$H_\infty$ Control of Multirate Sampled-Data Systems: 
A State-Space Approach

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Abstract

A state-space solution of the $H_\infty$ control problem for periodic multirate sampled-data systems is presented. The solution is characterized in terms of a pair of discrete algebraic Riccati equations with a set of associated matrix positive definiteness conditions and coupling criteria. The solution is derived using two different approaches. In the first approach, the solution is obtained by discretizing two coupled periodic Riccati equations with jumps which characterize the $H_\infty$-optimal controller. The second approach is based on the lifting technique. The $H_\infty$-optimal controller for a lifted representation of the multirate system is characterized by a two-Riccati equation solution, in which the so-called causality constraints are represented by a set of nonstandard positive definiteness conditions and coupling criteria.

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1 Introduction

During the last few years there has been an increased interest in the direct design of digital controllers using continuous-time performance measures. Such systems arise naturally when a digital controller is applied to a continuous-time plant. In contrast to standard continuous-time or discrete-time techniques, direct sampled-data methods address both the intersample behavior of the process as well as the effect the sampling frequency has on the performance.

Multirate systems arise when various measurement sampling functions and control signal hold elements operate at different rates. The development of a theory for such multirate systems started in the 1950’s with the pioneering work by Kranc [20], Kalman and Bertram [18] and Friedland [13]. More recent work in the field is concerned with, e.g., system analysis and stability [31, 12, 24, 25, 38] optimal control of multirate systems with a quadratic cost function [11, 8, 26, 29, 39] and $H_\infty$ control of multirate sampled-data systems [10, 39].

The $H_\infty$ problem for single-rate sampled-data systems has recently been studied in a number of papers [5, 9, 15, 17, 30, 32, 33, 34, 35, 36]. One common approach is to apply the lifting technique [5] to reduce the hybrid continuous/discrete-time problem to a norm-equivalent discrete-time $H_\infty$ problem to which standard methods can be applied. Another method is to state a solution directly for the hybrid problem [32, 33]. This leads to a solution in terms of Riccati equations with jumps. The approaches have recently been shown to give equivalent solutions [29, 37].

The lifting technique has also been applied to the multirate sampled-data problem [10, 39] where the periodically time-varying sampled-data system is lifted to give an equivalent time-invariant discrete system. The main difficulty encountered in this approach is caused by the causality constraint which is required for the lifted controller to correspond to a causal periodic controller. The causality constraint has limited the applicability of standard state-space solutions on the lifted system description. Previous solutions have therefore been derived through constrained model matching [10, 39].

In this work a state-space solution to the periodic multirate sampled-data $H_\infty$ problem is derived. The solution is obtained through two different approaches. The first approach is based on a direct Riccati-equation solution and the second approach is based on lifting. The direct method has to our best knowledge not previously been applied to the multirate sampled-data $H_\infty$ problem. The lifting method was used in [10, 39] but no time-domain solution was derived. The use of both approaches gives insight into the relationship between the two methods. Both approaches studied here generally give rise to the same synthesis equations and the same controller.

In the first approach the $H_\infty$-optimal controller is characterized by a pair of coupled
Riccati equations with jumps. Using the results in [29, 37] the Riccati equations can be discretized and the solution expressed by two periodic Riccati difference equations and discrete-time coupling conditions. The solutions to these periodic equations can then be expressed in terms of a pair of coupled standard algebraic Riccati equations. This direct approach allows for a very general problem statement including non-synchronized sampling and hold elements.

In the second approach we adopt the lifting technique to construct a time-invariant discrete-time system associated with the periodic multirate problem. A solution of the causally constrained $H_{\infty}$-optimal control problem is derived which consists of two standard discrete algebraic Riccati equations and a set of matrix positive definiteness criteria. An interesting feature of the result is that the algebraic Riccati equations are exactly those associated with the unconstrained discrete-time $H_{\infty}$ problem for the lifted time-invariant system description. Thus standard discrete-time solution methods can be applied. The additional causality constraint associated with the periodic multirate system appear in the solution as non-standard positive definiteness criteria and coupling conditions which are easy to check.

The paper is organized as follows. The next section contains the problem definition and a description of the notation used in the paper. In Section 3 a solution to the multirate sampled-data problem is derived through periodic Riccati equations. A state-space solution through the lifting technique is presented in Section 4. A simple example is used to illustrate the results in Section 5 and some concluding remarks are given in Section 6.

2 Problem statement

Consider a linear time-invariant finite-dimensional continuous-time plant described by

$$
\begin{align*}
\dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 u(t), \quad x(0) = 0 \\
z(t) &= C_1 x(t) + D_{12} u(t)
\end{align*}
$$

(1)

where $x(t) \in \mathbb{R}^n$ is the state vector, $w(t) \in \mathbb{R}^m$ is the process disturbance, $u(t) \in \mathbb{R}^m$ is the control signal and $z(t) \in \mathbb{R}^{p_1}$ is the controlled output.

The sampling of the measurements is assumed to be described by a periodic function $\mathcal{S}$ with the period $T_y$. The outputs may be sampled with different rates or possibly at non-equidistant time-instants. The discrete signal $y_d := \{ y_d(iT_y + \sigma_j), \ j = 0, 1, \ldots, N_y - 1, \ i = 0, 1, \ldots \}$ available for feedback is given by

$$
y_d(iT_y + \sigma_j) = C_2(\sigma_j) x(iT_y + \sigma_j) + D_{21}(\sigma_j) v_d(iT_y + \sigma_j),
$$

(2)

where $v_d(i)$ is a discrete measurement disturbance and $iT_y + \sigma_j$ ($\sigma_0 \geq 0 \Gamma \sigma_{j+1} > \sigma_j \Gamma \sigma_{N_y} = T_y$) are the times at which a measurement is available. It is assumed that the signal
$y_d(iT_y + \sigma_j)$ contains only those measurements obtained at time instant $iT_y + \sigma_j$. This means that the discrete output vector has a periodic and time-varying dimension.

The control signal is assumed to be generated by a zero-order hold function $H$ with the overall period $T_u$. The number of control signal updates during one period $T_u$ is $N_u$. The individual control signals may be updated at faster rates (with $T_u$ as a common multiple of the hold times) or the hold times may be periodically time-varying. The multirate zero-order hold is described by

$$u(t) = (I - H(\tau_i))u(kT_u + \tau_i) + H_u(\tau_i)u_d(kT_u + \tau_i^+),$$

$$t \in (kT_u + \tau_i, kT_u + \tau_{i+1}], \quad l = 0, 1, 2, \ldots, N_u - 1, \quad k = 0, 1, 2, \ldots$$

where $u_d := \{ u_d(kT_u + \tau_i^+), \quad l = 0, 1, 2, \ldots, N_u - 1, \quad k = 0, 1, 2, \ldots \}$ is a discrete-time control signal. The matrix $H(\tau_i) := \text{diag}\{\eta_1(\tau_i), \eta_2(\tau_i), \ldots, \eta_m(\tau_i)\}$ selects the control signals to be updated at time instant $kT_u + \tau_i$ ($\tau_i \geq 0 \Gamma_{N_u} > \pi \Gamma_{T_u} = \pi + T_u$). The functions $\eta_m(\tau_i)$ take the value 1 if the $m$th control signal is updated at time instant $kT_u + \tau_i$ and zero otherwise. The discrete control signal $u_d(kT_u + \tau_i^+)$ contains only those signals updated at time $kT_u + \tau_i$. The matrix $H_u(\tau_i)$ is therefore a submatrix of $H(\tau_i \Gamma)$ which consists of the nonzero columns of $H(\tau_i)$.

**Remark 2.1** The time-varying dimensions of the signals $y_d$ and $u_d$ guarantee the existence of matrix inverses which appear in the solution. For an alternative way to guarantee nonsingularity, see [11].

The system defined by (1) – (3) is assumed to be periodic with the period $T \Gamma$ such that $T$ is the least common multiple of the period of the sampler $T_y$ and the period of the hold function $T_u$. (The solution can however be extended to cover the case when the periods of $\mathcal{S}$ and $\mathcal{H}$ are asynchronously related.) The multirate sampled-data problem defined here is very general. Note that the only assumption on $\mathcal{S}$ and $\mathcal{H}$ is their periodicity otherwise the discrete events may occur at any (non-synchronized) instants. The solution presented in Section 3 can also be generalized to the case with non-periodic time-varying sampling. In connection with the lifting technique in Section 4 we will make some additional technically simplifying assumptions on the synchronization of the sampling and the hold elements.

The system (1) – (3) can be represented as a mixed continuous/discrete system with jumps $\Gamma$

$$\Sigma_G : \begin{cases} \dot{x}_e(t) &= A_x x_e(t) + B_{1e} u(t), \quad t \neq kT_u + \pi \\
x_e(kT_u + \tau_i^+) &= A_x(\tau_i) x_e(kT_u + \tau_i) + B_{1e}(\tau_i) u_d(kT_u + \tau_i^+) \\
z(t) &= C_{1e} x_e(t) \\
y_d(iT_y + \sigma_j) &= C_{2d}(\sigma_j) x_e(iT_y + \sigma_j) + D_{21}(\sigma_j) v_d(iT_y + \sigma_j) \end{cases}$$

4
where we have extended the state vector \( x(t) := [x'(t), u'(t)]' \) and
\[
A_\varepsilon := \begin{bmatrix} A & B_0 \\ 0 & 0 \end{bmatrix}, \quad B_{1\varepsilon} := \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \quad C_{1\varepsilon} := \begin{bmatrix} C_1 & D_{12} \end{bmatrix}
\]
(5)
\[
A_d(\tau) := \begin{bmatrix} I & 0 \\ 0 & I - H_d(\tau) \end{bmatrix}, \quad B_d(\tau) := \begin{bmatrix} 0 \\ H_d(\tau) \end{bmatrix}, \quad C_{2_d}(\sigma_j) := \begin{bmatrix} C_2(\sigma_j) & 0 \end{bmatrix}.
\]
(6)
We need the following definitions, cf. [32] and

**Definition 2.1** Let \( \Phi_{A_\varepsilon}^{t_2,t_1}(\cdot, \cdot) \) denote the state-transition matrix of the system \( \Sigma_d \). Then \( \Sigma_d \) is said to be exponentially stable if there exist \( c_1, c_2 > 0 \) such that
\[
\| \Phi(t_2,t_1) \| \leq c_1 e^{c_2(t_2-t_1)}, \text{ all } t_2 \geq t_1.
\]
(7)

**Definition 2.2** The system \( \Sigma_d \) is said to be stabilizable (respectively, detectable) if there exists a bounded function \( K_d(\tau) \) (respectively, \( L_d(\sigma_j) \)) such that \( \Phi_{A_\varepsilon}^{t_2,t_1} + B_d K_d \) (respectively, \( \Phi_{A_\varepsilon}^{t_2,t_1} + C_2 D_{12} \)) satisfies (7).

The following assumptions are made in the paper:

**A1** The system \( \Sigma_d \) is stabilizable.

**A2** The system \( \Sigma_d \) is detectable.

**A3** The matrix \( D_{12} D_{12} \) is non-singular.

**A4** The matrices \( D_{21}(\sigma_j) D_{11}(\sigma_j) \) are non-singular.

**A5** The continuous-time system \((A,B_1,C_1)\) is stabilizable and detectable.

The set of admissible controllers is assumed to consist of exponentially stabilizing discrete causal control laws \( u_d = K_d y_d \) such that \( u_d(kT_u + \tau_i) \) is a function of past measurements \( \{ y_d(kT_y + \sigma_j) : i T_y + \sigma_j \leq kT_u + \tau_i \} \) only.

The following worst-case performance measure induced by the disturbances \( w \in L_2[0,\infty) \) and \( v_d \in l_2(0,\infty) \) is introduced
\[
J_\infty(K) := \sup_{(w,v_d) \neq 0} \left[ \frac{\|z\|_{L_2}}{\sqrt{\|w\|_{L_2}^2 + \|v_d\|_{l_2}^2}} \right]^{1/2}
\]
(8)

The multirate sampled-data \( H_\infty \) control problem is defined as follows:

Find an admissible controller \( K \), if one exists, which achieves a specified attenuation level \( \gamma > 0 \), such that \( J_\infty(K) < \gamma \).
3 A Riccati equation solution

The following solution to the multirate sampled-data $H_\infty$ problem stated in terms of coupled periodic Riccati equations with jumps stands as a basis for the rest of the treatment in this section (Theorem 3.1). In Theorem 3.2 the Riccati equations are discretized and the solution is given in terms of discrete periodic Riccati equations. Theorem 3.3 shows the fact that solutions to the periodic equations can be obtained in terms of a pair of coupled algebraic Riccati equations.

**Theorem 3.1** Consider the multirate sampled-data system $\Sigma_G$. Suppose that the assumptions A1 $-$ A5 hold. Then there exists a discrete admissible controller $K$ which achieves the performance bound $J_\infty(K) < \gamma$ if and only if the following conditions are satisfied:

(i) There exists a bounded $T_u$-periodic symmetric positive semidefinite matrix function

\[ S_c(t) := \begin{bmatrix} S & S_1 \\ S_1^T & S_2 \end{bmatrix}(t), \ t \in [0, \infty) \]  

which satisfies the following periodic Riccati differential equation with jumps,

\[ -\dot{S}_c(t) = A_c^T S_c(t) + S_c(t) A_c + \gamma^{-2} S_c(t) B_1 \Gamma_c S_c(t) + C_1^T C_1, \ t \neq kT_u + \tau \]

\[ S_c(\tau) = A_c^T(\tau) S_c(\tau^+) [A_c(\tau) - B_c(\tau) K_d(\tau)] \]

\[ S_c(kT_u + \tau) = S_c(\tau) \text{ for } \tau = 0, 1, \ldots N_u - 1, \quad k = 0, 1, \ldots \]

where

\[ K_d(\tau) := [B_c^T(\tau) S_c(\tau^+) B_c(\tau)]^{-1} B_c^T(\tau) S_c(\tau^+) A_c(\tau) \]

such that the $T_u$-periodic system

\[ \dot{x}_c(t) = [A_c + \gamma^{-2} B_1 \Gamma_c S_c(t)] x_c(t), \ t \neq kT_u + \tau \]

\[ x_c(kT_u + \tau) = [A_c(\tau) - B_c(\tau) K_d(\tau)] x_c(kT_u + \tau) \]

is exponentially stable.

(ii) There exists a bounded $T_y$-periodic symmetric positive semidefinite matrix function $Q(\cdot)$ which satisfies the following periodic Riccati differential equation with jumps,

\[ \dot{Q}(t) = A Q(t) + Q(t) A^T + \gamma^{-2} Q(t) C_y^T C_y Q(t) + B_1 \Gamma_c, \ t \neq iT_y + \sigma \]

\[ Q(\sigma_j^+) = [I - L_d(\sigma_j) C_y(\sigma_j)] Q(\sigma_j) \]

\[ Q(iT_y + \sigma) = Q(\sigma_j) \quad j = 0, 1, \ldots N_y - 1, \quad i = 0, 1, \ldots \]

where

\[ L_d(\sigma_j) := Q(\sigma_j) C_y^T(\sigma_j) D_{21}(\sigma_j) D_{21}^T(\sigma_j) + C_y(\sigma_j) Q(\sigma_j) C_y(\sigma_j)^{-1} \]
such that the $T_y$-periodic system

$$\begin{align*}
\dot{x}(t) &= (A + \gamma^{-2}Q(t)C_1^TC_1)x(t), \quad t \neq iT_y + \sigma_j \\
x(iT_y + \sigma_j^+) &= [I - L_d(\sigma_j)C_3(\sigma_j)]x(iT_y + \sigma_j)
\end{align*}$$

(15)

is exponentially stable.

(iii)

$$\rho[S(t)Q(t)] < \gamma^2, \text{ all } t \in [0, \infty)$$

(16)

where $\rho(\cdot)$ denotes the spectral radius.

Moreover, when these conditions hold, the performance bound $J_{\infty}(K) < \gamma$ is achieved by the $T$-periodic controller

$$\begin{align*}
\dot{x}(t) &= \bar{A}(t)x(t) + \bar{B}_2(t)u(t) \quad t \neq iT_y + \sigma_j \\
\dot{x}(iT_y + \sigma_j) &= \dot{x}(iT_y + \sigma_j) + \bar{I}_d(\sigma_j)[y_d(iT_y + \sigma_j) - C_2(\sigma_j)\dot{x}(iT_y + \sigma_j)] \\
u_d(kT_u + \tau_i^+) &= -[H_1^T(\tau_i)S_2(\tau_i)H_u(\tau_i)]^{-1}H_1^T(\tau_i)\dot{x}(kT_u + \tau_i) \\
&\quad + S_2(\tau_i)[I - H(\tau_i)]u(kT_u + \tau_i)
\end{align*}$$

(17)

where $u(t)$ is given by (3),

$$\begin{align*}
\bar{A}(t) &:= A + \gamma^{-2}B_1B_1^TS(t) \\
\bar{B}_2(t) &:= B_2 + \gamma^{-2}B_1B_1^TS_1(t)
\end{align*}$$

(18)

$$L_d(\sigma_j) = N(\sigma_j)C_2^T(\sigma_j)[D_{21}(\sigma_j)D_{21}^T(\sigma_j) + C_2(\sigma_j)N(\sigma_j)C_2^T(\sigma_j)]^{-1}$$

(19)

and $N(t)$ is given by

$$N(t) = Q(t)(I - \gamma^{-2}S(t)Q(t))^{-1}.$$ 

(20)

**Proof:** The multirate sampled-data system defined by (4) can be regarded as a periodic time-varying system. The multirate $H_{\infty}$ problem can then be solved by adapting continuous-time and discrete-time $H_{\infty}$ theory of time-varying systems [19F27] to this case. This leads to a two-Riccati equation solution of the sampled-data problem. The theorem is a fairly straightforward generalization of the corresponding single-rate result in [30F33] and the dual-rate result in [36].

The periodic Riccati differential equations with jumps in (10) and (13) can be discretized to periodic Riccati difference equations. Similarly, the coupling condition (16) can be expressed as a discrete-time criterion. Since the sampling and hold functions may have different periods, the discretization points of the Riccati equations involved can occur at different...
time-instants. The solution can then be expressed in terms of two discrete periodic Riccati equations with different periods.

Introduce the $2(n + m_2) \times 2(n + m_2)$ matrix

$$\Pi(t) := \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{bmatrix}(t) := \exp \left( - \begin{bmatrix} A_{\tau} & \gamma^{-1} B_{1\tau} B_{1\tau}^{T} \\ -C_{1\tau}^{T} C_{1\tau} & -A_{\tau}^{T} \end{bmatrix} t \right)$$

(21)

where $A_{\tau}, B_{1\tau}$ and $C_{1\tau}$ are defined by (5) and define

$$\hat{A}_{\tau}(h_{i}, \pi) := \Pi_{11}(h_{i})^{-1} A_{d}(\pi)$$

$$\hat{B}_{1\tau}(h_{i}) \hat{B}_{1\tau}^{T}(h_{i}) := -\gamma^{2} \Pi_{11}(h_{i})^{-1} \Pi_{12}(h_{i}),$$

$$\hat{B}_{2\tau}(h_{i}, \pi) := \Pi_{11}(h_{i})^{-1} B_{d}(\pi)$$

$$\begin{bmatrix} \hat{C}_{1\tau}(h_{i}, \pi) \\ \hat{D}_{12}(h_{i}, \pi) \end{bmatrix} = \begin{bmatrix} A_{d}(\tau) \\ B_{1\tau}(\pi) \end{bmatrix} \Pi_{21}(h_{i}) \Pi_{11}(h_{i})^{-1} \begin{bmatrix} A_{d}(\tau) \\ B_{d}(\tau) \end{bmatrix}$$

(22)

where $h_{i} := \pi_{i+1} - \pi$. Note that the structure of the matrices in (22) associated with the augmented system (4) allows for the partition

$$\begin{bmatrix} \hat{A}_{\tau}(\cdot, \tau) =: \begin{bmatrix} \hat{A}(\cdot) & 0 \\ 0 & l \end{bmatrix} A_{d}(\tau) \\ \hat{B}_{1\tau}(\cdot) =: \begin{bmatrix} \hat{B}(1)(\cdot) \\ 0 \end{bmatrix} \end{bmatrix}$$

$$\begin{bmatrix} \hat{C}_{1\tau}(\cdot, \tau) =: [\hat{C}(1)(\cdot) & *] \end{bmatrix}$$

(23)

where $\hat{A}$ is $n \times n I$ and $\hat{B}_{1\tau}(\cdot) is n \times m_1$ and $\hat{C}$ is $p_1 \times n$. Denote further

$$\hat{C}_{2}(\cdot) := C_{2}(\cdot), \quad \hat{D}_{21}(\cdot) := D_{21}(\cdot).$$

(24)

It is assumed that the matrices $\Pi_{11}(\cdot)$ in (22) are invertible which is the case if there exist bounded solutions to (10) and (13) [see [37]].

**Theorem 3.2** Consider the multirate sampled-data system $\Sigma_{\Omega}$. Suppose the assumptions $A1 - A5$ hold. Then the conditions in Theorem 3.1 (i)–(iii) are equivalent to the following conditions:

(i) The matrices $\Pi_{11}(h_{i})$ are invertible and there is an $N_{u}$-periodic symmetric positive semidefinite matrix function $S_{\tau}(\pi)$ satisfying the following periodic Riccati difference equation

$$S_{\tau}(\pi) = \hat{A}_{\tau}^{T}(h_{i}, \pi) S_{\tau}(\pi_{i+1}) \hat{A}_{\tau}(h_{i}, \pi) + \hat{C}_{1\tau}^{T}(h_{i}, \pi) \hat{C}_{1\tau}(h_{i}, \pi) -$$

$$E_{S}(h_{i}, \pi) M_{\tau}^{-1}(h_{i}, \pi) E_{S}(h_{i}, \pi)$$

$$S_{\tau}(kT_{u} + \pi) = S_{\tau}(\pi), \quad l = N_{u} - 1, \ldots, 1, 0, \quad k = 0, 1, \ldots$$

(25)
where

\[
E_S(h_t, \pi_1) := \left[ \begin{array}{c}
\hat{B}_1'(h_t) \\
\hat{B}_2'(h_t)
\end{array} \right] S_{e}(\pi_{1+1}) \hat{A}_e(h_t, \pi) + \left[ \begin{array}{c}
0 \\
\hat{D}_1'(h_t, \pi)
\end{array} \right] \hat{C}_{1e}(h_t, \pi)
\]
\[
M_S(h_t, \pi_1) := \left[ \begin{array}{c}
-\gamma^2 I \\
0
\end{array} \right] \hat{D}_1'(h_t, \pi_1) \hat{D}_1(h_t, \pi) + \left[ \begin{array}{c}
\hat{B}_1'(h_t) \\
\hat{B}_2'(h_t)
\end{array} \right] S_{e}(\pi_{1+1}) \left[ \begin{array}{c}
\hat{B}_1'(h_t) \\
\hat{B}_2'(h_t)
\end{array} \right]',
\]

such that

\[
\gamma^2 I - \hat{B}_1'(h_t) S(\pi_{1+1}) \hat{B}_1(h_t) > 0,
\]

where \( S(\pi) \) and \( \hat{B}_1(h_t) \) are given by the partitions in (9) and (23), respectively, and the discrete \( N_u \)-periodic system

\[
x_e(kT_u + \pi_{1+1}) = A_{\pi}(h_t, \pi_1) x_e(kT_u + \pi_1)
\]

where

\[
A_{\pi}(h_t, \pi_1) := \hat{A}_e(h_t, \pi_1) - \left[ \begin{array}{c}
\hat{B}_1'(h_t) \\
\hat{B}_2'(h_t, \pi_1)
\end{array} \right] M_S(h_t, \pi_1)^{-1} E_S(h_t, \pi_1)
\]

is exponentially stable.

(ii) The matrices \( \Pi_1(s_j) \), where \( s_j := \sigma_{j+1} - \sigma_j \), are invertible and there is an \( N_y \)-periodic symmetric positive semidefinite matrix function \( Q(s_j) \) satisfying the following periodic Riccati difference equation

\[
Q(s_{j+1}) = \hat{A}(s_j) Q(s_j) \hat{A}'(s_j) + \hat{B}_1(s_j) \hat{B}_1'(s_j) - E_{Q}(s_j, s_j) M_Q^{-1}(s_j, s_j) E_Q'(s_j, s_j)
\]

\[
Q(iT_y + \sigma_j) = Q(s_j), \quad j = 0, 1, \ldots, N_y - 1, \quad i = 0, 1, \ldots
\]

where

\[
E_Q(s_j, s_j) := \hat{A}(s_j) Q(s_j) \left[ \begin{array}{cc}
\hat{C}_1'(s_j) & \hat{C}_2'(s_j)
\end{array} \right]
\]

\[
M_Q(s_j, s_j) := \left[ \begin{array}{c}
-\gamma^2 I \\
0
\end{array} \right] \hat{D}_1(s_j) \hat{D}_1'(s_j) + \left[ \begin{array}{c}
\hat{C}_1'(s_j) \\
\hat{C}_2'(s_j)
\end{array} \right] Q(s_j) \left[ \begin{array}{c}
\hat{C}_1'(s_j) \\
\hat{C}_2'(s_j)
\end{array} \right]',
\]

such that

\[
\gamma^2 I - \hat{C}_1'(s_j)[I - L_d(s_j) \hat{C}_2(s_j)] Q(s_j) \hat{C}_2'(s_j) > 0
\]

where \( L_d(\cdot) \) is given by (14) and the discrete \( N_y \)-periodic system

\[
x(iT_y + \sigma_{j+1}) = [\hat{A}(s_j) - E_Q(s_j, \sigma_j) M_Q^{-1}(s_j, \sigma_j) \left[ \begin{array}{c}
\hat{C}_1'(s_j) \\
\hat{C}_2'(s_j)
\end{array} \right]] x(iT_y + \sigma_j)
\]

is exponentially stable.

(iii)

\[
\rho[S(t)Q(t)] < \gamma^2, \text{ for } t = iT_y + \sigma_j, \text{ such that } t \leq T,
\]

\[
j = 0, 1, \ldots, N_y - 1, \quad i = 0, 1, \ldots
\]
where \( S(t) \) is given by (10) and \( T \) is the period of the multirate system \( \Sigma_G \). Moreover, when these conditions hold, the performance bound \( J_{\infty}(K) < \gamma \) is achieved by the controller in (17).

**Proof:** For the single-rate case a similar discretization of the equations corresponding to (10) and (13) was done in [29G37] and the proof is therefore not given in its full length here. The most obvious difference is the periodic time-varying nature of the resulting discrete-time equations (25) and (30). By the analysis in [29G37] the existence of bounded solutions to (10) and (13) between the discretization points is equivalent to the conditions (27) and (32) respectively and the stability criteria (12) and (15) are equivalent to the discrete-time criteria in (28) and (33) respectively.

An important step is the result that the coupling condition (16) can be discretized too. The discrete-time coupling condition is obtained from the observation that the quantity \( \rho[S(t)Q(t)] - \gamma^2 \) can change sign at the discontinuous jumps only i.e. at the instants \( \{ \tau_l \} \) and \( \{ \sigma_j \} \) [29G37]. From the equations describing the discontinuities in (10) and (13) we know that \( S(\tau^+) \geq S(\tau) \) and \( Q(\sigma_j) \geq Q(\sigma^+) \). By the property of the spectral radius function \( \rho(UV) \geq \rho(ZV) \) if \( U \geq Z \) where \( U, Z, V \) are positive definite matrices of compatible dimensions it then follows that if \( \rho(S(\sigma_j)Q(\sigma_j)) \leq \gamma^2 \) then \( \rho(S(t)Q(t)) \leq \gamma^2 \) is true in the whole interval \( t \in (\sigma_{j-1}, \sigma_j) \). Thus also the coupling condition can be discretized. The periodicity of the Riccati equations then give the set of coupling conditions (34).

**Remark 3.1** Observe that \( S(t) \) in (34) is obtained from the matrix exponential (21)

\[
S_c(t) = [\Pi_{21}(\tau - t) + \Pi_{22}(\tau - t)S_c(\tau)] [\Pi_{11}(\tau - t) + \Pi_{12}(\tau - t)S_c(\tau)]^{-1},
\]

where \( \tau_{l-1} < t \leq \tau_l \)

and the partition in (9).

**Remarks 3.2** An equivalent expression for the coupling condition (34) can be stated in terms of the periodicity of \( S_c(\cdot) \):

\[
\rho[S(t)Q(t)] < \gamma^2, \text{ for } t = kT + \tau^+_l, \text{ such that } t \leq T,
\]

\[
k = 0, 1, \ldots, N_u - 1, \quad l = 0, 1, \ldots
\]

where \( S(t) \) and \( Q(t) \) are given by (10) and (13), respectively.

**Remark 3.3** The controller (17) can be discretized by observing that the state transition matrix \( \Phi_A(\cdot, \cdot) \) associated with the matrix \( \Phi(\cdot) \) and the matrix \( \Gamma(t_2, t_1) := \int_{t_1}^{t_2} \Phi(t_2, \lambda) \tilde{B}_2(\lambda)d\lambda \) can be obtained through the matrix exponential in equation (21) [36].
Remark 3.4 Observe that no separate causality constraint problem arises in this solution, as opposed to the lifting solution [10, 39]. The controller (17) is periodically time-varying. A lifted time-invariant controller description can be obtained by lifting the periodic controller (17).

The periodic Riccati equations in (10) and (13) are discretized at those points only where a discrete event actually occurs. An alternative and perhaps more common formulation is to define a small time interval, say \( h \), which is a fraction of all sampling and hold intervals. This way of discretizing would result in two discrete Riccati equations with the same period. Further, the coupling condition would be expressed in terms of the discrete periodic solutions instead of the expressions (34) or (36). We would also need one matrix exponentiation only in order to obtain the matrices involved in the discrete Riccati equations. This way of discretizing may, however, lead to a very long period of the periodic equations since they may be discretized at times where no action actually occurs. In particular, a common sampling-time fraction may lead to a very long period for systems with non-synchronized sampling. I.e. when there are discrete time actions both at times \( \{ t+kT \} \) and \( \{ t+kT+\alpha \} \) for some \( \alpha < T \). The present approach is also applicable to the case when \( \alpha \) is irrational. This case is not possible to solve using a constant discretization step. Our problem formulation is stated in order to achieve the shortest possible period of the periodic Riccati difference equations in (25) and (30) with the possible expense of more matrix exponentiations. The alternative formulation with one single sampling-time fraction can be regarded as a special case of this description.

The following theorem shows that the periodic Riccati difference equations in (25) and (30) can be solved via standard discrete algebraic Riccati equations. See also [21] for an alternative generation of an algebraic Riccati equation.

Theorem 3.3 Assume that there exists a bounded stabilizing positive semidefinite matrix function \( P(\cdot) \) satisfying the following discrete N-periodic Riccati equation

\[
P(t_{k+1}) = A(t_k) P(t_k) A'(t_k) + R_t(t_k) - [A(t_k) P(t_k) C'(t_k) + R_{12}(t_k)]
\]
\[
\cdot [C(t_k) P(t_k) C'(t_k) + R_2(t_k)]^{-1} [C(t_k) P(t_k) A'(t_k) + R_{12}'(t_k)]
\]

\[
A(t_k) = A(t_{k+N}), \quad C(t_k) = C(t_{k+N}), \quad R_t(t_k) = R_t(t_{k+N})
\]

\[
k = 0, 1, 2, \ldots
\]

Then there exists a solution \( P := P(t_N) \) to the following algebraic Riccati equation

\[
P = \bar{A} P \bar{A}' + \bar{R}_1 - [\bar{A} P \bar{C}' + \bar{R}_1] [\bar{C} P \bar{C}' + \bar{R}_2]^{-1} [\bar{C} P \bar{A}' + \bar{R}_{12}]
\]

where

\[
\bar{A} := A_N, \quad \bar{C} := C_N
\]
\[
\bar{R}_1 := R_{1,N}, \quad \bar{R}_{12} := R_{12,N}, \quad \bar{R}_2 := R_{2,N},
\]

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where the matrices are defined by the following recursive equations

\[
A_k := A(t_k)A_{k-1}, \quad A_0 := A(t_0),
\]
\[
C_k := \begin{bmatrix} C_{k-1} \\ C(t_k)A_{k-1} \end{bmatrix}, \quad C_0 := C(t_0),
\]
\[
R_{1,k} := A(t_k)R_{1,k-1}A'(t_k) + R_1(t_k), \quad R_{1,0} := R_1(t_0),
\]
\[
R_{12,k} := \begin{bmatrix} A(t_k)R_{12,k-1}A'(t_k) + R_2(t_k) \\ C(t_k)R_{12,k-1}C'(t_k) + R_2(t_k) \end{bmatrix}, \quad R_{12,0} := R_2(t_0)
\]
\[
R_{2,k} := \begin{bmatrix} R_{2,k-1} \\ C(t_k)R_{2,k-1} \end{bmatrix}, \quad R_{2,0} := R_2(t_0).
\]

Proof: The result is obtained by induction cf. [7Γ12].

By Theorem 3.3 we can obtain the solutions to the periodic Riccati equations (25) and (30) at one point during their periods by solving algebraic Riccati-equations to which there exist algorithms with known convergence properties. The other solution points to (25) and (30) at the remaining time instants can then be determined using the original equations for from the recursion

\[
P(t_{k+1}) = A_kPA_k' + R_{1,k} - [A_kPC_k' + R_{12,k}][C_kPA_k' + R_{2,k}]^{-1}[C_kPC_k' + R_{12,k}].
\]

The solution to the multirate sampled-data \(H_\infty\) problem can thus be stated in terms of a pair of standard discrete algebraic Riccati equations together with the set of non-standard positive definiteness criteria (27)Γ(32) and the set of coupling conditions (34).

4 A lifting based solution

Periodic multirate control problems can be solved by the lifting technique in which the system is represented by a time-invariant lifted system. The lifted system has as its inputs and outputs the inputs and outputs during one period of the periodic system. The controller can then be designed for the time-invariant lifted system. However, the requirement that the controller should represent a causal periodic controller imposes conditions on the direct coupling term of the lifted controller. Due to these causality constraints the standard \(H_2\) and \(H_\infty\) synthesis methods are not directly applicable to the problem of designing controllers for periodic systems via lifting. Various special methods have therefore been studied for the problem of designing optimal controllers which satisfy a causality constraint [10Γ26Γ39Γ40].

In this section a two-Riccati equation solution is presented to the \(H_\infty\) problem for lifted systems with controller causality constraints. The result is closely related to the Riccati-equation based solution given in Section 3Γ and thus it also gives insight into the connection between the two solution procedures. A particular feature of the approach presented here is that lifting is applied separately to the periodic state feedback and estimation problems, which need not have the same periods.
In this section it is assumed that all hold elements are synchronized i.e., there exist times $kT_u$ such that the function $H(kT_u) = 1$ in (3). This enables us to formulate a particularly simple solution of the associated state-feedback problem.

4.1 A multirate state-feedback problem

Introduce the finite-dimensional $N_u$-periodic discrete system associated with the multirate system (4) [cf. [10139]]

$$
\begin{align*}
\dot{x}_e(kN_u + l + 1) &= \hat{A}_e(h_1, \pi) \dot{x}_e(kN_u + l) + \hat{B}_{1e}(h_1) \tilde{u}_d(kN_u + l) \\
&\quad + \hat{B}_{2e}(h_1, \pi) \tilde{u}_d(kN_u + l) \\
\tilde{z}_u(kN_u + l) &= \hat{C}_{1e}(h_1, \pi) \dot{x}_e(kN_u + l) + \hat{D}_{12}(h_1, \pi) \tilde{u}_d(kN_u + l),
\end{align*}
$$

\begin{align*}
l &= 0, 1, \ldots N_u - 1, \quad k = 0, 1, \ldots
\end{align*}

where $h_l = \pi_{l+1} - \pi_l \Gamma \tilde{u}_d(kN_u + l) := u_d(kT_u + \pi^r)\Gamma \hat{x}_e(kN_u + l) := [\hat{x}_e(kN_u + l) \ u'(kT_u + \pi)]^T := \hat{x}_e(kT_u + \pi)$ and $\tilde{u}_d(\cdot)$ denotes a discrete-time disturbance in $l_2$. The system matrices are given by (22). The system (42) is the equivalent discrete representation of (1) and (3) obtained by applying lifting to describe the sampled-data system in the intervals $[kT_u + \pi, kT_u + \pi + 1]$ [515139]. Thus a sampled state-feedback controller $K$ stabilizes the sampled-data system (4) and achieves the $H_\infty$-type performance bound $J_\infty(K) < \gamma$ if and only if it stabilizes (42) and the $l_2$-induced norm from $\tilde{u}_d$ to $\tilde{z}_u$ is less than $\gamma$.

In this section we apply the lifting technique to solve the problem of finding a causal stabilizing $N_u$-periodic state-feedback controller for the periodic system (42) such that the closed-loop system has the $l_2$-induced norm from $\tilde{u}_d$ to $\tilde{z}_u$ less than $\gamma$. Motivated by the multirate sampled-data problem, we consider the periodic discrete state-feedback problem only. However, the multirate full-information problem can be solved in a similar way.

The $N_u$-periodic system (42) has the state $\dot{x}_e(\cdot)$. However, as the lower part of the state $\dot{x}_e(\cdot)$ consists of the control inputs which are known to the controller any $N_u$-periodic state-feedback law $\tilde{u}_d = K_e \dot{x}_e$ can be written as

$$
\tilde{u}_d = K \tilde{x}_u,
$$

where $K$ is a dynamic $N_u$-periodic controller which includes the lower part of $\dot{x}_e(\cdot)$ as part of its state.

Introduce the lifted signals

$$
\begin{align*}
\bar{u}_u(k) &:= [u'_u(kN_u) \ a'_u(kN_u + 1) \ \cdots \ a'_u(kN_u + N_u - 1)]^T \\
\bar{u}_d(k) &:= [u'_d(kN_u) \ a'_d(kN_u + 1) \ \cdots \ a'_d(kN_u + N_u - 1)]^T \\
\bar{z}_u(k) &:= [z'_u(kN_u) \ a'_u(kN_u + 1) \ \cdots \ a'_u(kN_u + N_u - 1)]^T.
\end{align*}
$$

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Assuming that $H(kT_u) = I$ the associated time-invariant lifted system is given by

$$\bar{x}_u(k + 1) = A_u \bar{x}_u(k) + B_{1,u} \bar{u}_u(k) + B_{2,u} \bar{u}_d(k)$$

$$\bar{x}_u(k) = C_{1,u} \bar{x}_u(k) + D_{11,u} \bar{u}_u(k) + D_{12,u} \bar{u}_d(k)$$

(45)

where $\bar{x}_u(k) := \bar{x}(kN_u)$ and

$$A_u := \hat{A}(T_u)$$

$$B_{1,u} := [\hat{A}(\tau_{N_u} - \tau_1) \hat{B}_1(h_0) \hat{A}(\tau_{N_u} - \tau_2) \hat{B}_1(h_1) \cdots \hat{B}_1(h_{N_u-1})]$$

$$B_{2,u} := [\hat{A}(\tau_{N_u} - \tau_1) \hat{B}_2(h_0) H_u(\tau_0) \hat{A}(\tau_{N_u} - \tau_2) \hat{B}_2(h_1) H_u(\tau_1) \cdots$$

$$\cdots \hat{B}_2(h_{N_u-1}) H_u(\tau_{N_u-1})]$$

$$C_{1,u} := \begin{bmatrix}
\hat{C}_1(h_0) \\
\hat{C}_1(h_1) \hat{A}(\tau_2 - \tau_1) \\
\vdots \\
\hat{C}_1(h_{N_u-1}) \hat{A}(\tau_{N_u-1} - \tau_2) \\
\frac{\hat{A}(\tau_{N_u} - \tau_1)}{\hat{A}(\tau_{N_u} - \tau_2)}
\end{bmatrix}$$

$$D_{11,u} := [D_{11,u}(p,q)], \quad p, q = 0, 1, \ldots, N_u - 1$$

$$D_{12,u} := [D_{12,u}(p,q)], \quad p, q = 0, 1, \ldots, N_u - 1$$

where

$$D_{11,u}(p,q) := \begin{cases}
0, & \text{if } p \leq q \\
\hat{C}_1(h_p) \hat{A}(\tau_p - \tau_{q+1}) \hat{B}_1(h_q), & \text{if } p > q
\end{cases}$$

$$D_{12,u}(p,q) := \begin{cases}
0, & \text{if } p < q \\
\hat{D}_{12}(h_p, \tau_q), & \text{if } p = q \\
\hat{C}_1(h_q) \hat{A}(\tau_p - \tau_{q+1}) \hat{B}_2(h_q) H_u(\tau_q), & \text{if } p > q
\end{cases}$$

(47)

and $\hat{A}(\cdot) \hat{B}_1(\cdot) \hat{B}_2(\cdot) \hat{C}_1(\cdot)$ and $\hat{D}_{12}(\cdot, \cdot)$ are defined by (22) and (23). Note that for $p > q$

$$\hat{A}(\tau_p - \tau_q) = \hat{A}(h_{p-1}) \hat{A}(h_{q-2}) \cdots \hat{A}(h_1).$$

(48)

Notice that the lifted system (45) has an $n$-dimensional state vector. This is due to the assumption $H(kT_u) = I$ i.e. that all inputs are updated at time instants $kT_u$. If the state of the lifted systems were defined at time instants where only part of the inputs are subject to change then the state of the lifted system would include those elements of $u$ which are not updated at these time instants i.e. the augmented state in (4).

From the construction of the lifted system (45) it follows that an $N_u$-periodic state-feedback controller $K$ for (42) can be characterized as a time-invariant full-information controller $K_L := [K_x \ K_w] : l_2 \oplus l_2 \rightarrow l_2$ for (45) such that

$$\bar{u}_d(k) = ([K_x \ K_w] \begin{bmatrix}
\bar{x}_u \\
\bar{u}_u
\end{bmatrix})(k) := \begin{bmatrix}
(K \bar{x}_u)(kN_u) \\
\vdots \\
(K \bar{x}_u)(kN_u + N_u - 1)
\end{bmatrix}$$

(49)
By construction, the controller (43) stabilizes the periodic system (42) if and only if the controller (49) stabilizes (45) and the $l_2$-induced norm of the closed-loop system (42)$\Gamma$(43) is equal to the $H_\infty$ norm of the closed loop (45)$\Gamma$(49).

The problem of constructing a stabilizing periodic state-feedback controller for (42) which achieves a specified $l_2$-induced norm bound can thus be reduced to an $H_\infty$-optimal full-information control problem for the time-invariant lifted system (45). However, the equivalent $H_\infty$ problem defined for (45) is not a standard $H_\infty$ problem, because the requirement that (43) should be causal imposes corresponding causality constraints on the structure of the lifted controller in (49). In particular, partitioning $K_w$ in compliance with $u_d$ and $w_d$, the direct coupling term of $K_w$ should have zeros on and above the main block diagonal.

For future reference, it is convenient to introduce a connection between the Riccati-equation solution in Section 3 and the lifted time-invariant system (45).

**Theorem 4.1** Consider the multirate system (4) and the associated discrete $N_u$-periodic Riccati equation (25). Let $H_u(kT_u) = I$. Then

$$S_t(kT_u) = \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix} \quad (50)$$

where $S$ is the symmetric positive semidefinite solution of the discrete algebraic Riccati equation

$$S = A_u'SA_u + C_{1,u}'C_{1,u} - E_S'M_S^{-1}E_S \quad (51)$$

where

$$E_S := \begin{bmatrix} B_{1,u}' \\ B_{2,u}' \end{bmatrix} S A_u + \begin{bmatrix} D_{11,u}' \\ D_{12,u}' \end{bmatrix} C_{1,u}$$

$$M_S := \begin{bmatrix} B_{1,u}' \\ B_{2,u}' \end{bmatrix} S \begin{bmatrix} B_{1,u}' \\ B_{2,u}' \end{bmatrix}' + \begin{bmatrix} -\gamma^2 I + D_{11,u}'D_{11,u} & D_{11,u}'D_{12,u} \\ D_{12,u}'D_{11,u} & D_{12,u}'D_{12,u} \end{bmatrix} \quad (52)$$

**Proof:** The result follows from (25)$\Gamma$Theorem 3.3 and the structures of the matrices in (22).

The following theorem gives a state-space solution to the causally constrained lifted full-information $H_\infty$ problem associated with the multirate $H_\infty$-optimal state-feedback problem.

**Theorem 4.2** Consider the lifted system (44), (45) associated with the $N_u$-periodic system (42). There exists a stabilizing full-information controller $K_L$ described by (49) such that the associated $N_u$-periodic controller $\hat{K}$ is causal and such that the closed-loop system (45), (49) has $H_\infty$-norm less $\gamma$ if and only if there exists a stabilizing solution $S$ to the discrete algebraic Riccati equation (51) such that

$$\gamma^2 I - \hat{B}_1(h_l)S\hat{B}_1(h_l) > 0, \quad l = N_u - 1, \ldots, 0 \quad (53)$$
where the matrices \( \{ S(\tau) \} \) are defined recursively according to (25) with the partition in (9), and
\[
S_\tau(\tau_{N,u}) := \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix}.
\] (54)

In this case, a stabilizing causal \( N_u \)-periodic state-feedback controller \( K \) which achieves the norm bound is given by
\[
\hat{u}_d(kN_u + l) = -V^{-1}_{\tau}(h_\tau, \tau)|\tilde{B}_{x}(h_\tau, \tau)S_\tau(h_\tau, \tau)\tilde{A}_\tau(h_\tau, \tau)
+ \tilde{D}_{12}(h_\tau, \tau)\tilde{C}_{12}(h_\tau, \tau]|\hat{x}_e(kN_u + l)
\]
\[
l = 0, 1, \ldots, N_u - 1, \; k = 0, 1, \ldots
\] (55)

where \( \hat{x}_e(kN_u + l) = [\tilde{x}_u'(kN_u + l) \quad \tilde{u}(kN_u + l)] \),
\[
S_\tau(\tau) := S(\tau)[I - \gamma^{-2}\tilde{B}_{12}(h_\tau)\tilde{B}_{12}^*(h_\tau)S_\tau(\tau)]^{-1},
\] (56)

and
\[
V_{\tau}(h_\tau, \tau) := \tilde{B}_{2x}^*(h_\tau, \tau)S_\tau(\tau)\tilde{B}_{2x}(h_\tau, \tau) + \tilde{D}_{12}^*(h_\tau, \tau)\tilde{D}_{12}(h_\tau, \tau).
\] (57)

**Proof:** By standard discrete \( H_\infty \) control theory, a necessary condition for the existence of a full-information controller \( K \) which achieves the \( H_\infty \) performance bound for the system (45) is that the discrete algebraic Riccati equation (51) has a stabilizing symmetric positive semidefinite solution. In this case with \( \tilde{x}_u(0) = 0 \) we have the expansion [14]
\[
\|\tilde{x}_u\|_2^2 - \gamma^2\|\bar{u}_u\|_2^2 = \sum_{k=0}^{\infty} L_k(\bar{u}_u(k), \bar{u}_d(k))
\] (58)

where
\[
L_k(\bar{u}_u(k), \bar{u}_d(k)) := \left[ \begin{bmatrix} \bar{u}_u - \bar{u}_u^0 \\ \bar{u}_d - \bar{u}_d^0 \end{bmatrix} \right]'(k)M_S\left[ \begin{bmatrix} \bar{u}_u - \bar{u}_u^0 \\ \bar{u}_d - \bar{u}_d^0 \end{bmatrix} \right](k)
\] (59)

and
\[
\begin{bmatrix} \bar{u}_u^0 \\ \bar{u}_d^0 \end{bmatrix}(k) := -M_S^{-1}E_S\tilde{x}_u(k)
\] (60)

Moreover, the contribution to the quadratic cost (58) from the discrete time instant \( k \) can be expressed as
\[
\zeta^2_u(k)\tilde{x}_u(k) - \gamma^2\tilde{u}_u(k)\tilde{u}_u(k) = L_k(\bar{u}_u(k), \bar{u}_d(k)) + \tilde{x}_u(k)S\tilde{x}_u(k)
- \tilde{x}_u(k + 1)S\tilde{x}_u(k + 1)
\] (61)

From the definitions of the signals \( \tilde{x}_u(k) \) and \( \bar{u}_u(k) \) and the system equation (42), which describes the system behavior between the discrete time instants \( kN_u \) and \( kN_u + N_u \), we have similarly the expansion
\[ z_u'(k) z_u(k) - \gamma^2 \tilde{w}_u'(k) \tilde{w}_u(k) = \sum_{l=0}^{N_u-1} [z_u'(kN_u + l) \tilde{z}_u(kN_u + l) - \gamma^2 \tilde{w}_u'(kN_u + l) \tilde{w}_u(kN_u + l)] \]
\[ = \sum_{l=0}^{N_u-1} l_t(\tilde{u}_u(kN_u + l), \tilde{u}_d(kN_u + l)) + z_u'(kN_u) S \tilde{z}_u(kN_u) \]
\[ - \tilde{z}_u'(kN_u + N_u) S \tilde{z}_u(kN_u + N_u) \tag{62} \]

where
\[ l_t(\tilde{u}_u, \tilde{u}_d) := \begin{bmatrix} \tilde{u}_u - \tilde{u}_u^0 \\ \tilde{u}_d - \tilde{u}_d^0 \end{bmatrix}' M_S(h_t, \pi) \begin{bmatrix} \tilde{u}_u - \tilde{u}_u^0 \\ \tilde{u}_d - \tilde{u}_d^0 \end{bmatrix} \tag{63} \]

and
\[ \begin{bmatrix} \tilde{w}_u^0 \\ \tilde{w}_d^0 \end{bmatrix} (kN_u + l) := -M_S(h_t, \pi)^{-1} E_S(h_t, \pi) \tilde{z}_e(kN_u + l) \tag{64} \]

The result of the theorem then follows from (58)-(61) and (62) by standard discrete $H_\infty$ control theory [4T14T31].

**Remark 4.1** Notice that the solution of the causally constrained $H_\infty$ state-feedback problem is obtained in terms of the same algebraic Riccati equation (51) which is associated with the lifted system (45) and $H_\infty$ problem without causality constraints. The constraints appear only in the nonstandard form of the positive definiteness conditions (53) and the controller structure (55).

**Remark 4.2** Theorem 4.2 gives a causal $N_u$-periodic controller which achieves the specified $l_1$-induced norm bound for the system (42). The corresponding lifted time-invariant controller $K_L$ in (49) can be recovered by applying lifting to the periodic controller (55). It is easy to see that the lifted version of (55) is a static full-information controller of the form $K_L = [K_x \ K_u]$, where $K_u$ has zeros on and above the main block diagonal.

### 4.2 A multirate estimation problem

An analogous dual result is obtained for the multirate estimation problem. Define the finite-dimensional $N_y$-periodic discrete-time system associated with the multirate system (4) \[ \begin{align*}
\dot{x}_y(iN_y + j + 1) &= \hat{A}(s_j) \dot{x}_y(iN_y + j) + \hat{B}_1(s_j) \tilde{u}_y(iN_y + j) \\
\dot{z}_y(iN_y + j) &= \hat{C}_1(s_j) \tilde{y}_y(iN_y + j) \\
\dot{y}_d(iN_y + j) &= \hat{C}_2(s_j) \tilde{r}_d(iN_y + j) + \hat{D}_n(s_j) \tilde{y}_d(iN_y + j)
\end{align*} \tag{65} \]

where $j = 0, 1, \ldots N_y - 1$ and $i = 0, 1, \ldots$
where \( s_j := \sigma_{j+1} - \sigma_j \Gamma_{\hat{y}_y}(iN_y + j) := x(iT_y + \sigma_j) \Gamma_{\hat{y}_y}(iN_y + j) := y_d(iT_y + \sigma_j) \Gamma_{\hat{y}_d}(iN_y + j) := v_d(iT_y + \sigma_j) \) and \( \hat{u}_y(\cdot) \) denotes a discrete disturbance in \( l_2 \). The system matrices are defined by (23) and (24). The multirate \( H_\infty \)-optimal estimation problem consists of finding an \( N_y \)-periodic stable causal estimator

\[
\hat{z}_y = \mathcal{F} \check{y}_d
\]

which achieves the performance bound

\[
\sup_{(\tilde{u}_y, \tilde{e}_y) \neq 0} \frac{||\hat{z}_y - \check{z}_y||_{l_2}}{||\tilde{u}_y||_{l_2}^{1/2}} < \gamma
\]

The time-invariant system obtained by applying lifting to (65) over the period \( N_y \) of the sampling function is given by

\[
\begin{align*}
\tilde{x}_y(i + 1) &= A_y \tilde{x}_y(i) + B_{1,y} \tilde{u}(i) \\
\tilde{z}_y(i) &= C_{1,y} \tilde{x}_y(i) + D_{1,y} \tilde{u}(i) \\
\check{y}_d(i) &= C_{2,y} \tilde{x}_y(i) + D_{2,y} \tilde{u}(i)
\end{align*}
\]

where we have defined \( \check{z}_y(i) := \hat{z}_y(iN_y) \Gamma_{\check{u}} := [\hat{u}'(iN_y) \check{u}'(iN_y + 1) \ldots \check{u}'(iN_y + N_y - 1)]' \), and the lifted signals

\[
\begin{align*}
\tilde{u}(i) := [\hat{u}'(iN_y) \check{u}'(iN_y + 1) \ldots \check{u}'(iN_y + N_y - 1)]' \\
\tilde{z}_y(i) := [\hat{z}'(iN_y) \check{z}'(iN_y + 1) \ldots \check{z}'(iN_y + N_y - 1)]' \\
\check{y}_d(i) := [\check{y}'(iN_y) \check{y}'(iN_y + 1) \ldots \check{y}'(iN_y + N_y - 1)]'.
\end{align*}
\]

The system-matrices of the lifted system are given by

\[
\begin{align*}
A_y := & \hat{A}(T_y) \\
B_{1,y} := & [\hat{A}(\sigma_{N_y} - \sigma_1) \check{B}_1(s_0) \hat{A}(\sigma_{N_y} - \sigma_2) \check{B}_1(s_1) \ldots \check{B}_1(s_{N_y - 1})] \\
C_{1,y} := & \begin{bmatrix} 
\hat{C}_1(s_0) \\
\hat{C}_1(s_1) \hat{A}(\sigma_1 - \sigma_0) \\
\vdots \\
\hat{C}_1(s_{N_y - 1}) \hat{A}(\sigma_{N_y - 1} - \sigma_0)
\end{bmatrix} \\
C_{2,y} := & \begin{bmatrix} 
\hat{C}_2(s_0) \\
\hat{C}_2(s_1) \hat{A}(\sigma_1 - \sigma_0) \\
\vdots \\
\hat{C}_2(s_{N_y - 1}) \hat{A}(\sigma_{N_y - 1} - \sigma_0)
\end{bmatrix} \\
D_{11,y} := & [D_{11,y}(p, q)], \ p, q = 0, 1, \ldots, N_y - 1 \\
D_{21,y} := & [D_{21,y}(p, q)], \ p, q = 0, 1, \ldots, N_y - 1
\end{align*}
\]

where

\[
\begin{align*}
D_{11,y}(p, q) := \begin{cases} 
0, & \text{if } p \leq q \\
\hat{C}_1(s_p) \hat{A}(\sigma_p - \sigma_{q+1}) \check{B}_1(s_q), & \text{if } p > q
\end{cases} \\
D_{21,y}(p, q) := \begin{cases} 
0, & \text{if } p < q \\
\hat{C}_2(s_p) \hat{A}(\sigma_p - \sigma_{q+1}) \check{B}_1(s_q), & \text{if } p > q
\end{cases}
\end{align*}
\]
where we have introduced
\[
\begin{bmatrix}
\tilde{B}_1(s_j) \\
\tilde{D}_{21}(\sigma_j)
\end{bmatrix} :=
\begin{bmatrix}
\hat{B}_1(s_j) \\
0
\end{bmatrix}
\begin{bmatrix}
0 \\
\hat{D}_{21}(s_j)
\end{bmatrix}.
\] (72)

The $N_y$-periodic estimator (66) is equivalent to a time-invariant lifted estimator $\mathcal{F}_L : l_2 \to l_2$ described by
\[
\tilde{z}_y(i) = (\mathcal{F}_L \tilde{y}_d)(i) :=
\begin{bmatrix}
(\mathcal{F} \tilde{y}_d)(iN_y) \\
\vdots \\
(\mathcal{F} \tilde{y}_d)(iN_y + N_y - 1)
\end{bmatrix}.
\] (73)

Thus $\mathcal{F}_L$ achieves the performance bound
\[
\sup_{\alpha \neq 0} \frac{\|\tilde{z}_y - \tilde{z}_y\|_2}{\|\tilde{a}\|_2} < \gamma
\] (74)
for the system (68) if and only if $\mathcal{F}$ achieves the bound (67) for the system (65). The multirate $H_\infty$-optimal estimation problem can thus be described in terms of a time-invariant $H_\infty$-optimal estimation problem for the lifted system (68) subject to a causality constraint imposed by the causality of (66) and the structure of (73). More specifically the direct coupling term of $\mathcal{F}_L$ is required to have a lower block-triangular structure.

The following auxiliary result gives the solution to the periodic Riccati equation (30) in terms of an algebraic Riccati equation associated with the lifted system (68).

**Theorem 4.3** Consider the multirate system (4) and the associated discrete $N_y$-periodic Riccati equation (30). Then $Q := Q(I_{N_y} + \sigma_0)$ is the symmetric positive semidefinite solution of the discrete algebraic Riccati equation
\[
Q = A_yQA_y' + B_{1,y}B_{1,y}' - E_M^{-1}E_Q
\] (75)
where
\[
E_Q := A_y Q \begin{bmatrix} C_{1,y} & C_{2,y} \end{bmatrix}' + B_{1,y} \begin{bmatrix} D_{11,y} & D_{12,y} \end{bmatrix}'
\]
\[
M_Q := \begin{bmatrix} C_{1,y} & C_{2,y} \end{bmatrix} Q \begin{bmatrix} C_{1,y}' & C_{2,y}' \end{bmatrix} + \begin{bmatrix} -\gamma^2 I + D_{11,y}D_{11,y}' & D_{11,y}D_{21,y}' \\ D_{21,y}D_{11,y}' & D_{21,y}D_{21,y}' \end{bmatrix}. \] (76)

The following theorem gives a state-space solution to the causally constrained lifted $H_\infty$ estimation problem associated with the multirate $H_\infty$-optimal estimation problem.

**Theorem 4.4** Consider the lifted system described by equations (68) and (69) associated with the $N_y$-periodic system (65). There exists a stable estimator $\mathcal{F}_L$ described by (73) such that the associated $N_y$-periodic estimator $\mathcal{F}$ is causal and such that the $H_\infty$ performance
bound (74) is achieved if and only if the there exists a stabilizing solution $Q$ to the discrete algebraic Riccati equation (75) such that

$$
\gamma^2 I - \hat{C}_1(s_j)[I - L_d(\sigma_j)\hat{C}_2(\sigma_j)]Q(\sigma_j)\hat{C}_1'(s_j) > 0 \quad j = 0, 1, \ldots, N_y - 1
$$

(77)

where $L_d$ is given by (14) and the matrices $\{Q(\sigma_j)\}$ are defined recursively according to (30) with

$$
Q(\sigma_0) = Q.
$$

(78)

In this case, a stable causal $N_y$-periodic estimator $\mathcal{F}$ which achieves the norm bound (67) is given by

$$
\begin{align*}
\hat{x}_y(iN_y + j + 1^-) &= \hat{A}(s_j)\hat{x}_y(iN_y + j) \\
\hat{x}_y(iN_y + j) &= \hat{A}(s_j)\hat{x}_y(iN_y + j^-) + L_d(\sigma_j)[\gamma_d(iT_y + s_j) - \hat{C}_2(\sigma_j)\hat{x}_y(iN_y + j^-)] \\
\hat{z}_y(iN_y + j) &= \hat{C}_1(\sigma_j)\hat{x}_y(iN_y + j)
\end{align*}
$$

(79)

**Proof:** The proof is analogous to the proof of Theorem 4.2.

In analogy with the state-feedback problem the time-invariant lifted estimator $\mathcal{F}_L$ associated with (79) can be determined by applying lifting to the periodic estimator (79).

### 4.3 The multirate output-feedback problem

The causally constrained lifted state-feedback controller and estimator results of Theorems 4.2 and 4.4 can be combined to provide a lifting-based solution of the multirate output feedback problem. Although the result can be stated for general periodic sampling and hold functions it will for simplicity be assumed that the sampling and hold functions operate synchronously.

**Theorem 4.5** Consider the multirate sampled-data system described by (1)-(3). Assume that the sampling and hold functions operate synchronously, i.e. $N_u = N_y =$: $N$ and $\tau_l = \sigma_l$, $l = 0, 1, \ldots, N$. There exists a causal $N$-periodic controller $K$ which stabilizes the sampled-data system and which achieves the performance bound $J(K) < \gamma$ if and only if the following conditions are satisfied:

(a) The conditions of Theorem 4.2 are satisfied, 

(b) the conditions of Theorem 4.4 are satisfied, and

(c) $\rho[S(\tau)Q(\tau)] < \gamma$, $l = 0, 1, \ldots, N - 1$. 

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Moreover, when conditions (a)-(c) hold, a stabilizing multirate controller which achieves the $H_{\infty}$ performance bound is given by

\begin{align*}
\dot{x}_e(kN+l+1^-) &= A_{e1}(h_t, \tau)\dot{x}_e(kN+l) \\
\dot{x}_e(kN+l) &= \dot{x}_e(kN+l^-) \\
&+ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \hat{L}_d(\tau)[\tilde{y}_d(kN+l) - C_d(\tau)\dot{x}_e(kN+l^-)] \\
\hat{u}_d(kN+l) &= -V_{S}^{-1}(h_t, \tau)\tilde{B}_{e1}(h_t, \tau)S_{e}^{-1}(\tau_{i+1})\tilde{A}_e(h_t, \tau) \\
&+ \hat{D}'(h_t, \tau)\hat{C}_1(h_t, \tau)\dot{x}_e(kN_u+l)
\end{align*}

where $A_{e1}(\cdot, \cdot)$ is defined in (29), $V_S(\cdot, \cdot)$ is given by (57), and $\hat{L}_d(\cdot)$ is given by (19).

**Proof:** The first part of the proof follows directly from Theorems 4.1 - 4.4 and standard $H_{\infty}$ control theory. The equivalence of the controllers (17) and (80) can be shown in analogy with the single-rate case [28T37].

**Remark 4.3** Notice that Theorem 4.5 gives the multirate $H_{\infty}$-optimal controller in terms of the discrete algebraic Riccati equations of Theorems 4.1 and 4.3, which are the Riccati equations of the standard $H_{\infty}$ problem associated with the lifted system. The causality relations of the multirate system appear in the solution in the form of the nonstandard positive definiteness conditions (53) and (77) and the coupling conditions in Theorem 4.5 (c). Theorem 4.5 gives a periodic controller which achieves the $H_{\infty}$ performance bound for the multirate system. The corresponding time-invariant lifted controller can be determined by lifting the periodic controller (80) over the period $N$.

## 5 Example

The following simple numerical example is used to illustrate some of the results in this paper. Consider the system

\begin{align*}
\dot{x}(t) &= -0.8x(t) + w(t) + 2u(t) \\
z(t) &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t).
\end{align*}

Assume that there are two measurement devices that give sampled data with two different rates. Both devices measure the state but the more frequently available measurement is corrupted by a larger disturbance

\begin{align*}
y_{d1}(is) &= x(is) + 0.7v_{d1}(is), \quad i = 0, 1, 2, \ldots \\
y_{d2}(js) &= x(js) + 0.1v_{d2}(js), \quad j = 0, 2, 4, \ldots
\end{align*}
where \( s = 1.5 \) such that \( y_d(is) = [y_{d1}(is) \ y_{d2}(is)]' \) for \( i = 0, 2, 4, \ldots \) and \( y_d(is) = y_{d1}(is) \) for \( i = 1, 3, 5, \ldots \). The control signal

\[
u(t) = u_d(kh^+), \quad t \in (kh, kh + h], \quad k = 0, 1, 2, \ldots
\]

is updated with the period \( h = 2 \). The sampling function has the period \( T_y = 3 \) and the single-rate hold function has the period \( T_u = 2 \). The period of the hybrid multirate system defined by \( (81)-(83) \) is \( T = 6 \).

By separate discretization of the feedback and the estimation problems we obtain a time-invariant discrete Riccati-equation \( (25) \) associated with the feedback problem and a 2-periodic discrete Riccati-equation \( (30) \) associated with state estimation\(\text{cf. Theorem 4.3.} \)

The latter one can further be lifted to a time-invariant equation. Direct application of the lifting technique in order to construct a time-invariant description of the hybrid system \( (81)-(83) \) with a constant discretization interval would require the interval to be \( 0.5 \Gamma \) and the associated periodic discrete Riccati-equations would have the period \( 12 \).

The solutions \( S(t) \) and \( Q(t) \) to \( (10) \) and \( (13) \) respectively together with the spectral radius condition \( (16) \) are illustrated in Figure 1. Note that although the spectral radius condition must be satisfied for all times it is sufficient to check the coupling condition at three points during the period \( 2 \) namely at \( \{kh^+, k = 0, 1, 2\} \text{cf. Remark 3.2.} \)

Alternatively we may apply \( (34) \) which gives four coupling conditions at \( \{is, i = 0, 1, 2, 3\} \). The optimal cost is \( \gamma_{\text{inf}} = 0.971 \) and the solutions to the discrete Riccati equations \( (25) \) and \( (30) \) are

\[
S(kT_u) = 0.507 \quad Q(iT_y + s) = 0.810 \quad Q(iT_y) = 0.971.
\]

6 Conclusion

A state-space solution to the multirate sampled-data \( H_\infty \) problem has been derived. The solution is expressed in terms of a pair of discrete algebraic Riccati-equations together with a set of matrix positive-definiteness conditions and a set of coupling constraints. The result has been derived both through a direct Riccati-equation approach and via the lifting technique. Both methods generally lead to the same equations and the same controller. Although this paper has focused on the \( H_\infty \) problem a similar relationship between lifting and a two-Riccati solution can be shown to hold in the multirate Linear Quadratic/\( H_2 \) problem as well.

The results of this paper provide a closed state-space solution to the lifted time-invariant \( H_\infty \) problem with causality constraints. A particular feature of the solution is that the algebraic Riccati equations are the same as those associated with the unconstrained lifted \( H_\infty \) problem. The causality constraint of the multirate problem appear here as a set of additional nonstandard positive-definiteness constraints and a set of coupling conditions.
References


Figure 1: The solutions $S(t)$, $Q(t)$, and the coupling condition $S(t)Q(t)$ for $\gamma = \gamma_{\text{inf}}$ over one period $[0, T]$ of the multirate system (81) - (83).