

Data-based Uncertainty Modeling by Convex Optimization Techniques

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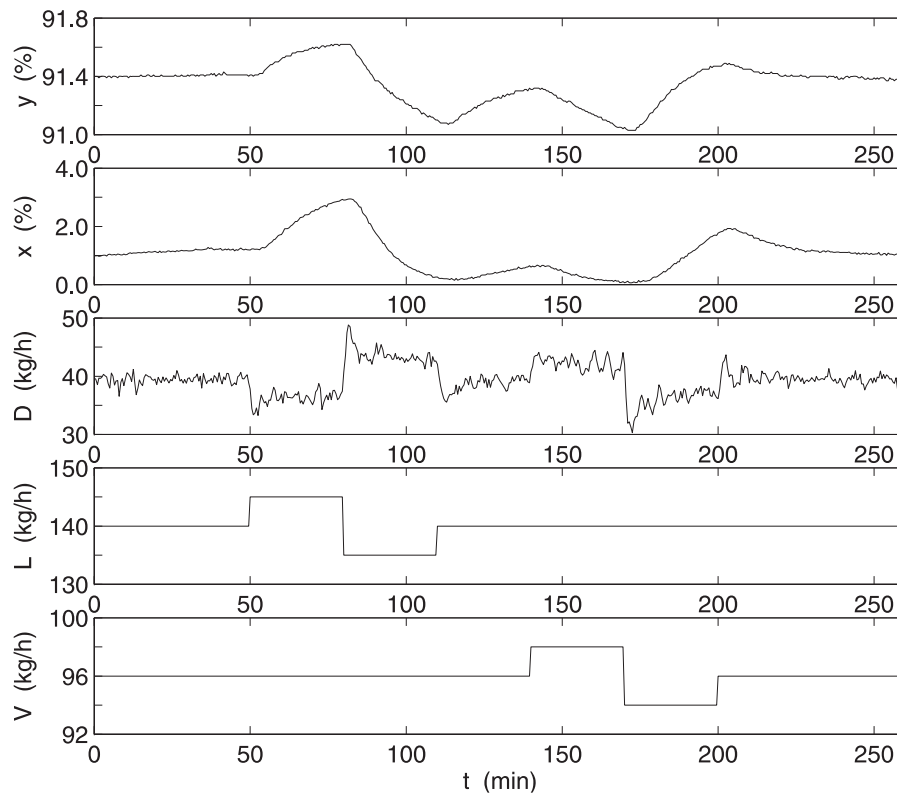
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Outline

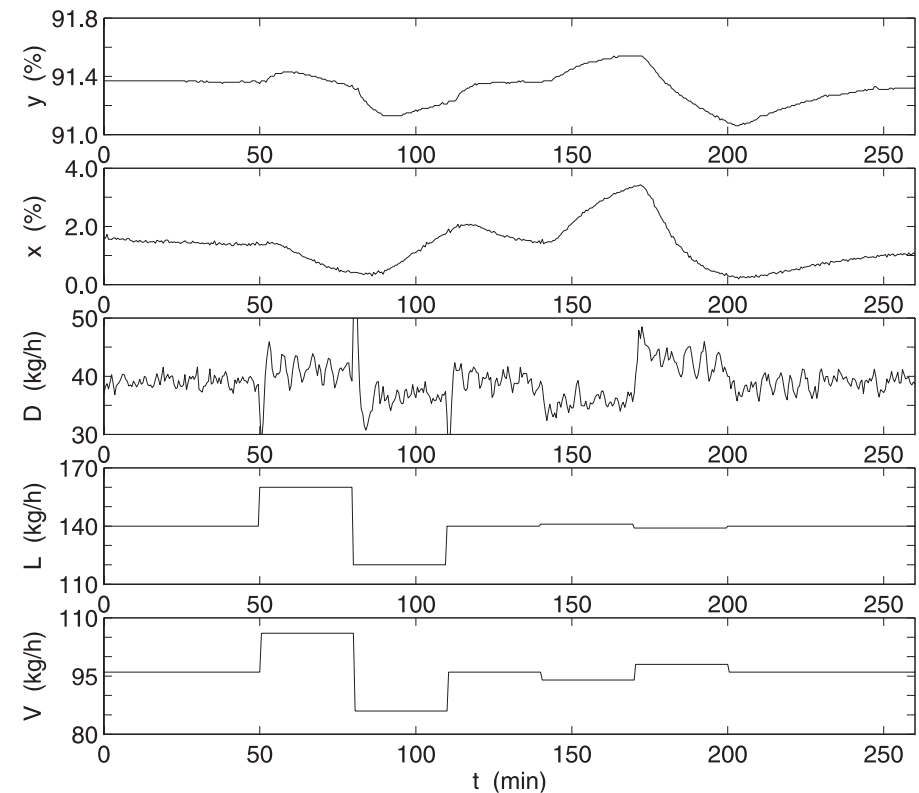
- ◆ Background: Identification of nonlinear MIMO systems
 - Determine a set of models!
- ◆ Problem formulation
 - We have a set of models or sets of data
 - We want an uncertainty description
- ◆ Various options
 - Model matching or input-output matching?
 - Choice of objective function
 - Decisions regarding uncertainty model (type, structure)
- ◆ A solution by convex optimization and LMIs
- ◆ An application to distillation modeling
- ◆ Conclusion

Problems in Identification

Directionality

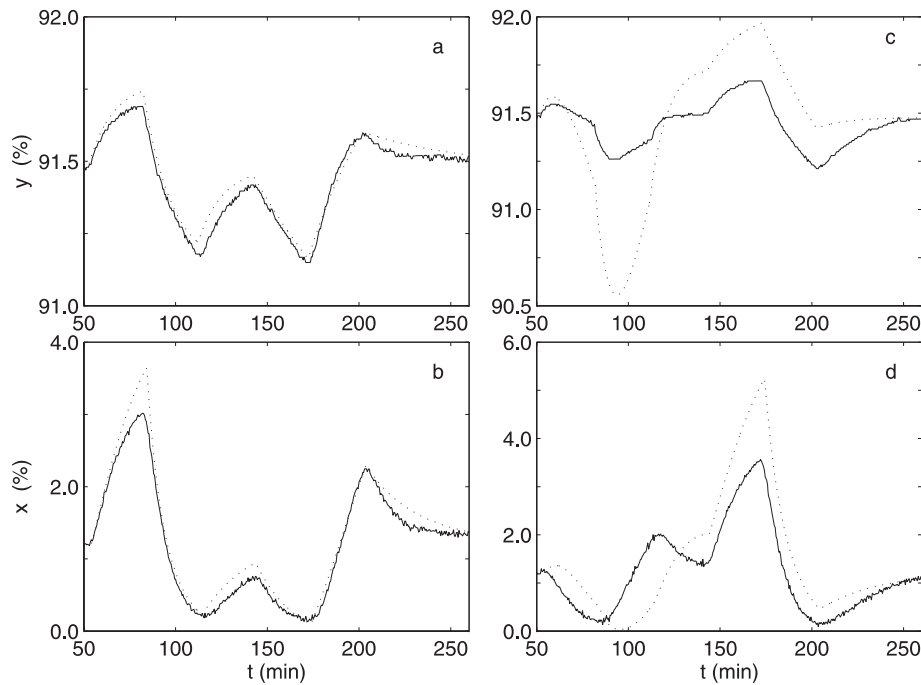


Conventional step excitation



Low- and high-gain excitation

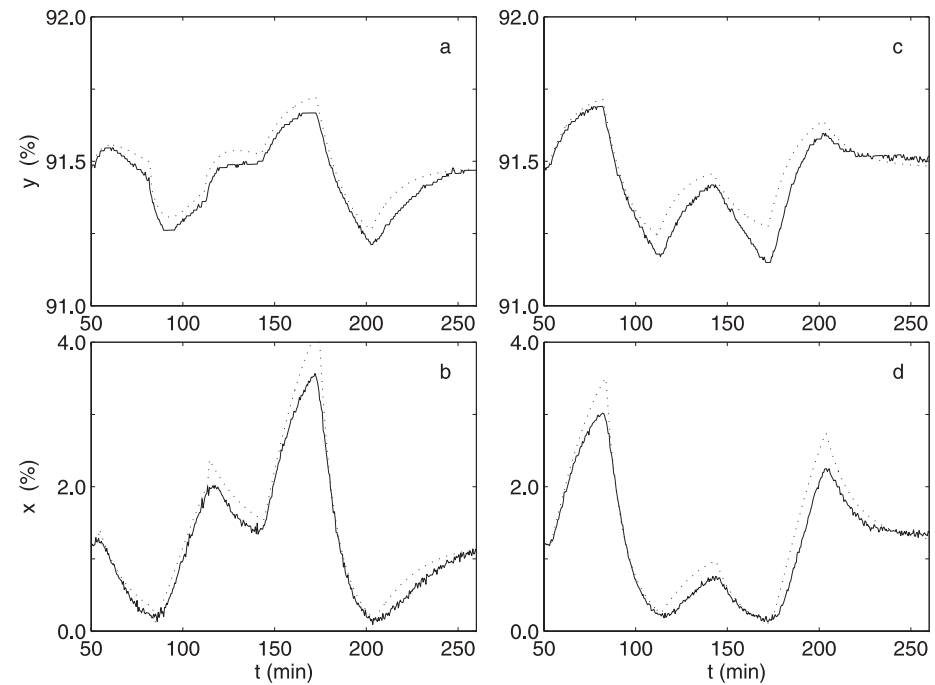
Problems in Identification – Directionality



Left: Model fit to conventional step excitation.

Right: Prediction of low- and high-gain responses.

Experiments (—), model outputs (···).

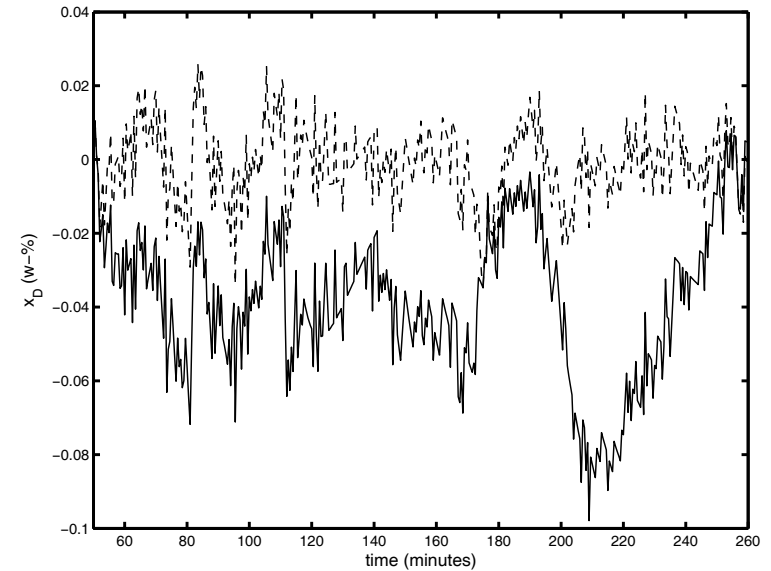
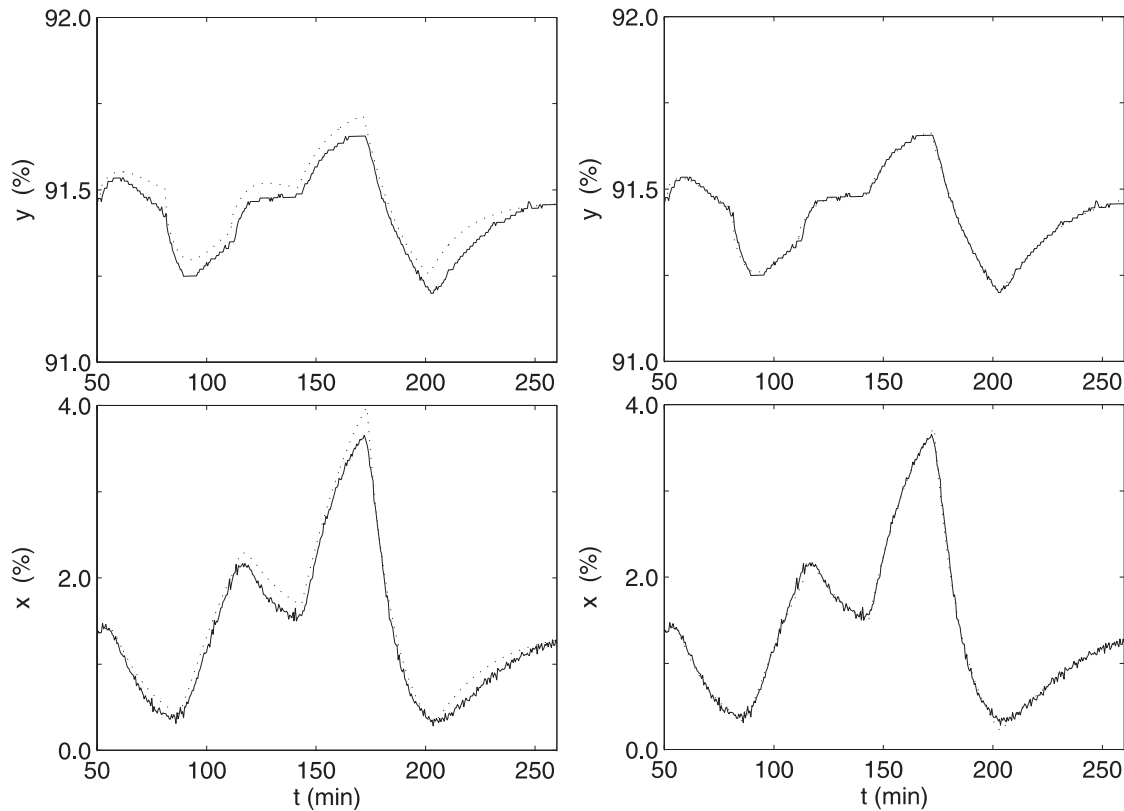


Left: Model fit to low- and high-gain excitation.

Right: Prediction of conventional step responses.

Experiments (—), model outputs (···).

Nonlinearity



Residual of single-model fit (below) and multi-model fit (above).

Left: Single-model fit $\Rightarrow$ **bias** due to nonlinearity

Right: Multi-model fit $\Rightarrow$ **separation of noise** and “difficult” dynamics

Problem Formulation

- ◆ Uncertainty modeling in the frequency domain
 - by considering a number of “relevant” frequencies $\omega \in \Omega$

- ◆ Data available from N experiments

$$\{u_k(s), G_k(s)\} \Rightarrow \{u_k(j\omega), G_k(j\omega) \mid \omega \in \Omega\}, \quad k = 1, \dots, N$$

or
$$\{u_k(j\omega), y_k(j\omega) \mid \omega \in \Omega\}, \quad y_k(j\omega) := G_k(j\omega)u_k(j\omega)$$

In addition, a nominal model $\hat{G}_0(s)$ may be known.

- ◆ We want to derive an uncertainty model

$$\tilde{G}(s) := G_0(s) + W_1(s)\Delta(s)W_2(s), \quad \|\Delta\|_\infty \leq 1$$

- which can reproduce all known data for some allowed $\Delta = \Delta_k$
- as “tightly” as possible
- subject to various structural (or other) constraints

Problem Formulation

- ◆ The work presented here allows
 - only **unstructured uncertainty**, i.e., Δ is unstructured
 - additive as well as multiplicative input and output uncertainty
 - arbitrarily structured weights W_1 and W_2 ; calculation of an optimal G_0
- ◆ As techniques to guarantee “data reproduction” we consider both
 - **model matching** (requires $\tilde{G} = G_k$ for some $\Delta = \Delta_k, \forall k$)
 - **input-output matching** (requires $\tilde{G}u_k = G_k u_k = y_k, \Delta = \Delta_k, \forall k$)
- ◆ As measures of uncertainty (to be minimized) we consider both
 - the **worst-case output deviation** $(\tilde{G} - G_k)u, \bar{\sigma}(\Delta) \leq 1, \|u\|_2 \leq 1$
 - the “**volume**” of the n -dimensional region covered by $y = \tilde{G}u, \bar{\sigma}(\Delta) \leq 1$, for the worst-case input $u, \|u\|_2 \leq 1$
- ◆ All calculation are done “frequency-by-frequency”; transfer functions are **not** determined in this work

The Region of Uncertainty

- For an arbitrary input u , the uncertainty of the output $y = \tilde{G}u$ is given by the deviation

$$e = y - G_0 u = (\tilde{G} - G_0)u = W_1 \Delta W_2 u$$

- An unstructured Δ can produce any vector

$$x = \Delta W_2 u \quad \text{such that} \quad \|x\|_2 \leq \|W_2 u\|_2$$

- When x varies over its range of possible values, the deviation $e = W_1 x$ covers an n -dimensional **ellipsoidal region**.
- For the worst-case input u , the “volume” of this region is proportional to

$$J = \det(W_1 W_1^*)^{1/2} \|W_2\|_2^n \|u\|_2^n$$

- The norm $\gamma_2 = \|W_2\|_2$ can be arbitrarily fixed. Thus, the volume of the uncertainty region can be minimized by minimizing $\det(W_1 W_1^*)^{1/2}$.

The Region of Uncertainty

A convex objective function

- Unfortunately, $\det(W_1 W_1^*)^{1/2}$ is not a convex function.
- A standard way of “convexifying” a determinant is to use the logarithm of the determinant. However, $\log \det(W_1 W_1^*)^{1/2}$ is not convex, either.

- On the other hand,

$$\log \det Y^{-1}, \quad Y = (W_1 W_1^*)^{-1/2} \succ 0$$

is a convex function if Y is used as a decision variable.

Restricting or minimizing the norm of W_1

- It may be desirable to introduce a restriction $\|W_1\|_2 \leq \gamma_1$. We then have

$$W_1 W_1^* = Y^{-2} \preceq \gamma_1^2 I \Leftrightarrow \gamma_1 I - Y^{-1} \succeq 0 \Leftrightarrow \begin{bmatrix} Y & I \\ I & \gamma_1 \end{bmatrix} \succeq 0$$

- If desired, the worst-case output uncertainty can be minimized instead of the volume of the uncertainty region by minimizing γ_1 .

Data Matching

Model matching

It is required that $\tilde{G} = G_k$ for some allowed $\Delta = \Delta_k$, or equivalently, that

$$E_k := G_k - G_0 = W_1 \Delta_k W_2, \quad \bar{\sigma}(\Delta_k) \leq 1$$

Such a Δ_k exists iff (Poolla *et al.*, 1994; Chen and Gu, 2000)

$$\begin{bmatrix} W_1 W_1^* & E_k \\ E_k^* & W_2^* W_2 \end{bmatrix} \succcurlyeq 0 \Leftrightarrow \begin{bmatrix} I & Y E_k \\ (Y E_k)^* & X \end{bmatrix} \succcurlyeq 0, \quad \begin{aligned} Y &:= (W_1 W_1^*)^{-1/2} \\ X &:= W_2^* W_2 \end{aligned}$$

Optimization with respect to G_0 can be included by rewriting the LMI as

$$\begin{bmatrix} I & Y G_k - Z \\ (Y G_k - Z)^* & X \end{bmatrix} \succcurlyeq 0, \quad Z := Y G_0 \quad (\text{A})$$

The norm restriction $\|W_2\|_2 \leq \gamma_2$ is equivalent to $0 \preccurlyeq X \preccurlyeq \gamma_2^2 I$. (B)

Uncertainty minimization: Minimize $\log \det Y^{-1}$ (or γ_1) with respect to $Y \in \mathcal{Y}$, $X \in \mathcal{X}$ and $Z \in \mathcal{Z}$ subject to (A) and (B) for $\forall \omega \in \Omega$ and $\forall k$.

Input-output matching

It is required that $\tilde{G}u_k = G_k u_k = y_k$ for some allowed $\Delta = \Delta_k$, or

$$e_k := y_k - G_0 u_k = W_1 \Delta_k W_2 u_k, \quad \bar{\sigma}(\Delta_k) \leq 1$$

Such a Δ_k exists iff

$$\begin{bmatrix} W_1 W_1^* & e_k \\ e_k^* & u_k^* W_2^* W_2 u_k \end{bmatrix} \succcurlyeq 0 \iff \begin{bmatrix} I & Y e_k \\ (Y e_k)^* & u_k^* X u_k \end{bmatrix} \succcurlyeq 0, \quad \begin{aligned} Y &:= (W_1 W_1^*)^{-1/2} \\ X &:= W_2^* W_2 \end{aligned}$$

Optimization with respect to G_0 can be included by rewriting the LMI as

$$\begin{bmatrix} I & Y y_k - Z u_k \\ (Y y_k - Z u_k)^* & u_k^* X u_k \end{bmatrix} \succcurlyeq 0, \quad Z := Y G_0 \quad (\text{C})$$

Uncertainty minimization: Minimize $\log \det Y^{-1}$ (or γ_1) with respect to $Y \in \mathcal{Y}$, $X \in \mathcal{X}$ and $Z \in \mathcal{Z}$ subject to (B) and (C) for $\forall \omega \in \Omega$ and $\forall k$.

Application to Distillation Modeling



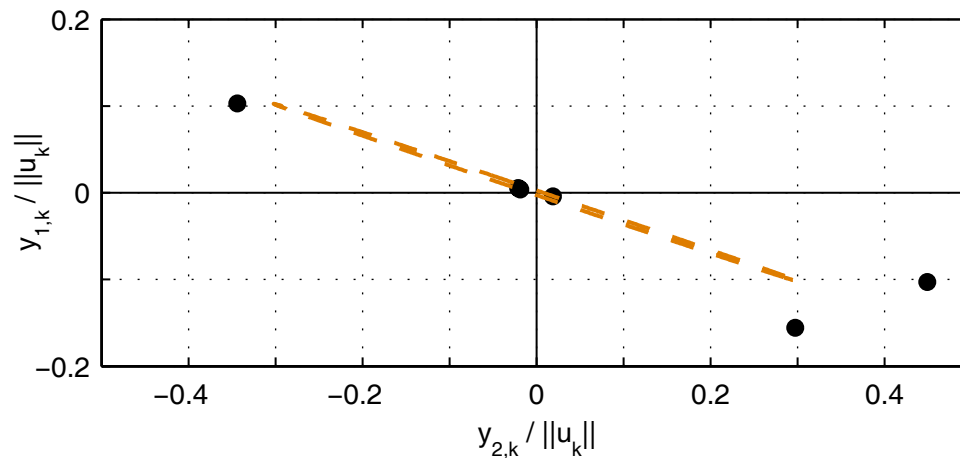
TABLE I
Nominal Steady-State Model

$$\hat{G}_0(0) = \begin{bmatrix} -0.04229 & 0.09349 \\ 0.11733 & -0.27858 \end{bmatrix}$$

TABLE II
Steady-State Data of Individual Experiments

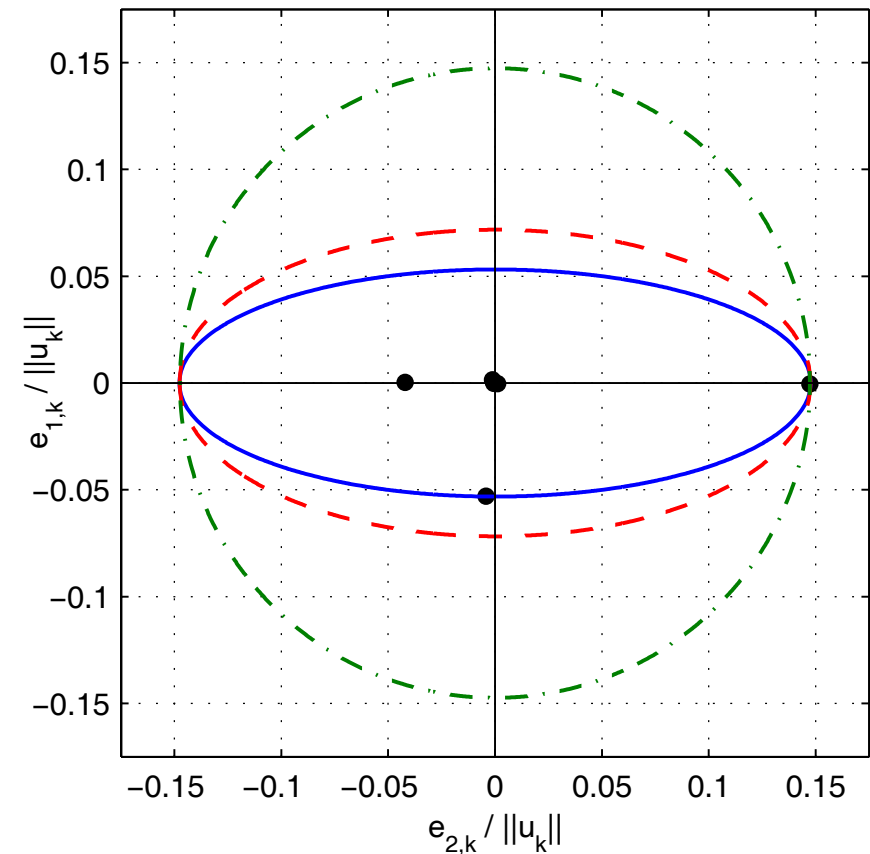
Exp. #	u_1	u_2	y_1	y_2
1	10.0	5.0	0.06180	-0.23315
2	-20.0	-10.0	-0.09280	0.42640
3	10.0	5.0	0.04135	-0.20590
4	0.5	-1.0	-0.11513	0.50204
5	-1.0	2.0	0.22997	-0.76869
6	0.5	-1.0	-0.17393	0.33254

Application to Distillation

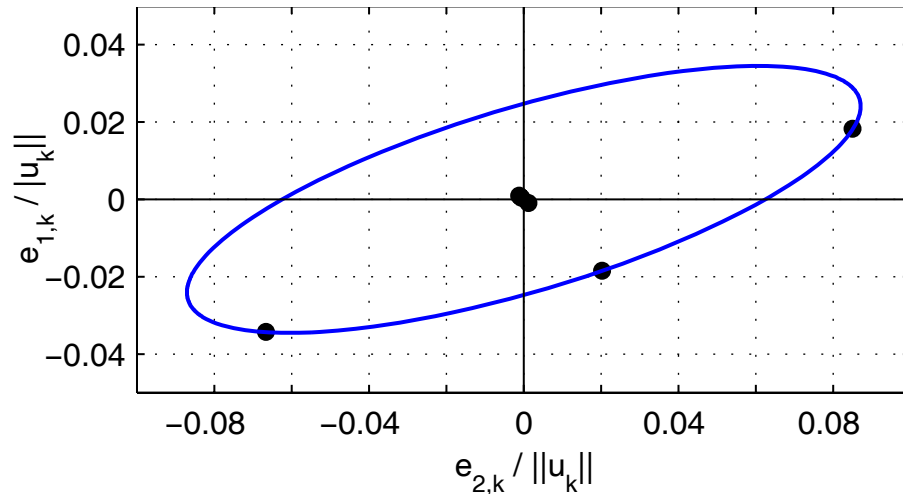


Above: Experimental outputs (●) and region covered by **original nominal model** (- -).

Right: Deviation between experimental and nominal model outputs (●); uncertainty regions with **smallest area** (—), **smallest norm** (- -) and **best scalar weight** (- · -) for a multiplicative output uncertainty model $\tilde{G} = G_0 + W_1 \Delta G_0$ using original nominal model.

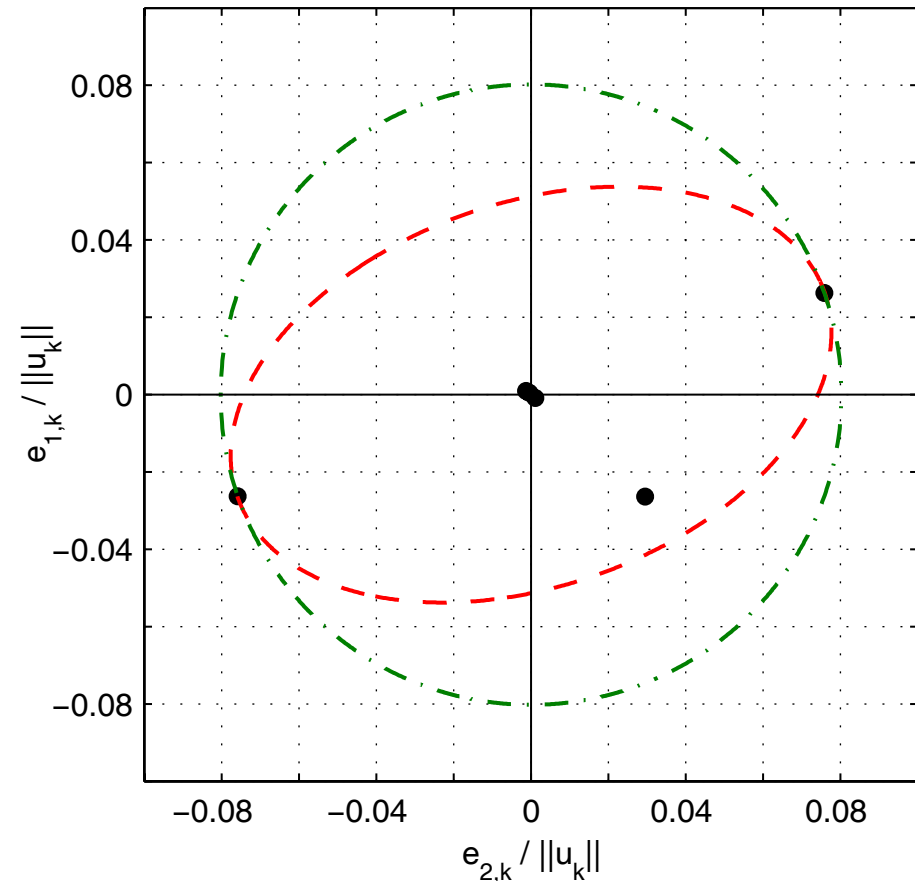


Result with optimized nominal model



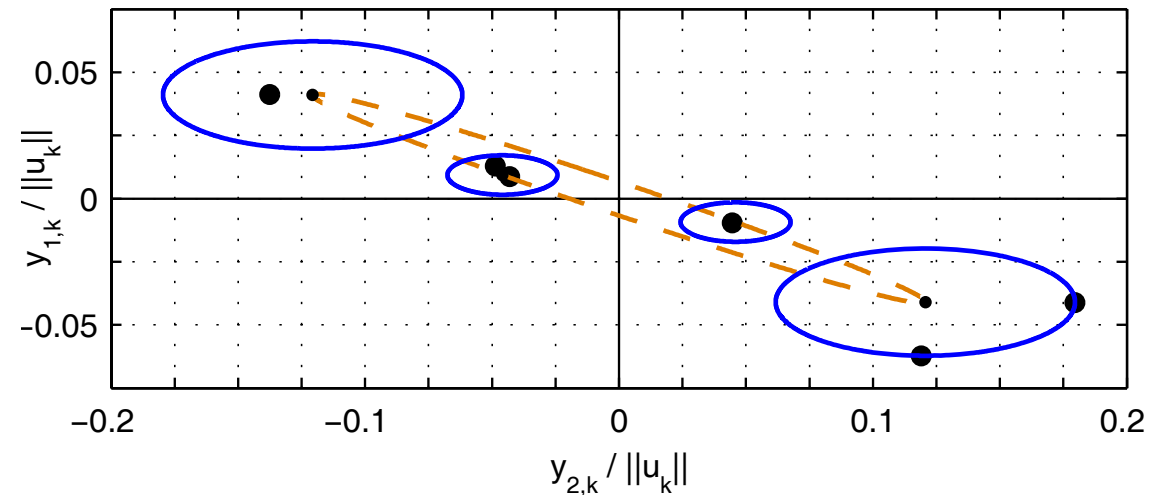
Above: Uncertainty region with **smallest area**.

Right: Uncertainty region with **smallest norm** (- -) and **best scalar weight** (- · -).



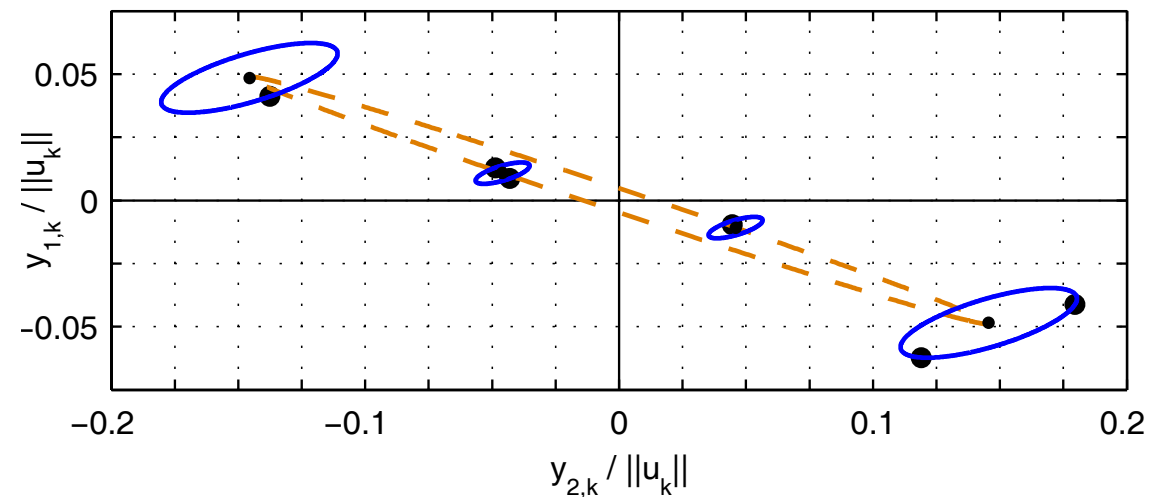
Another view

Experimental outputs ($\bullet$), region covered by nominal model (---) and uncertainty regions with **smallest area** (—) at experimental points using **original nominal model**.



Ditto with **optimized nominal model**.

To improve the visualization the data have been transformed to reduce the condition number of the nominal model.



Conclusion

- ◆ A procedure based on **convex optimization and LMIs** for deriving norm-bounded uncertainty models for MIMO systems from experimental data has been described.
- ◆ Data may be available as (i) sets of input-output data or (ii) a set of models.
 - Generally **input-output data** gives a **less conservative** model.
- ◆ By employing **data-matching** techniques it can be ensured that the uncertainty model can reproduce known data in the frequency domain.
- ◆ The output region covered by the uncertainty model for the worst-case input can be minimized by minimizing (i) the n -dimensional volume or (ii) the largest distance of the region.
 - Results indicate that the **volume should be minimized**.

Conclusion

- ◆ The procedure as presented here
 - applies to **unstructured uncertainty**
 - allows arbitrary structures of uncertainty weight matrices
 - does not require a nominal model to be known initially
 - can handle additive as well as multiplicative input and output uncertainty (in principle, any type of LFT uncertainty)
- ◆ An application to distillation modeling was presented
 - A **clear improvement** of the uncertainty model was obtained **by optimizing the nominal model**.