective is minimized or a set of objectives (usually a sum of objectives) is minimized. Constraints can be included in the optimization as appropriate.

3.1 Gradient-based optimization

In order to achieve efficient numerical optimization, it is advantageous to have gradient expressions of the costs which are minimized. In the sequel gradient expressions needed for solving the formulated problems are given. Some of the expressions are of a more general and efficient form than previous ones which have been introduced e.g. in [70].

Some basic formulae which are frequently used to obtain the results in the sequel can be found in Appendix A. An example of deriving a gradient can be found in Appendix B.

3.1.1 The continuous-time $H_2$ problem — cost and gradients

Consider the system $T_{2a0o}$ given in (2.6), and the $H_2$ cost defined by (2.7) and (2.8), where

\[
A_f := A + B_2 K C_2, \quad B_f := B_0 + B_2 K D_{20} \\
C_f := C_0 + D_{02} K C_2, \quad D_f := D_{02} K D_{20} = 0
\]  

(3.1)

The cost is given by

\[
J_2 := \text{tr} \left[ B^T_f Q B_f \right]
\]  

(3.2)

where $Q$ is obtained from the Lyapunov equation

\[
A^T_f Q + QA_f + C^T_f C_f = 0.
\]  

(3.3)
Through first-order expansion we obtain as an intermediate result
\[
\frac{\partial J_2}{\partial A_f} = 2Q_1, \quad \frac{\partial J_2}{\partial B_f} = 2Q_2, \quad \frac{\partial J_2}{\partial C_f} = 2C_f
\]  
(3.4)
where \( P \) is obtained from the Lyapunov equation
\[
A_f P + PA_f^T + B_f B_f^T = 0
\]  
(3.5)
and, thus,
\[
\frac{\partial J_2}{\partial K} = B_2^T \frac{\partial J_2}{\partial A_f} C_2^T + B_2^T \frac{\partial J_2}{\partial B_f} D_2 + D_2^T \frac{\partial J_2}{\partial C_f} C_2^T.
\]  
(3.6)

### 3.1.2 The discrete-time \( H_2 \) problem — cost and gradients

Consider the system \( T_{z,w_0} \) given in (2.6), and the \( H_2 \) cost defined by (2.7) and (2.9). The \( H_2 \) cost is given by
\[
J_2 = \text{tr} [B_f^T Q B_f]
\]  
(3.7)
where \( Q \) is obtained from the discrete-time Lyapunov equation
\[
Q = A_f^T Q A_f + C_f^T C_f.
\]  
(3.8)
As an intermediate result, we obtain
\[
\frac{\partial J_2}{\partial A_f} = 2Q_1, \quad \frac{\partial J_2}{\partial B_f} = 2Q_2, \quad \frac{\partial J_2}{\partial C_f} = 2C_f
\]  
(3.9)
where \( P \) is obtained from the Lyapunov equation
\[
P = A_f P A_f^T + B_f B_f^T
\]  
(3.10)
and the total gradient is obtained as
\[
\frac{\partial J_2}{\partial K} = B_2^T \frac{\partial J_2}{\partial A_f} C_2^T + B_2^T \frac{\partial J_2}{\partial B_f} D_2 + D_2^T \frac{\partial J_2}{\partial C_f} C_2^T
\]  
(3.11)

### 3.1.3 The \( H_\infty \) auxiliary cost

Consider the criterion in Result 2.2 for testing the \( H_\infty \) norm of a system \( T_{x_1,w_1} \). A stationary Riccati equation (2.13) or (2.15) is solved and a matrix \( A_S \) as defined in (2.14) or (2.16) is tested for stability.

In order to enforce the stability of \( A_S \) numerically in an optimization, it is convenient to introduce an auxiliary penalty function \( J_{aux} \), which approaches infinity as \( A_S \) approaches instability.

Such a penalty function is given [70] by solving the stationary Lyapunov equation

1. In continuous time:
\[
A_S^T X + X A_S + I = 0.
\]  
(3.12)
2. In discrete time:

\[ X = A_S^T X A_S + I. \]  

(3.13)

and by defining

\[ J_{\text{aux}} := \text{tr}[X] \]  

(3.14)

### 3.1.4 Continuous-time \( H_\infty \) design — auxiliary cost and gradients

Consider the closed-loop system

\[ T_{z_1 w_1} = \begin{bmatrix} A_f & B_f \\ C_f & D_f \end{bmatrix} \]  

(3.15)

where

\[ A_f := A + B_2 K C_2, \quad B_f := B_1 + B_2 K D_{21} \]

\[ C_f := C_1 + D_{12} K C_2, \quad D_f := D_{12} K D_{21} \]  

(3.16)

For convenience, define the following matrices,

\[ G_S := (\gamma^2 I - D_f^T D_f)^{-1} \]

\[ H_S := G_S (B_f^T S + D_f^T C_f). \]  

(3.17)

Then the stability matrix as defined in (2.14) takes the form

\[ A_S = A_f + B_f H_S \]  

(3.18)

and the Riccati equation (2.13) is given by

\[ A_f^T S + S A_f + (B_f^T S + D_f^T C_f)^T G_S (B_f^T S + D_f^T C_f) + C_f^T C_f = 0 \]  

(3.19)

The auxiliary cost is given by

\[ J_{\text{aux}} = \text{tr}[X] \]  

(3.20)

where \( X \) is obtained from the Lyapunov equation

\[ A_S^T X + X A_S + I = 0. \]  

(3.21)

We obtain

\[
\frac{\partial J_{\text{aux}}}{\partial A_f} = 2XP + 2S(R + R^T)
\]

\[
\frac{\partial J_{\text{aux}}}{\partial B_f} = 2XP H_S^T + 2SPXB_f G_S + 2S(R + R^T)H_S^T
\]

\[
\frac{\partial J_{\text{aux}}}{\partial C_f} = 2D_f G_S B_f^T XP + 2(C_f + D_f H_S)(R + R^T)
\]

\[
\frac{\partial J_{\text{aux}}}{\partial D_f} = 2(C_f + D_f H_S)XP B_f G_S + 2D_f G_S B_f^T XPH_S^T + 2(C_f + D_f H_S)(R + R^T)H_S^T
\]  

(3.22)
where $P$ and $R$ are obtained from the Lyapunov equations

$$A_s P + P A_s^T + I = 0 \quad (3.23)$$
$$A_s R + R A_s^T + P X B_f G S B_f^T = 0 \quad (3.24)$$

and, thus, the gradient with respect to the controller $K$ is given by

$$\frac{\partial J_{aux}}{\partial K} = B_2^T \frac{\partial J_{aux}}{\partial A_f} C_2^T + B_2^T \frac{\partial J_{aux}}{\partial B_f} D_{21}^T + D_{12}^T \frac{\partial J_{aux}}{\partial C_f} C_2^T + D_{12}^T \frac{\partial J_{aux}}{\partial D_f} D_{21}^T \quad (3.25)$$

and the gradient with respect to $\gamma$ by

$$\frac{\partial J_{aux}}{\partial \gamma} = \text{tr}[-2\gamma H S P X B_f G S - 2\gamma G S B_f^T X P H S^T - 2\gamma H S (R + R^T) H S^T] \quad (3.26)$$

(The derivation of $\partial J_{aux}/\partial K$ can be found in Appendix B).

### 3.1.5 Discrete-time $H_\infty$ design — auxiliary cost and gradients

Consider the closed loop system $T_{z_1 w_1}$, defined as in (3.15) and (3.16). Introducing the auxiliary matrices

$$G_s := (\gamma^2 I - D_f^T D_f - B_f^T S B_f)^{-1} \quad (3.27)$$
$$H_s := G_s (B_f^T S A_f + D_f^T C_f)$$

yields

$$A_s := A_f + B_f H_s \quad (3.28)$$

for the stability matrix defined in (2.16) and the Riccati equation as defined in (2.15) takes the form

$$S = A_f^T S A_f + (B_f^T S A_f + D_f^T C_f)^T G_s (B_f^T S A_f + D_f^T C_f) + C_f^T C_f \quad (3.29)$$

The auxiliary cost is given by

$$J_{aux} = \text{tr}[X] \quad (3.30)$$

where $X$ is obtained from the discrete-time Lyapunov equation

$$X = A_s^T X A_s + I. \quad (3.31)$$

The following intermediate gradient expressions can be derived,

- $\frac{\partial J_{aux}}{\partial A_f} = 2 X A_s P + 2 S B_f G_s B_f^T X A_s P + 2 S A_s (R + R^T)$
- $\frac{\partial J_{aux}}{\partial B_f} = 2 X A_s P H_s^T + 2 S A_s P A_s^T X B_f G_s + 2 S B_f G_s B_f^T X A_s P H_s^T + 2 S A_s (R + R^T) H_s^T$ \quad (3.32)
- $\frac{\partial J_{aux}}{\partial C_f} = 2 D_f G_s B_f^T X A_s P + 2 (C_f + D_f H_s) (R + R^T)$
- $\frac{\partial J_{aux}}{\partial D_f} = 2 (C_f + D_f H_s) P A_s^T X B_f G_s + 2 D_s G_s B_f^T X A_s P H_s^T + 2 (C_f + D_f H_s) (R + R^T) H_s^T$
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where \( P \) and \( R \) are obtained from the Lyapunov equations

\[
P = A_s P A_s^T + I \\
R = A_s R A_s^T + A_s P A_s^T X B_f G_s B_f^T.
\]

(3.33)

From the above expressions the gradient with respect to the controller \( K \) is obtained,

\[
\frac{\partial J_{\text{aux}}}{\partial K} = B_2^T \frac{\partial J_{\text{aux}}}{\partial A_f} C_2^T + B_2^T \frac{\partial J_{\text{aux}}}{\partial B_f} D_{21}^T + D_{12}^T \frac{\partial J_{\text{aux}}}{\partial C_f} C_2^T + D_{12}^T \frac{\partial J_{\text{aux}}}{\partial D_f} D_{21}^T
\]

(3.34)

as well as the gradient with respect to \( \gamma \),

\[
\frac{\partial J_{\text{aux}}}{\partial \gamma} = \text{tr}[-2\gamma H_s P A_s^T X B_f G_s - 2\gamma G_s B_f^T X A_s P H_s^T - 2\gamma H_s (R + R^T) H_s^T].
\]

(3.35)

3.1.6 Robust \( H_2 \) synthesis in continuous time

Costs

The plant of Figure 2.2,

\[
P = \begin{bmatrix}
A & B_0 & B_1 & B_2 \\
C_0 & 0 & 0 & D_{02} \\
C_1 & 0 & 0 & D_{12} \\
C_2 & D_{20} & D_{21} & 0
\end{bmatrix}
\]

(3.36)

and the feedback \( K \) yield the closed-loop plant

\[
T_{zw} := \begin{bmatrix}
A_P & B_{P0} & B_{P1} \\
C_{P0} & 0 & 0 \\
C_{P1} & 0 & 0
\end{bmatrix}
\]

(3.37)

where

\[
A_P := A + B_2 K C_2, \quad B_{P0} := B_0 + B_2 K D_{20}, \quad B_{P1} := B_1 + B_2 K D_{21} \\
C_{P0} := C_0 + D_{02} K C_2 \\
C_{P1} := C_1 + D_{12} K C_2.
\]

(3.38)

where, for the sake of simplicity, the system has been assumed to be such that all direct feedthrough matrices \( D \) are equal to zero.

Introduce the following system to describe the augmented plant in Figure 2.3,

\[
\begin{bmatrix}
\dot{z}_0 \\
\dot{z}_\Delta
\end{bmatrix} = \begin{bmatrix}
A_R & B_{R0} & B_{R1} \\
C_{R0} & 0 & 0 \\
C_{R1} & 0 & 0
\end{bmatrix} \begin{bmatrix}
w_0 \\
w_\Delta
\end{bmatrix}.
\]

(3.39)

It is obtained by augmenting the closed-loop plant \( T_{zw} \) in (3.37) by the weighting filter \( D \) given by

\[
D := \begin{bmatrix}
A_D & B_D \\
C_D & D_D
\end{bmatrix}.
\]

(3.40)
yielding the system matrices

\[
A_R := \begin{bmatrix}
A_D - B_D D_D^{-1} C_D & 0 & 0 \\
-B_P D_D^{-1} C_D & A_P & 0 \\
0 & B_D C_P & A_D \\
\end{bmatrix},
\]

\[
B_{R0} := \begin{bmatrix}
0 \\
B_{P0} \\
0 \\
\end{bmatrix}, \quad B_{R1} := \begin{bmatrix}
B_D D_D^{-1} \\
B_P D_D^{-1} \\
0 \\
\end{bmatrix} \tag{3.41}
\]

\[
C_{R0} := \begin{bmatrix}
0 & C_{P0} & 0 \\
\end{bmatrix}, \quad C_{R1} := \begin{bmatrix}
0 & D_DC_P & C_D \\
\end{bmatrix}.
\]

Let \( J_{RLQ}(K, D) \) denote the representation of the worst-case cost \( \mathcal{L}(P, K, d) \) as obtained with the dynamic filter \( D \). Its value is obtained by solving the continuous-time Riccati equation

\[
A_R^T S + S A_R + \gamma^{-2} SB_{R1} B_{R1}^T S + C_{R0}^T C_{R0} + C_{R1}^T C_{R1} = 0 \tag{3.42}
\]

where the matrix

\[
A_S := A_R + \gamma^{-2} B_{R1} B_{R1}^T S \tag{3.43}
\]

should be asymptotically stable. The value of the robust \( H_2 \) cost is then

\[
J_{RLQ} = \text{tr}\{SB_{R0}B_{R0}^T\}. \tag{3.44}
\]

In some cases it can be useful to have an auxiliary cost to guarantee that \( A_S \) is a stability matrix. We therefore introduce the cost

\[
J_{R, \text{aux}} := \text{tr}\{R\} \tag{3.45}
\]

where \( R \) is given by the continuous Lyapunov equation

\[
A_S^T R + RA_S + I = 0. \tag{3.46}
\]

Robust \( H_2 \) synthesis is then done by minimizing \( J_{RLQ} \), or alternatively a weighted sum of \( J_{RLQ} \) and \( J_{R, \text{aux}} \) denoted \( J_R \). For this purpose, expressions for the gradients are derived in the sequel.

**Gradient expressions**

The gradients of the cost functions, with respect to the matrices in (3.41), are

\[
\frac{\partial J_{RLQ}}{\partial A_R} = 2S M_1, \quad \frac{\partial J_{RLQ}}{\partial B_{R0}} = 2SB_{R0}, \quad \frac{\partial J_{RLQ}}{\partial B_{R1}} = 2SM_1 M_0^T \\
\frac{\partial J_{RLQ}}{\partial C_{R0}} = 2C_{R0} M_1, \quad \frac{\partial J_{RLQ}}{\partial C_{R1}} = 2C_{R1} M_1 \\
\frac{\partial J_{R, \text{aux}}}{\partial A_R} = 2RM_2 + 2S(M_3 + M_3^T), \quad \frac{\partial J_{R, \text{aux}}}{\partial B_{R0}} = 0 \\
\frac{\partial J_{R, \text{aux}}}{\partial B_{R1}} = 2RM_2 M_0^T + 2\gamma^{-2} SM_2 RB_{R1} + 2S(M_3 + M_3^T)M_0^T \\
\frac{\partial J_{R, \text{aux}}}{\partial C_{R0}} = 2C_{R0}(M_3 + M_3^T), \quad \frac{\partial J_{R, \text{aux}}}{\partial C_{R1}} = 2C_{R1}(M_3 + M_3^T) \tag{3.47}
\]
3.1. GRADIENT-BASED OPTIMIZATION

where \(M_0, M_1, M_2\) and \(M_3\) are given by

\[
M_0 = \gamma^{-2} B_{R1}^T S \\
A_S M_1 + M_1 A_S^T + B_{R0} B_{R0}^T = 0 \\
A_S M_2 + M_2 A_S^T + I = 0 \\
A_S M_3 + M_3 A_S^T + \gamma^{-2} M_2 R B_{R1} B_{R1}^T = 0.
\] (3.48)

The gradients of the total cost function \(J_R\) with respect to the matrices of the feedback plant (3.37) and those of the weighting filter \(D\) in (3.40) are then

\[
\frac{\partial J_R}{\partial A_D} = \begin{bmatrix} I & 0 & 0 \end{bmatrix} \frac{\partial J_R}{\partial A_R} \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} I & 0 & 0 \end{bmatrix} \frac{\partial J_R}{\partial A_R} \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix}
\]

\[
\frac{\partial J_R}{\partial B_D} = -\begin{bmatrix} I & 0 & 0 \end{bmatrix} \frac{\partial J_R}{\partial A_R} \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} (D_D^{-1} C_D)^T + \begin{bmatrix} I & 0 & 0 \end{bmatrix} \frac{\partial J_R}{\partial A_R} \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} C_{P1}^T
\]

\[
\frac{\partial J_R}{\partial C_D} = -B_{R1}^T \frac{\partial J_R}{\partial A_R} \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} + \frac{\partial J_R}{\partial C_{R1}} \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix}
\]

\[
\frac{\partial J_R}{\partial D_D} = B_{R1}^T \frac{\partial J_R}{\partial A_R} \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} (D_D^{-1} C_D)^T - B_{R1}^T \frac{\partial J_R}{\partial B_{R1}} (D_D^{-1})^T + \frac{\partial J_R}{\partial C_{R1}} \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} C_{P1}^T
\]

\[
\frac{\partial J_R}{\partial A_P} = \begin{bmatrix} 0 & I & 0 \end{bmatrix} \frac{\partial J_R}{\partial A_R} \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix}
\]

\[
\frac{\partial J_R}{\partial B_{P1}} = -\begin{bmatrix} 0 & I & 0 \end{bmatrix} \frac{\partial J_R}{\partial A_R} \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} (D_D^{-1} C_D)^T + \begin{bmatrix} 0 & I & 0 \end{bmatrix} \frac{\partial J_R}{\partial B_{R1}} (D_D^{-1})^T
\]

\[
\frac{\partial J_R}{\partial B_{P0}} = \frac{\partial J_R}{\partial C_{R0}} \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix}
\]

\[
\frac{\partial J_R}{\partial C_{P1}} = B_D^T \begin{bmatrix} 0 & 0 & I \end{bmatrix} \frac{\partial J_R}{\partial A_R} \begin{bmatrix} I \\ 0 \end{bmatrix} + D_D^T \frac{\partial J_R}{\partial C_{R1}} \begin{bmatrix} 0 \\ I \end{bmatrix}
\]

where \(\partial J_R/\partial \ast\) is a weighted sum of \(\partial J_{RLQ}/\partial \ast\) and \(\partial J_{R,aux}/\partial \ast\) in the same way that \(J_R\) is a weighted sum of \(J_{RLQ}\) and \(J_{R,aux}\).

Finally, the gradient with respect to the feedback \(K\) is given by

\[
\frac{\partial J_R}{\partial K} = B_2^T \frac{\partial J_R}{\partial A_P} C_2^T + B_2^T \frac{\partial J_R}{\partial B_{P0}} D_{20}^T + B_2^T \frac{\partial J_R}{\partial B_{P1}} D_{21}^T \\
+ D_{02}^T \frac{\partial J_R}{\partial C_{P0}} C_2^T + D_{12}^T \frac{\partial J_R}{\partial C_{P1}} C_2^T.
\] (3.49)
3.1.7 Robust $H_2$ synthesis in discrete time

Costs

As in the case of continuous-time synthesis, the plant matrices describing $P$ in Figure 2.2 are given in (2.1). The closed-loop plant is obtained as

$$ T_{zw} = \begin{bmatrix} A_P & B_{P0} & B_{P1} \\ C_{P0} & D_{P0} & D_{P1} \\ C_{P1} & 0 & 0 \end{bmatrix} $$

(3.50)

where

$$ A_P := A + B_2 K C_2, \quad B_{P0} := B_0 + B_2 K D_{20}, \quad B_{P1} := B_1 + B_2 K D_{21} $$

(3.51)

$$ C_{P0} := C_0 + D_{20} K C_2, \quad D_{P0} := D_{20} K D_{20}, \quad D_{P1} := D_{20} K D_{21} $$

Notice that the direct feedthrough matrices $D_{P0}$ and $D_{P1}$ are allowed to assume values other than zero in this case.

The closed-loop plant (3.50) is augmented by a weighting filter $D$ of the form given in (3.40), yielding the plant depicted in Figure 23,

$$ \begin{bmatrix} z_0 \\ z_{\Delta} \end{bmatrix} = \begin{bmatrix} A_R & B_{R0} & B_{R1} \\ C_R & D_{R0} & D_{R1} \end{bmatrix} \begin{bmatrix} w_0 \\ w_{\Delta} \end{bmatrix} $$

(3.52)

with the matrices

$$ A_R := \begin{bmatrix} A_D - B_D D_D^{-1} C_D & 0 & 0 \\ -B_{P1} D_D^{-1} C_D & A_P & 0 \\ 0 & B_D C_{P1} & A_D \end{bmatrix} $$

(3.53)

$$ B_{R0} := \begin{bmatrix} 0 \\ B_{P0} \\ 0 \end{bmatrix}, \quad B_{R1} := \begin{bmatrix} B_D D_D^{-1} \\ B_{P1} D_D^{-1} \\ 0 \end{bmatrix} $$

$$ C_R := \begin{bmatrix} -D_{P1} D_D^{-1} C_D & C_{P0} & 0 \\ 0 & D_D C_{P1} & C_D \end{bmatrix} $$

$$ D_{R0} := \begin{bmatrix} D_{P0} \\ 0 \end{bmatrix}, \quad D_{R1} := \begin{bmatrix} D_{P1} D_D^{-1} \\ 0 \end{bmatrix} $$

The worst-case cost is now given by

$$ J_{RLQ} := \text{tr} \{ B_{R0}^T S B_{R0} \} + \text{tr} \{ D_{R0}^T D_{R0} \} $$

(3.54)

where $S$ is obtained from the discrete-time Riccati equation

$$ S = A_R^T S A_R + (B_{R1}^T S A_R + D_{R1}^T C_R)^T (\gamma^2 I - D_{R1}^T D_{R1} - B_{R1}^T S B_{R1})^{-1} (B_{R1}^T S A_R + D_{R1}^T C_R) + C_R^T C_R $$

(3.55)

such that

$$ \gamma^2 I - D_{R1}^T D_{R1} - B_{R1}^T S B_{R1} > 0 $$

(3.56)
and the matrix
\[ A_S := A_R + B_R (\gamma^2 I - D_{R1}^T D_{R1} - B_{R1}^T S B_{R1})^{-1} (B_{R1}^T S A_R + D_{R1}^T C_R) \] (3.57)
is asymptotically stable.

The auxiliary cost corresponding to (3.45) is given by
\[ J_{R, aux} := \text{tr} \{ R \}, \] (3.58)
where \( R \) is given by the discrete-time Lyapunov equation
\[ R = A_S^T R A_S + I. \] (3.59)

Also, the weighted cost \( J_R \) is introduced as before.

**Gradient expressions**

The gradients of the cost functions, with respect to the matrices in the augmented system (3.52), are

\[ \frac{\partial J_{RLQ}}{\partial A_R} = 2S A_S M_1, \quad \frac{\partial J_{RLQ}}{\partial B_{R0}} = 2SB_{R0}, \quad \frac{\partial J_{RLQ}}{\partial C_R} = 2(C_R + D_{R1} M_0) M_1 \]
\[ \frac{\partial J_{RLQ}}{\partial D_{R0}} = 2D_{R0}, \quad \frac{\partial J_{RLQ}}{\partial D_{R1}} = 2(C_R + D_{R1} M_0) M_1 M_0^T \]

\[ \frac{\partial J_{R, aux}}{\partial A_R} = 2RA_S M_2 + 2SB_{R1} M B_{R1}^T R A_S M_2 + 2SA_S (M_3 + M_3^T), \quad \frac{\partial J_{R, aux}}{\partial B_{R0}} = 0 \]
\[ \frac{\partial J_{R, aux}}{\partial B_{R1}} = 2RA_S M_2 M_0^T + 2SB_{R1} M B_{R1}^T R A_S M_2 M_0^T + 2SA_S (M_3 + M_3^T) M_0^T \]
\[ \frac{\partial J_{R, aux}}{\partial C_R} = 2D_{R1} M B_{R1}^T R A_S M_2 + 2(C_R + D_{R1} M_0) (M_3 + M_3^T), \quad \frac{\partial J_{R, aux}}{\partial D_{R0}} = 0 \]
\[ \frac{\partial J_{R, aux}}{\partial D_{R1}} = 2(C_R + D_{R1} M_0) M_2 A_S^T R B_{R1} M + 2D_{R1} M B_{R1}^T R A_S M_2 M_0^T \]
\[ + 2(C_R + D_{R1} M_0) (M_3 + M_3^T) M_0^T \]

where
\[ M = (\gamma^2 I - D_{R1}^T D_{R1} - B_{R1}^T S B_{R1})^{-1} \]

\[ M_0 = M(B_{R1}^T S A_R + D_{R1}^T C_R) \]
\[ M_1 = A_S M_1 A_S^T + B_{R0} B_{R0}^T \]
\[ M_2 = A_S M_2 A_S^T + I \]
\[ M_3 = A_S M_3 A_S^T + A_S M_2 A_S^T R B_{R1} M B_{R1}^T. \] (3.60)

The gradients of the cost function \( J_R \) with respect to the matrices of the weighting filter \( D \) and the closed-loop plant \( T_{z,w} \) are

\[ \frac{\partial J_R}{\partial A_D} = \begin{bmatrix} I & 0 & 0 \end{bmatrix} \frac{\partial J_R}{\partial A_R} \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & I \end{bmatrix} \frac{\partial J_R}{\partial A_R} \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} \]
\[
\begin{align*}
\frac{\partial J_R}{\partial B_D} &= -\left[ I \ 0 \ 0 \right] \frac{\partial J_R}{\partial \mathbf{A}_R} \left[ I \ 0 \right]^T (D_D^{-1}C_D)^T + \left[ 0 \ 0 \ I \right] \frac{\partial J_R}{\partial \mathbf{A}_R} \left[ I \ 0 \right]^T C_{P_1}^T \\
&\quad + \left[ I \ 0 \ 0 \right] \frac{\partial J_R}{\partial \mathbf{B}_{R1}} (D_D^{-1})^T \\
\frac{\partial J_R}{\partial C_D} &= -B_{R1} \frac{\partial J_R}{\partial \mathbf{A}_R} \left[ I \ 0 \ 0 \right] \left[ I \ 0 \right]^T (D_D^{-1}C_D)^T - B_{R1} \frac{\partial J_R}{\partial \mathbf{B}_{R1}} (D_D^{-1})^T \\
&\quad + D_{R1} \frac{\partial J_R}{\partial \mathbf{C}_R} \left[ I \ 0 \ 0 \right] (D_D^{-1}C_D)^T + \left[ 0 \ 0 \ I \right] \frac{\partial J_R}{\partial \mathbf{C}_R} \left[ I \ 0 \right]^T C_{P_1}^T - D_{R1} \frac{\partial J_R}{\partial \mathbf{D}_{R1}} (D_D^{-1})^T \\
\frac{\partial J_R}{\partial \mathbf{A}_P} &= \left[ 0 \ 0 \ I \right] \frac{\partial J_R}{\partial \mathbf{A}_R} \left[ I \ 0 \ 0 \right] , \quad \frac{\partial J_R}{\partial \mathbf{B}_{P0}} = \left[ 0 \ 0 \ I \right] \frac{\partial J_R}{\partial \mathbf{B}_{R0}} \\
\frac{\partial J_R}{\partial \mathbf{B}_{P1}} &= -\left[ 0 \ 0 \ I \right] \frac{\partial J_R}{\partial \mathbf{A}_R} \left[ I \ 0 \ 0 \right]^T (D_D^{-1}C_D)^T + \left[ 0 \ 0 \ I \right] \frac{\partial J_R}{\partial \mathbf{C}_R} \left[ I \ 0 \right]^T C_{P_1}^T \frac{\partial J_R}{\partial \mathbf{D}_{P1}} (D_D^{-1})^T \\
\frac{\partial J_R}{\partial C_{P0}} &= \left[ I \ 0 \ 0 \right] \frac{\partial J_R}{\partial \mathbf{A}_R} \left[ I \ 0 \ 0 \right] \\
\frac{\partial J_R}{\partial \mathbf{C}_{P1}} &= B_D \left[ 0 \ 0 \ I \right] \frac{\partial J_R}{\partial \mathbf{A}_R} \left[ I \ 0 \ 0 \right]^T + D_D \left[ 0 \ 0 \ I \right] \frac{\partial J_R}{\partial \mathbf{C}_R} \left[ I \ 0 \ 0 \right] \\
\frac{\partial J_R}{\partial \mathbf{D}_{P0}} &= \left[ I \ 0 \ 0 \right] \frac{\partial J_R}{\partial \mathbf{A}_R} \left[ I \ 0 \ 0 \right] \\
\frac{\partial J_R}{\partial \mathbf{D}_{P1}} &= -\left[ I \ 0 \ 0 \right] \frac{\partial J_R}{\partial \mathbf{A}_R} \left[ I \ 0 \ 0 \right]^T (D_D^{-1}C_D)^T + \left[ I \ 0 \ 0 \right] \frac{\partial J_R}{\partial \mathbf{D}_{R1}} (D_D^{-1})^T.
\end{align*}
\]

Finally, the gradient with respect to the feedback \( K \) is

\[
\frac{\partial J_R}{\partial K} = B_2 \frac{\partial J_R}{\partial \mathbf{A}_P} C_2^T + B_2 \frac{\partial J_R}{\partial \mathbf{B}_{P0}} D_{20}^T + B_2 \frac{\partial J_R}{\partial \mathbf{B}_{P1}} D_{21}^T + D_{20} \frac{\partial J_R}{\partial \mathbf{C}_{P0}} C_2^T \\
+ D_{12} \frac{\partial J_R}{\partial \mathbf{C}_{P1}} C_2^T + D_{20} \frac{\partial J_R}{\partial \mathbf{D}_{P0}} D_{20}^T + D_{20} \frac{\partial J_R}{\partial \mathbf{D}_{P1}} D_{21}^T. \quad (3.61)
\]

### 3.2 Parametric solutions of stated problems

Having introduced the necessary costs and gradient expressions, we are now ready to solve the problems introduced before. Notice that in all cases it is assumed that we
have an initial stable controller which satisfies the constraints pertinent to the method in question (see e.g. [70] for details on how to find an initial stable controller).

The $H_2$ problem

In order to solve Problem 2.1, it is sufficient to directly minimize the cost $J_2$ as given in (2.7). Improved performance is obtained by using the gradient expressions of sections 3.1.1 or 3.1.2. The cost $J_2$ approaches infinity as the closed loop approaches instability. However, since the cost is infinite only at the stability boundary, some additional numerical checks are necessary in order to ensure that the stability boundary is not crossed during optimization.

The $H_\infty$ minimization problem

In order to solve Problem 2.2 by minimizing the $H_\infty$ norm, an approach is to minimize a cost the upper bound of the $H_\infty$ norm, $\gamma$, e.g. by minimizing

$$J_\gamma := \gamma^2,$$ \hfill (3.62)

and to use the auxiliary cost $J_{aux}$ in (3.14) as a weighted penalty cost in order to ensure that the value of $\gamma$ is allowable.

Some additional numerical test are necessary in order to ensure that the stability boundary is not crossed.

The $H_\infty$ norm existence problem

A solution to Problem 2.3 can be obtained by solving Problem 2.2 and interrupting the optimization as an allowable value of the $H_\infty$ norm has been obtained.

Notice that the value of $\gamma$ used in the optimization may be larger than the $H_\infty$ norm which is sought for.

The $H_\infty$ norm satisfaction problem

Problem 2.4 has relevance as a subproblem in a multiobjective control problem. The auxiliary cost $J_{aux}$ can be used as a weighted penalty cost in conjunction with other costs in order to ensure that the $H_\infty$ norm bound holds.

The $H_2/H_\infty$ problem I

Problem 2.5 can be minimized by simultaneous minimization of a weighted sum of the costs $J_2$, $J_\gamma$ and $J_{aux}$. This problem rarely has practical significance.
The $H_2/H_\infty$ problem II

Problem 2.6 is a standard problem, where an $H_2$ performance measure is optimized whilst the $H_\infty$ constraint ensures that the closed loop is robustly stable.

The problem can be solved by minimizing a weighted sum of the cost $J_2$ and the penalty cost $J_{aux}$.

The multimodel $H_2$ problem

Problem 2.7 can be solved by minimizing a weighted sum of costs $J_2(P_n,K)$, $n = 1, \ldots, N$.

The multimodel $H_2/H_\infty$ problem

Problem 2.8 can be solved by minimizing a weighted sum of costs $J_2(P_n,K)$ and $J_{aux}(P_n,K)$, $n = 1, \ldots, N$.

The robust $H_2$ problem

Problem 2.9 is solved by direct minimization of the cost $J_{RLQ}(P,K,D)$ as given in (3.44) with respect to the controller $K$ and the filter $D$.

In some cases the optimizations display better convergence if the auxiliary cost $J_{R,aux}(P,K,D)$ is used in conjunction with $J_{RLQ}$.

The multimodel robust $H_2$ problem

In the study [58] an additional problem has been studied, where a robust $H_2$ cost is minimized for a set of plants $P_n$, $n = 1, \ldots, N$. This leads to a problem where a sum of costs $J_{RLQ}(P_n,K,D_n)$ is minimized with respect to the controller $K$ and $N$ different weighting filters $D_n$. The auxiliary costs $J_{R,aux}(P_n,K,D_n)$ can be used to ensure that the solution stays within the allowed region for all models $P_n$.

3.3 Considerations in parametric design

3.3.1 Selection of weights

Usually when designing controllers by the methods described in this work, a number of weighting filters are introduced to weight conflicting design objectives against one
another. In order to obtain good values for these weighting filters, the usual approach is to use trial and error, where different sets of weights are tested by alternating control synthesis and simulation. A drawback with parametric design is that the control synthesis requires longer computation time than the analytical methods. Thus, the weight tuning may demand a significant amount of time.

Besides filters related to weighting different objectives against each other, in some methods there are also the weights related to the auxiliary costs. A good way to select the magnitude of these is usually to perform some form of weight contraction. First, a relative large number is selected for the penalty weight. For this weight, an optimization, possibly with a limited number of iterations, is performed. Then, the magnitude of the weight is reduced and a new optimization is done, etc., until the weight is so small that new optimizations will not affect the result.

This scheme is useful since setting the penalty weight to be very small at the start will produce an ill-conditioned optimization which may not converge properly, due to the resulting steep boundary between allowed and non-allowed function space. Conversely, if the optimization is done only with a large weighting filter, the value of the penalty cost at the optimum may dominate the total cost too much.

3.3.2 The form of the controller

The theoretical solutions of the $H_2$ and the $H_\infty$ problems yield controllers without direct feedthrough, i.e. controllers whose matrix $D_e$ is zero. Even so, the optimal solution of the problem may be a controller with direct feedthrough. The solver approximates this solution by including very fast dynamics in the controller, i.e. very large elements in its $A_e$ matrix. An example of such a problem is the example in [91].

When the optimum is obtained by a controller with direct feedthrough, and a zero $D_e$ matrix is imposed on the controller structure, the parametric optimization procedure will generally run into problems. It is then a better idea to allow the matrix $D_e$ of the controller to attain non-zero values.

However, some of the design restrictions may prohibit this. In particular, the matrix $D_{02}D_eD_{20}$ is usually required to be equal to zero in the basic $H_2$ problem. If this is not already the case, a simple way to satisfy the restriction is to eliminate direct feedthrough from the plant input to the controlled output by an appropriate filter which turns $D_{02}$ into a zero matrix (this has been done e.g. in the example with the diesel engine in Section 3.4.4).

Another factor which may hinder the parametric optimization from converging properly is under-definition of the control objectives. It may then be a good idea to have an additional term in the cost, for example

$$J_{supp} = \text{tr}[K^TK]$$

which suppresses the magnitude of the elements of the controller $K$.  

(3.63)
3.3.3 Initial controller

The methods described here have in common that the optimization must start from an initial controller which satisfies the given stability and $H_\infty$-bound criteria. If a stabilizing controller is not readily available, it is usually not hard to find one by trial and error. Alternatively, a procedure based on optimization can be used [70].

3.4 Augmenting for controllers

3.4.1 Augmenting for a state-space controller

The parametric design methods described above have been given in a form where the controller $K$ is assumed not to contain any dynamics, i.e. to consist of only a direct feedthrough in the control law

$$ u(t) = Ky(t). \quad (3.64) $$

However, in most cases we should like to perform parametric design of controllers of higher order than zero. This can be done by including the necessary dynamics into the system, so that the controller is still expressed as in (3.64).

Augmenting for a controller: continuous time

It is assumed that the continuous-time system for which to design a controller is given by

$$ x_p(t) = A_{p}x_p(t) + B_{p1}w_1(t) + B_{p2}u_p(t) $$
$$ z_1(t) = C_{p1}x_p(t) + D_{p12}u_p(t) $$
$$ y_p(t) = C_{p2}x_p(t) + D_{p21}w_1(t) \quad (3.65) $$

where, for simplicity of illustration, only the matrices related to the $H_\infty$ control problem are shown. Further, it is assumed that the controller is to have the state-space representation

$$ \dot{x}_c(t) = A_c x_c(t) + B_c y_p(t) $$
$$ u_p(t) = C_c x_c(t) + D_c y_p(t) \quad (3.66) $$

where $x_c$ is of suitable dimension.

An augmented system can be formed by making the signal definitions

$$ x(t) := \begin{bmatrix} x_p(t) \\ x_c(t) \end{bmatrix}, \quad u(t) := \begin{bmatrix} \dot{x}_c(t) \\ u_p(t) \end{bmatrix}, \quad y(t) := \begin{bmatrix} x_c(t) \\ y_p(t) \end{bmatrix}. \quad (3.67) $$

Then, the system matrices are rewritten as

$$ A = \begin{bmatrix} A_p & 0 \\ 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} B_{p1} \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & B_{p2} \\ I & 0 \end{bmatrix}, $$
$$ C_1 = \begin{bmatrix} C_{p1} & 0 \end{bmatrix}, \quad D_{12} = \begin{bmatrix} 0 & D_{p12} \end{bmatrix}, $$
$$ C_2 = \begin{bmatrix} 0 & I \\ C_{p2} & 0 \end{bmatrix}, \quad D_{21} = \begin{bmatrix} 0 \\ D_{p21} \end{bmatrix}. \quad (3.68) $$
and
\[
K = \begin{bmatrix}
A_C & B_C \\
C_C & D_C
\end{bmatrix}
\] (3.69)

forming the system
\[
\begin{align*}
\dot{x}(t) &= A x(t) + B_1 w_1(t) + B_2 u(t) \\
z_1(t) &= C_1 x(t) + D_{12} u(t) \\
y(t) &= C_2 x(t) + D_{21} w_1(t)
\end{align*}
\] (3.70)

with the direct feedback
\[
u(t) = K y(t),
\] (3.71)

cf. (2.1).

The system defined in (3.67)–(3.71) is equivalent to the one defined in (3.65) and (3.66). The expressions for costs and gradients can readily be used by performing the above transformation of the system. In this way, no augmentation is needed during optimization.

**Augmenting for a controller: discrete time**

In analogy with the case in continuous time described above, the signals of a discrete-time system
\[
\begin{align*}
x_P(t+1) &= A_p x_P(t) + B_{p1} w_1(t) + B_{p2} u_P(t) \\
z_1(t) &= C_{p1} x_P(t) + D_{p12} u_P(t) \\
y_P(t) &= C_{p2} x_P(t) + D_{p21} w_1(t)
\end{align*}
\] (3.72)

and a discrete-time controller
\[
\begin{align*}
x_C(t+1) &= A_C x_C(t) + B_C y_P(t) \\
u_P(t) &= C_C x_C(t) + D_C y_P(t).
\end{align*}
\] (3.73)

can be expanded according to
\[
\begin{align*}
x &= \begin{bmatrix} x_P(t) \\ x_C(t) \end{bmatrix}, \\
u &= \begin{bmatrix} x_C(t+1) \\ u_P(t) \end{bmatrix}, \\
y &= \begin{bmatrix} x_C(t) \\ y_P(t) \end{bmatrix}
\end{align*}
\] (3.74)

Then, defining the system matrices as in (3.68) and (3.69) yields the equivalent system
\[
\begin{align*}
x(t+1) &= A x(t) + B_1 w_1(t) + B_2 u(t) \\
z_1(t) &= C_1 x(t) + D_{12} u(t) \\
y(t) &= C_2 x(t) + D_{21} w_1(t)
\end{align*}
\] (3.75)

with the direct feedback
\[
u(t) = K y(t),
\] (3.76)

cf. (2.1).
3.4.2 Structurally fixed state-space controllers

When optimizing a state-space controller using the previously described augmentation, it is always possible to reduce the number of optimized elements in the controller by masking out fixed elements from the vector of optimization variables. If this is done, the corresponding vector of gradients must be formed by the gradient elements which correspond to the optimized elements. This can be done for instance when an optimal decentralized control scheme is sought for.

Especially in the case of a SISO controller, optimizing on all elements in the state-space matrices involves a large number of redundant optimization variables. By fixing the controller to some canonical transfer function form, the number of optimized variables can be reduced to the minimum. However, experience has shown that the above procedure does not necessarily lead to the best result, since fixing the structure of the controller may lead to optimization problems with worse condition. Thus, for low-order controllers, the simplest and safest approach is to optimize on full state-space matrices.

3.4.3 Augmenting for a PID-type controller

On occasion, it is of interest to perform parametric design on a controller of traditional PID structure. A straightforward way to accomplish this is to construct an output signal which contains the process output, an integral of the process output and an approximated derivative or, in discrete time, a difference of the process output. The controller can then once again be expressed as in (3.64), where

\[
F = \begin{bmatrix} K_P & K_I & K_D \end{bmatrix}.
\] (3.77)

In the following, an example of such a controller is described.

3.4.4 Example: a parametrically designed PID controller with feed forward for a diesel engine

The methodology of parametric control design for a simple controller structure is illustrated by a simulated example with a large-scale diesel engine. The model of the diesel engine has been described in detail in the Master’s thesis of Nuortio [56]. The thesis was written for Wärtsilä NSD Corporation.

The engine is modelled in continuous time. The linear model at a given operating point is of order six. The property which is being controlled is the engine’s speed of rotation \( \omega_e \) by manipulating the relative fuel rack position. The engine is subject disturbances in the load torque \( M_{\text{load}} \), of which measurements are available.

It is proposed in [56] that the engine be controlled by a combined feedforward and feedback scheme, possibly by a controller with PID structure. For design of a parametrically tuned LQ controller with this structure, the plant is therefore augmented as in Figure 3.1. The plant output \( y \) is set to be a vector of signals, where one signal is the
measurement of the disturbance, another one an integral of the measured plant output $\omega$, one is its derivative and the last one is the measured plant output itself.

Thus, the block denoted $F_p$ represents the assumed measurement dynamics of the load disturbance, and $F_d$ represents a filter which approximates a derivative. The filter $F$ is an integrator, which may be approximated by the stable function $1/(s + \epsilon)$ in order to facilitate the numerical solution of the problem. In addition to the integrator at the output, one integrator is required to represent the step-disturbance problem as an initial-value problem. Given the augmentation of Figure 3.1, the PID controller with a feed-forward component is represented by a simple one-by-four matrix whose constant elements are the sole variables being optimized.

In addition to forming the signal $y$, the signals of the output of the augmented process are also used in conjunction with a filtered value of the signal $u$ to form the controlled output $z$ (omitted from the figure). The problem of finding suitable weights for the signals in $z$ transforms the tuning of the elements of the controller into the tuning of a corresponding set of weights. However, it is relatively simple to find weights which yield good performance.

At present, the engines of Wärtsilä NSD Corp. have been controlled by a PID-type gain-scheduled controller. The response obtained by this controller at the given operating point by introducing a load disturbance is shown in Figure A. At the time of the disturbance occurring, the setpoint of the rotation speed $\omega$ is modified as a function of the disturbance measurement. This is the so-called 'speed droop', cf. the figure. Some oscillation is visible in the output. This is due to the model having an under-damped mode.

An example of the control result with a parametrically designed PID controller with a feedforward component is shown in Figure 3.3. The overshoot of the controlled signal has been decreased by using information about the disturbance.

For comparison, the feedforward component is left out and an LQ-optimal PID controller, based on feedback only, is synthesized. For this a separate set of weights is required. The result is shown in Figure 3.4. The overshoot is now considerably larger, and comparable with the original PID controller, a fact which shows that the original controller is well tuned at the operating point in question.
Figure 3.2: Control result obtained by Wärtsilä’s original PID-type controller. The dashed line shows the setpoint including the speed droop.

3.5 Augmenting for disturbances

3.6 Initial-value problems

As previously mentioned, the $H_2$ norm gives an expression of the quadratic cost of the output of a system which is subject to a pulse or an initial-value disturbance. For the closed-loop and initial-value system

$$
\begin{align*}
\sigma x & = A_f x, \ x(0) = x_0 \\
x_0 & = C_f x
\end{align*}
$$

(3.78)

the $H_2$ cost, i.e. the square of the $L_2$ norm of $x_0$, integrated over time, is given by

$$
J_{2, init} = \text{tr} \left\{ x_0^T Q x_0 \right\}
$$

(3.79)

where $Q$ is obtained from the Lyapunov equation (3.3) in the continuous-time case and (3.8) in the discrete-time case.
Thus, recalling from e.g. (3.1) that \( B_f = B_0 + B_2 K D_{20} \), it is clear that the \( H_2 \) synthesis methods described previously can be used as such for the initial-value problem, by setting

\[
B_0 = x_0, \quad D_{20} = 0. \tag{3.80}
\]

Further, a sum of costs for a set of initial values \( x_{0,i}, i = 1, \ldots, N \) can be optimized for by setting

\[
B_0 = \begin{bmatrix} x_{0,1} & \cdots & x_{0,N} \end{bmatrix}, \quad D_{20} = 0. \tag{3.81}
\]

The initial-value problem enables the design of \( H_2 \)-optimal control for a large class of disturbances which can be expressed as initial-value differential equations, as well as many types of tracking problems. A common example is described in the sequel, where the augmentation for a step disturbance or setpoint step change tracking problem is described.
Figure 3.4: Result obtained by LQ-optimally tuned PID (feedback-only) controller.

### 3.6.1 Augmenting for step disturbances

Consider an input–output description of a plant \( G \), given by the state-space equations

\[
\begin{align*}
    x_G(t) &= A_G x_G(t) + B_G u_G(t) + B_s s(t) \\
    y_G(t) &= C_G x(t) + D_s s(t).
\end{align*}
\]

(3.82)

where

\[
s(t) = \begin{cases} 
    0, & t < 0 \\
    1, & t \geq 0
\end{cases}
\]

(3.83)

The signal \( B_s s(t) \) is thus a step disturbance starting at time \( t = 0 \), which affects one or more states of the process and \( D_s s(t) \) may represent e.g. a setpoint change applied at time \( t = 0 \), or some other form of measurement step disturbance.

We should like to obtain a control law which minimizes the \( H_2 \)-type cost

\[
    J_{2,\text{step}} := \int_0^\infty y_G(t)^T W_y y_G(t) + u_G(t)^T W_u u_G(t) \, dt.
\]

(3.84)

The control should be integrating in order to remove the steady-state offset, thus causing \( J_{2,\text{step}} \) to be finite. For this reason, the time derivative of \( u_G \) has been used in the cost \( J_{2,\text{step}} \), rather than \( u_G \) itself.
3.7 Augmenting for Uncertainty

Differentiation of (3.82) yields the equivalent initial-value system

\[
\begin{align*}
\dot{x}_G(t) &= A_G x_G(t) + B_G \dot{u}_G(t), \quad x_G(0) = B_s \\
y_G(t) &= C_G x_G(t), \quad y_G(0) = D_s
\end{align*}
\]  

(3.85)

An expanded model \( P \) can then be introduced, where the signals of the \( P \) are defined as

\[
\begin{align*}
x_P &= \begin{bmatrix} \dot{x}_G \\ y_G \end{bmatrix}, \quad u_P = \dot{u}_G, \quad y_P = y_G, \quad z_0 = \begin{bmatrix} \sqrt{W_y y_G} \\ \sqrt{W_u u_G} \end{bmatrix}
\end{align*}
\]  

(3.86)

and the system matrices of \( P \) as

\[
\begin{align*}
A_P := \begin{bmatrix} A_G & 0 \\ C_G & 0 \end{bmatrix}, \quad B_{P2} := \begin{bmatrix} B_G \\ 0 \end{bmatrix}, \quad x_P(0) = \begin{bmatrix} B_s \\ D_s \end{bmatrix} \\
C_{P0} := \begin{bmatrix} 0 \\ \sqrt{W_y} \end{bmatrix}, \quad D_{P0} := \begin{bmatrix} 0 \\ \sqrt{W_u} \end{bmatrix}, \quad C_{P2} := \begin{bmatrix} 0 & I \end{bmatrix}
\end{align*}
\]  

(3.87)

This yields the equivalent initial-value system

\[
\begin{align*}
x_P(t) &= A_P x_P(t) + B_{P2} u_P(t) \\
z_0(t) &= C_{P0} x_P(t) + D_{P0} u_P(t) \\
y_P(t) &= C_{P2} x_P(t)
\end{align*}
\]  

(3.88)

Thus, in compliance with Section 3.6, the step disturbance problem can be solved in the \( H_2 \) framework given in this work by setting

\[
B_{P0} = x_P(0), \quad D_{P20} = 0.
\]  

(3.89)

This minimizes the cost \( J_s = \int_0^\infty z_0(t)^T z_0(t) dt \) which is equivalent to (3.84). Solving the step disturbance \( H_2 \) problem yields an incremental controller, whose output \( u_P = \dot{u}_G \) must be integrated in order to obtain the control law for the original plant \( G \).

By increasing the number of columns of \( B_{P0} \), the cost \( J_{2, \text{step}} \) is expanded to a weighted sum of step disturbance costs, which enables a tradeoff between optimality for different setpoint changes.

The step disturbance control problem can be analogously derived for the discrete-time case.

3.7 Augmenting for Uncertainty

Several types of uncertainty can be handled within the framework described so far, the most basic ones being unstructured, norm-bounded additive or multiplicative uncertainty.

The uncertainty descriptions can be obtained as a guess based on knowledge of the process or specifications, by using knowledge of errors that result from approximations or simplifications or by identifying both a process and an uncertainty model at the stage of process identification. If a multimodel description is available, the uncertainty can be estimated from inter-model distances. In many cases, the uncertainty description is obtained as a combination of all above approaches.

In Chapter 4 is described a way to determine an uncertainty description from multimodel data.
Chapter 4

Estimation of uncertainty from a multimodel set

In the publications in parts II, III and IV, an uncertainty description based on a multimodel description has been used when robust controllers have been designed. In the sequel follows a brief description of the underlying ideas of this approach.

The material is freely adapted from the conference paper [11] and the article [12].

4.1 Notation

Below are briefly introduced the fundamental concepts in this chapter. The notation of the norms roughly follows [84].

The representation of a matrix operator $G$ in the frequency domain is denoted $G(\omega)$ and a signal $y$ transformed into the frequency domain is denoted $y(\omega)$. The complex conjugate transpose of a vector $x$ is denoted $x^\star$. The singular values of a matrix $A$ are denoted $\sigma_k(A)$, where the $k$th singular value is obtained as the square root of the $k$th eigenvalue of $A^*A$. The largest singular value is denoted $\sigma(A)$. The euclidean norm of a vector $x$ is defined as

$$\|x\|_2 = \sqrt{x^\star x}. \quad (4.1)$$

The norm of an operator $A$ induced by the vector norm $\|\cdot\|_2$ is defined by

$$\|A\|_2 = \max_{\|x\|_2=1} \|Ax\|_2 \quad (4.2)$$

and its value is equal to the maximum singular value $\sigma(A)$. As a matrix norm, the induced norm satisfies the product inequality

$$\|AB\|_2 \leq \|A\|_2 \|B\|_2. \quad (4.3)$$
Further, the induced norm of a matrix $A$ which is a product of two vectors $x$ and $y$ according to $A = xy^*$ has the useful property
\[ \|A\|_2 = \|x\|_2\|y\|_2. \] (4.4)

The $H_\infty$ norm of a transfer function $G$ is equal to
\[ \|G\|_\infty = \sup_\omega \|G(\omega)\|_2 = \sup_\omega (\hat{\varphi}[G(\omega)]). \] (4.5)

## 4.2 Uncertainty estimation from a multimodel set

In the following is presented a method for determining an uncertainty description from a set of linear models. The method yields weighting filters which are suitable for $H_\infty$-based synthesis. A particularity of the method is that it requires a set of inputs associated with the model set. In the form presented here, the method yields output multiplicative uncertainty descriptions, where the weighting filters are applied to the output of the $H_\infty$-bounded perturbation operator. However, it is straightforward to extend the method to include additive uncertainty descriptions of similar structure. The method is applicable both to discrete-time and to continuous-time data and/or models.

### 4.3 Problem setup

It is assumed that we have a set of linear models,
\[ \mathcal{G}_{MM} := \{G_n, \ n = 0, \ldots, N\} \] (4.6)
describing the input–output behaviour of the plant in question. The model $G_0$ is defined to be nominal, i.e. it is central with respect to the set $\mathcal{G}_{MM}$, according to some suitable criterion. If no nominal model is available, it can be determined from the models $G_n, \ n = 1, \ldots, N$. Another assumption is that we have access to the input signals $u_n, \ n = 1, \ldots, N$, which were used to determine the models $G_n$ at the phase of identification.

We are interested in using a norm-bounded perturbation operator $\Delta$ in conjunction with the nominal model $G_0$ to describe the model set. More exactly, we should like the input–output behaviour of the set
\[ \mathcal{G}_{NB}(W) := \{(I + W\Delta)G_0, \ \|\Delta\|_\infty \leq 1\} \] (4.7)
to encompass all of the set $\mathcal{G}_{MM}$ without being too conservative, i.e. while being as small as possible. Here $W$ is a weight which has been introduced in order to shape the norm-bounded and unstructured perturbation $\Delta$. The uncertainty representation (4.7) is of output multiplicative structure.
Observing that the $H_\infty$-norm bound $\|\Delta\|_\infty \leq 1$ is equivalent to
\[
\|\Delta(\omega)\|_2 \leq 1, \text{ all } \omega,
\] (4.8)
we can analyse the problem frequency by frequency by output matching. Then, the requirements above can be translated into the following problems

**Problem 4.1** Find an ideal, infinite-dimensional weighting function $W_\infty(\omega)$ with a given structure and with the following properties.

1. For all frequencies $\omega$ and all models $n$ there exists a perturbation $\Delta_n(\omega)$ with induced norm less than or equal to one, such that
\[
e_n(\omega) := G_n(\omega)u_n(\omega) - G_0(\omega)u_n(\omega) = W_\infty(\omega)\Delta_n(\omega)G_0(\omega)u_n(\omega).
\] (4.9)
I.e., the input-output behaviour of the set $\mathcal{G}_{NB}(W_\infty)$ encloses that of the model set $\mathcal{G}_{MM}$ for a given set of input signals.

2. If, at any frequency $\omega$, the magnitude of $W_\infty(\omega)$ is reduced by a scalar factor, then, for some model $n$, there exists no $\Delta_n(\omega)$ with induced norm less than or equal to one which satisfies (4.9).
I.e., the set $\mathcal{G}_{NB}(W_\infty)$ is as small as possible.

**Problem 4.2** Find a finite-dimensional weighting function $W$ which approximates $W_\infty$, whose magnitude is minimal in some sense, and which satisfies property 1 in Problem 4.1 above.

Using the input data $u_n(\omega)$ as above is advantageous in that the uncertainty description becomes less conservative since vectors are compared instead of matrices. The approach can also be interpreted as associating a given model with a given direction. If no input data were available, a feasible approach might be to use a direction mapping as a substitute. In the SISO or SIMO cases the influence of the inputs is eliminated.

In the sequel, the frequency arguments are omitted for the sake of notational brevity. The following proposition provides a solution to both problems.

**Proposition 4.1** Let $W_\infty$ be an infinite-dimensional weighting function whose structure is defined by a (possibly frequency-dependent) matrix $W_0$ according to
\[
W_\infty := w_\infty W_0
\] (4.10)
where $w_\infty$ is a scalar scaling filter. If the magnitude of $w_\infty$ satisfies
\[
|w_\infty| = \max_n \frac{\|W_0^{-1}e_n\|_2}{\|G_0u_n\|_2}
\] (4.11)
then the weighting function $W_\infty$ has the properties required to be a solution of Problem 4.1.
The filter \( W_0 \) has been introduced in order to have a means of manipulating the directionality properties of the uncertainty. This can lead to a less conservative uncertainty description which still accounts for the model differences.

**Proof of Proposition 4.1.**

*Property 1 of \( W_\infty \):* take any \( n \) and select

\[
\Delta_n = \frac{W_\infty^{-1}e_n(G_0u_n)^*}{\|G_0u_n\|^2_2}.
\]  

(4.12)

Substitution into (4.9) yields

\[
e_n = W_\infty \Delta_n G_0u_n = W_\infty \frac{W_\infty^{-1}e_n(G_0u_n)^*}{\|G_0u_n\|^2_2} G_0u_n = e_n \frac{(G_0u_n)^*G_0u_n}{\|G_0u_n\|^2_2} = e_n.
\]  

(4.13)

Further, by utilization of (4.4), the induced norm of the perturbation \( \Delta_n \) is obtained as

\[
\|\Delta_n\|_2 = \frac{\|W_\infty^{-1}e_n(G_0u_n)^*\|_2}{\|G_0u_n\|^2_2} = \frac{\|W_\infty^{-1}e_n\|_2}{\|G_0u_n\|^2_2} = \|w_\infty\|^{-1}\frac{\|W_0^{-1}e_n\|_2}{\|G_0u_n\|^2_2}
\]  

(4.14)

which is equal to one for the maximizing value of \( n \) and less than or equal to one for other values of \( n \).

*Property 2 of \( W_\infty \):* assume that

\[
|w_\infty| < \max_n \frac{\|W_0^{-1}e_n\|_2}{\|G_0u_n\|^2_2}
\]  

(4.15)

Further, assume that for every \( n \) there is a perturbation \( \Delta_n \) such that \( \|\Delta_n\|_2 \leq 1 \) and \( e_n = W_\infty \Delta_n G_0u_n \). For the maximizing value of \( n \), substitution of \( e_n \) into (4.15) yields

\[
|w_\infty| < \frac{\|W_0^{-1}W_\infty \Delta_n G_0u_n\|_2}{\|G_0u_n\|^2_2} = |w_\infty| \frac{\|\Delta_n G_0u_n\|_2}{\|G_0u_n\|^2_2} \leq |w_\infty| \frac{\|\Delta_n\|_2}{\|G_0u_n\|^2_2} \leq |w_\infty|.
\]  

(4.16)

and so the assumption cannot hold.

Strictly speaking, (4.12) mostly defines an unstable operator \( \Delta_n \), if \( G_0 \) is MIMO and if the equality holds over the whole imaginary axis. However, (4.12) is motivated as a minimal expression for \( \Delta_n \), which in practice is never attained. Firstly, the number of numerically treated frequencies will be finite. Secondly, the approximating finite-order filter \( W \) will be be larger than \( W_\infty \), allowing a larger class of perturbations.

### 4.4 Constructing an uncertainty description numerically

Proposition 4.1 provides a means to construct an output multiplicative uncertainty description from the model set by numerical calculation.
First, the relevant range of frequencies \( \omega \) is selected, and the models and input signals are transformed into the frequency domain.

A basic way to proceed towards a solution of Problem 4.2 is then to select an optionally frequency-dependent weight \( W_0 \) with positive real parameters and diagonal structure, and to calculate the quantity \( |w_\infty(\omega)| \) as a function of frequency. This, in combination with the absolute values of the diagonal elements of \( W_0 \), yields the absolute values of the parameters of the diagonal of \( W_\infty \). Then a finite-dimensional diagonal filter \( W \) which approximates \( W_\infty \) can be calculated by minimizing the magnitude of \( W \) subject to the requirement that the absolute values of its elements should be greater than those of \( W_\infty \) at all relevant frequencies.

In the sequel, two examples of diagonal weighting functions are briefly discussed.

### 4.4.1 Diagonal weight with equal elements

The simplest way to select the structure of the uncertainty weight is to set \( W_0 = I \), i.e., to set the elements on the diagonal to be identical. This is the form of the weight \( W_0 \) which makes the least assumptions about the system. The following property then holds.

**Corollary 4.1** The weight \( W_0 = I \) gives the weight \( W_\infty \) as defined in (4.10) and (4.11) with the smallest induced norm (maximum singular value).

**Proof:**

\[
\|W_\infty(W_0)\|_2 = \max_n \frac{\|W_0^{-1}e_n\|_2}{\|G_0u_n\|_2} \geq \max_n \frac{\|W_0^{-1}e_n\|_2}{\|G_0u_n\|_2} = \max_n \frac{\|W_0^{-1}e_n\|_2}{\|G_0u_n\|_2} = \|W_\infty(I)\|_2
\]

(4.17)

where \( W_\infty(W_0) \) refers to the weight \( W_\infty \) as given by an arbitrary \( W_0 \), whereas \( W_\infty(I) \) refers to the \( W_\infty \) which is obtained by setting \( W_0 = I \). The first equality follows from the definition of \( W_\infty \), the inequality is general for all matrix norms and the second-to-last equality is particular to the induced norm.

### 4.4.2 Diagonal weight with minimized span

In order for a control scheme based on the uncertainty description not to be overly conservative, it is of interest to minimize the magnitude of \( W_\infty \) in some sense. Despite the weight with equal elements on the diagonal having minimal induced norm, this might not be the most appropriate selection. Instead, it might be of interest to minimize e.g. the square sum of the singular values of \( W \), i.e. to solve the following problem.

**Problem 4.3** Find the real and diagonal weight \( W_0 \) which, at each observed frequency, is a solution to

\[
\min_{W_0} \sum_k \sigma_k^2(W_\infty) = \min_{W_0} \text{tr}\{W_\infty W_\infty\}
\]

(4.18)

where \( W_\infty \) is defined as in (4.10) and (4.11).
The cost \( \sum_k \sigma_k^2(W_\infty) \) is connected with the amount of model space that the multiplicative uncertainty \( W_\infty \Delta \) covers, and should thus be relevant for obtaining nonconservative control. In [12], an alternative cost expression of the type \( \prod_k \sigma_k(W_\infty) \) is proposed.

The problem is best solved by some form of numerical optimization.

4.4.3 Design of more complex weights

As noted above, an attempt to minimize the magnitude of \( W_\infty \) will lead to numerical optimization. On the other hand, the aim is to obtain a finite-dimensional weight \( W \), which has to be adapted to \( W_\infty \), also by some form of optimization, or trial and error. Thus, a more to-the-point approach would be to perform all optimization in one single step.

In order to provide a basis for such a procedure, Proposition 4.1 is reformulated below as a test whether a given, finite-dimensional weight \( W \) is a valid uncertainty weight for the model set \( G_{MM} \).

**Proposition 4.2** Let \( W \) be a (finite-dimensional) weighting function. Then, at each frequency, there exists a perturbation \( \Delta_n, \| \Delta_n \|_2 \leq 1 \) such that \( e_n = W \Delta_n G_0 u_n \) if and only if

\[
\frac{\| W^{-1} e_n \|_2}{\| G_0 u_n \|_2} \leq 1
\]

(4.19)

The proof is almost identical to that of Proposition 4.1.

By regarding a range of frequencies instead of a single one, Proposition 4.2 will lead to an \( H_\infty \)-type numerical test. On the other hand, forming the sum over all frequencies of the cost introduced in Problem 4.3 and minimizing this expression will naturally lead to an \( H_2 \)-type problem. Consequently, the problem of simultaneous determination of an optimal weight \( W_\infty \) and adaptation of a finite-dimensional weighting filter \( W \) can be stated in terms of an \( H_2 \)-type problem with a multimodel \( H_\infty \)-type constraint.

**Problem 4.4 (Shaping problem for multimodel uncertainty filter)**

For the multimodel set \( G_{MM} \) described above and the observed frequencies \( \omega_e \), find a stable and invertible uncertainty filter \( W \), which minimizes the \( H_2 \)-type cost

\[
\sum_{\omega_e} \text{tr}[W(\omega_e)^* W(\omega_e)]
\]

(4.20)

subject to the \( H_\infty \)-type constraints

\[
\frac{\| W(\omega_e)^{-1} e_n(\omega_e) \|_2}{\| G_0(\omega_e) u_n(\omega_e) \|_2} \leq 1, \text{ all } n, \ n = 1, \ldots, N, \text{ and all } \omega_e.
\]

(4.21)
The problem defined above is an alternative to the problems 4.1 and 4.2, the major difference being that the finite-dimensional weight \( W \) is calculated directly as a function of the frequency-dependent data. The numerical solution of the problem does not require that the structure of \( W \) be fixed. In a practical situation, it might be beneficial to augment the above problem by weighting functions, which reflect the relative importance of the frequencies. The weighting functions may allow the \( H_{\infty} \) norms to attain values greater than one at frequencies which are unimportant from a control design perspective, as well as allowing the magnitude of \( W \) to be greater at frequencies where added conservatism does not greatly matter.

In addition to solving Problem 4.4 by optimization with respect to \( W \), the area covered by the norm-bounded model set \( G_{NB} \) can be further decreased by optimization with respect to \( G_0 \). In this way a nominal model \( G_0 \) is obtained which is as central with respect to the model set \( G_{MM} \) as possible.

### 4.4.4 Case study: uncertainty weights for a binary distillation column

![Graph](image1.png)

Figure 4.1: Singular values of diagonal uncertainty weights for distillation column.

Figure 4.1 depicts the result of designing discrete-time uncertainty weights for the binary distillation column described in [11, 12, 30, 59, 61, 63, 64]. The number of models in the model set \( G_{MM} \) is \( N = 6 \), in addition to the nominal model \( G_0 \). The weights are designed numerically by solving Problem 4.4 by constrained optimization. The
frequency range over which the problem is solved is the one shown in the figure, where the number of evaluated frequencies is 100. The frequencies are spaced equidistantly in a logarithmic sense. This gives lower frequencies a larger weight in the $H_2$-type cost (4.20). Apart from this, all frequencies are given equal weights in the norm expressions (4.20) and (4.21). The design methodology and the obtained weighting functions differ slightly from previously reported ones [11, 61, 63, 64].

Two diagonal weighting functions $W$ are calculated: one with two identical elements on the diagonal (represented by the symbol ‘0’ in the graphs) and one where the two elements on the diagonal are used as separate optimization variables (represented by the symbol ‘+’). In both cases the transfer functions are of first order. The upper graph gives the absolute value of a weighting function’s first diagonal element as a function of frequency, and the lower graph the absolute value of its second diagonal element.

For comparison, the graphs additionally contain the frequency responses of two ideal, infinite-dimensional weighting functions $W_\infty$. One has been calculated using Proposition 4.1 with $W_0 = I$ (solid line). The other one has been calculated, also by using Proposition 4.1, from real and diagonal matrices $W_0$ which are numerical solutions to Problem 4.3 (dashed line).

Since the weighting functions are diagonal, the two absolute values of their diagonal elements are equal to their singular values. The figure shows that the magnitude of the first diagonal element or singular value in a weight $W$ can be greatly reduced without causing significant degradation of the second one. The graph also shows that first-order weighting functions provide good approximations of the ideal weights for the distillation column models.
Chapter 5

Concluding remarks

The underlying assumptions of this thesis were roughly the following.

Let $G_{MM}$ denote a set consisting of linear process models plus, possibly, associated linear norm-bounded uncertainty descriptions. Then

1. Weakly nonlinear processes can be represented as a set $G_{MM}$ and controlled by a linear feedback controller adapted to $G_{MM}$.

2. Uncertain and time-invariant processes can be represented by $G_{MM}$ and controlled by a linear feedback controller adapted to $G_{MM}$.

3. Strongly nonlinear processes can be represented by $G_{MM}$ and controlled by a nonlinear feedback controller based on a set of linear controllers which have been adapted to $G_{MM}$.

Assumptions 1 and 2 were studied in [60] and [58] in Part II and, particularly, in the papers [63] and [61] in Part III, where very good control results were obtained by linear multimodel control.

In order to construct nonlinear controllers related to assumption 3, the straightforward method of using so-called gain scheduling was in the main focus. The papers [58] and [60] showed that forming a well-functioning gain-scheduled controller for the pH example process was not a trivial problem. In fact, the result reported in the articles was not quite acceptable.

In order to improve the result of gain scheduling, the pH process was revisited in the later papers [62, 65] (Part IV), where the problem was approached in a more rigorous way. Note that the method with a local controller network largely corresponds to the less successful earlier attempts in [58, 60], and this method was not very successful in the later study either. However, by handling the problem in an LPV framework and using time-varying Riccati equations, the control result was improved and at last as good as, or even slightly better than, that of MPC.
It could be claimed, however, that as time-varying Riccati equations are used, the underlying concept of the method gradually shifts away from the original idea of gain scheduling. In the final result, there is not much left of the notion of interpolating between linear controllers. However, the pH example process was selected to be very difficult to control. For processes which relatively slowly shift from one operating regime to another, and for which the scheduling variable can be taken to be external, simple scheduling based on interpolation may still be a good choice.

Finally, the results of the studies in this thesis show that multimodel parametric methods for $H_2/H_\infty$ control form a well-functioning framework for practical robust and optimal controller design.