Mixed $H_2/H_\infty$ control: Synthesis equations for the discrete-time case

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Abstract

A discrete mixed $H_2/H_\infty$ optimal control problem is studied, where an upper bound on an $H_2$ cost is minimized subject to an $H_\infty$ norm bound on the closed-loop transfer function. New synthesis equations for the optimal controller are obtained, which are the discrete-time counterparts of continuous-time synthesis equations given in the literature. The synthesis equations are obtained by deriving a set of optimality conditions for the parameters of an optimal controller parameterization. The sufficiency and necessity of the optimality conditions is also studied.

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1 Introduction

In this note a discrete-time version of the mixed $H_2/H_\infty$ problem due to Bernstein and Haddad [1] is studied. The problem consists of the minimization of an upper bound on an $H_2$ cost subject to an $H_\infty$ norm bound, in the special case when the $H_2$ and the $H_\infty$ loops have different outputs but share the same inputs. In the continuous-time case, the solution to this problem and its dual, where the outputs of the two loops are assumed equal, is known. The optimal controller can be characterized by a set of synthesis equations, consisting of coupled nonlinear matrix equations [1, 8, 3], or it can be formulated as a convex programming problem [6].

The discrete-time mixed $H_2/H_\infty$ problem has been much less studied than the continuous-time version. Kaminer et al. [5] have shown that the optimal controller can be parameterized as a compensator whose state dimension does not exceed the plant dimension, and they also give a convex parameterization of the optimal control problem. The synthesis of optimal fixed-order and state-feedback controllers have been studied in References [4, 2]. However, the discrete-time counterparts of the synthesis equations existing for the continuous-time problem [1, 8, 3] are still lacking, cf. for example the discussion in References [5, 2].

The purpose of this note is to derive synthesis equations for the discrete-time mixed $H_2/H_\infty$ problem. This is achieved by using the controller parameterization of Kaminer et al. [5], and by deriving optimality conditions for the associated optimization problem. The synthesis equations obtained in this way consist of a set of coupled nonlinear matrix equations. We also provide necessary and sufficient conditions for the optimality of a controller which satisfies the given synthesis equations.

2 Preliminaries

In this section the mixed $H_2/H_\infty$ optimal control problem is stated, and the optimal controller parameterization [5], on which the synthesis equations given in Section 3 will be based, is introduced.

2.1 Problem formulation

Consider the linear time-invariant finite-dimensional discrete-time plant $\mathcal{G}$ described by

\begin{align*}
    x_p(k+1) &= Ax_p(k) + B_1 w(k) + B_2 u(k), x_p(0) = 0 \\
    z_0(k) &= C_0 x_p(k) + D_0 u(k) \\
    z_1(k) &= C_1 x_p(k) + D_1 u(k) \\
    y(k) &= C_2 x_p(k) + D_2 w(k).
\end{align*}

(1)
The following standing assumptions are made on the system:

- The pairs \((A, B_1)\) and \((A, B_2)\) are stabilizable.
- The pair \((C_2, A)\) is detectable.
- \(D_2 D'_2 > 0\).

We assume a stabilizing linear compensator \(C\) of the form

\[
x_c(k+1) = A_c x_c(k) + B_c y(k), \quad x_c(0) = 0
\]

\[
u(k) = C_c x_c(k) + D_c y(k).
\]

Denote the closed-loop system \((1), (2)\) by

\[
x(k+1) = F x(k) + G w(k), \quad x(0) = 0
\]

\[
z_0(k) = H_0 x(k) + J_0 w(k)
\]

\[
z_1(k) = H_1 x(k) + J_1 w(k).
\]

Let \(T_{z_1 w}\) and \(T_{z_0 w}\) denote the closed-loop transfer functions from \(w\) to \(z_1\) and \(z_0\), respectively. Assuming that the closed-loop system \((3)\) is asymptotically stable, then the \(H_\infty\) norm bound

\[
\|T_{z_1 w}\|_\infty < \gamma
\]

is satisfied if and only if there exists a symmetric bounded positive semidefinite \((b.p.s.d.)\) matrix \(X\) which satisfies the equations

\[
X = FXF' + (FX H'_1 + GJ'_1) M(X)^{-1} (FX H'_1 + GJ'_1)' + GG'
\]

\[
M(X) := \gamma^2 I - J_1 J'_1 - H_1 X H'_1 > 0
\]

such that \(F + (FX H'_1 + GJ'_1) M(X)^{-1} H_1\) is asymptotically stable. The mixed \(H_2/H_\infty\) cost is defined as \([1, 5]\)

\[
J(\mathcal{G}, C) := \text{tr}(H_0 X H'_0 + J_0 J'_0),
\]

and it provides an upper bound on the \(H_2\) cost,

\[
\|T_{z_0 w}\|_2^2 \leq J(\mathcal{G}, C).
\]

The mixed \(H_2/H_\infty\) control problem consists of finding an internally stabilizing compensator \(C\) such that the \(H_\infty\) bound \((4)\) is achieved, and which minimizes the cost \((7)\).
2.2 Controller parameterization

The synthesis equations for the optimal mixed $H_2/H_\infty$ controller will be based on a parameterization of the optimal controller which is due to Kaminer et al. [5], and which is briefly summarized below.

The existence of an internally stabilizing controller for the plant (1) which achieves the $H_\infty$ bound (4) implies that there exists a b.p.s.d. stabilizing solution $Q$ to the discrete algebraic Riccati equation associated with the estimation part of the $H_\infty$ optimal control problem, and an associated variable transformation, which takes the system (1) to the auxiliary system $\tilde{\mathcal{G}}$ given by

$$
\begin{align*}
\dot{x}(k+1) &= \tilde{A}\tilde{x}(k) + \tilde{B}_1\tilde{w}(k) + \tilde{B}_2u(k) , \quad \tilde{x}(0) = 0 \\
\tilde{z}_0(k) &= \tilde{C}_0\tilde{x}(k) + \tilde{D}_{01}\tilde{w}(k) + \tilde{D}_{02}u(k) \\
\tilde{z}_1(k) &= \tilde{C}_1\tilde{x}(k) + \tilde{D}_{11}\tilde{w}(k) + \tilde{D}_{12}u(k) .
\end{align*}
$$

(9)

See [5] for explicit expressions for the matrices in (9). Moreover, there exists a stabilizing controller (2) which achieves the performance bound (4) for the plant (1) if and only if there exists a static-gain full-information controller

$$
u(k) = K_1\tilde{x}(k) + K_2\tilde{w}(k)$$

(10)

which achieves the $H_\infty$ performance bound $\|T_{\tilde{z}\tilde{w}}\|_\infty < \gamma$ for the system (9). In this case, the corresponding compensator $\mathcal{C}(K_1, K_2)$ in the original variables stabilizes (1) and achieves the bound (4), and is defined by (2) with

$$
\begin{align*}
A_c &= \tilde{A} + \tilde{B}_2C_c - \tilde{B}_1C_2 \\
B_c &= (\tilde{B}_1 + \tilde{B}_2K_2)(C_2QC_2' + D_2D_2')^{-1/2} \\
C_c &= K_1 - D_1C_2 \\
D_c &= K_2(C_2QC_2' + D_2D_2')^{-1/2} .
\end{align*}
$$

(11)

Kaminer et al. [5] have shown that the compensator $\mathcal{C}(K_1, K_2)$ has the optimal structure, i.e.,

$$
\inf_{\mathcal{C}} J(\mathcal{G}, \mathcal{C}) = \inf_{K_1, K_2} J(\mathcal{G}, \mathcal{C}(K_1, K_2))
$$

(12)

where the infima are taken with respect to internally stabilizing controllers which obey the $H_\infty$ bound (4). Moreover, the mixed $H_2/H_\infty$ cost achieved with $\mathcal{C}(K_1, K_2)$ is given by [5]

$$
J(\mathcal{G}, \mathcal{C}(K_1, K_2)) = \text{tr}(C_0QC_0') - \text{tr}[C_0QC_2'(C_2QC_2' + D_2D_2)^{-1}C_2QC_0'] + J(\tilde{\mathcal{G}}, K_1, K_2),
$$

(13)
where $J(\tilde{G}, K_1, K_2)$ denotes the mixed $H_2/H_\infty$ cost for the closed-loop system (9), (10). An explicit expression for the cost $J(\tilde{G}, K_1, K_2)$ is obtained by introducing the closed-loop system (9), (10),

$$
\begin{align*}
\tilde{x}(k+1) &= \bar{A}\tilde{x}(k) + \bar{B}_1\tilde{w}(k) \\
\tilde{z}_0(k) &= \bar{C}_0\tilde{x}(k) + \bar{D}_0\tilde{w}(k) \\
\tilde{z}_1(k) &= \bar{C}_1\tilde{x}(k) + \bar{D}_1\tilde{w}(k)
\end{align*}
$$

where

$$
\begin{align*}
\bar{A} &= \bar{A} + \bar{B}_2K_1, & \bar{B}_1 &= \bar{B}_1 + \bar{B}_2K_2, & \bar{C}_0 &= \bar{C}_0 + \bar{D}_0K_1, \\
\bar{D}_0 &= \bar{D}_{01} + \bar{D}_{02}K_2, & \bar{C}_1 &= \bar{C}_1 + \bar{D}_{12}K_1, & \bar{D}_1 &= \bar{D}_{11} + \bar{D}_{12}K_2.
\end{align*}
$$

By Section 2.1 the system (14) has $H_\infty$ norm from $\tilde{w}$ to $\tilde{z}_1$ less than $\gamma$ if and only if there exists a b.p.s.d. solution $Y$ to the equations

$$
Y = \bar{A}Y\bar{A} + (\bar{A}Y\bar{C}_1' + \bar{B}_1\bar{D}_1')M(Y)^{-1}(\bar{A}Y\bar{C}_1' + \bar{B}_1\bar{D}_1')' + \bar{B}_1\bar{B}_1'
$$

$$
M(Y) := \gamma^2I - \bar{D}_1\bar{D}_1' - \bar{C}_1Y\bar{C}_1' > 0
$$

such that the matrix

$$
A_Y := \bar{A} + (\bar{A}Y\bar{C}_1' + \bar{B}_1\bar{D}_1')M(Y)^{-1}\bar{C}_1
$$

is asymptotically stable. The $H_2/H_\infty$ cost $J(\tilde{G}, K_1, K_2)$ is then given by

$$
J(\tilde{G}, K_1, K_2) = \text{tr}(\bar{C}_0Y\bar{C}_0 + \bar{D}_0\bar{D}_0').
$$

### 3 Main results

The controller parameterization $C(K_1, K_2)$ and the relations (12) and (13) take the optimal mixed $H_2/H_\infty$ controller to a parametric optimization problem. In this section we present optimality conditions for the mixed $H_2/H_\infty$ cost (18), which provide a set of synthesis equation for the optimal controller.

**Theorem 3.1** Consider the closed-loop system (9), (10) and the cost $J(\tilde{G}, K_1, K_2)$. Suppose that the closed-loop system (14) is asymptotically stable and has $H_\infty$ norm from $\tilde{w}$ to $\tilde{z}_1$ less than $\gamma$. Then

$$
\frac{\partial J(\tilde{G}, K_1, K_2)}{\partial K_1} = 2[\bar{B}_1'P(\bar{A} + H\bar{C}_1) + \bar{D}_0'(\bar{C}_0 + \bar{D}_0K_1)]Y
$$

(19)
and
\[
\frac{\partial J(\tilde{G}, K_1, K_2)}{\partial K_i} = 2[(\bar{B}_2 + H \bar{D}_{12})'P(\bar{B}_1 + H \bar{D}_1) + \bar{D}_{02}'(\bar{D}_{01} + \bar{D}_{02}K_2)]
\]

where
\[
H := (\bar{A}Y\bar{C}'_1 + B_1\bar{D}'_1)M(Y)^{-1}
\]

and \( P \) is the symmetric positive semidefinite solution to the equation
\[
P = A'_YPA_Y + \bar{C}'_0\bar{C}_0.
\]

**Proof:** Observe that the equation (15) for \( Y \) can be written
\[
Y = (\bar{A} + \Delta \bar{C}_1)'(\bar{A} + \Delta \bar{C}_1)Y
- [\Delta (\gamma^2 I - \bar{D}_1\bar{D}'_1) - \bar{B}_1\bar{D}'_1](\gamma^2 I - \bar{D}_1\bar{D}'_1)^{-1}[\Delta (\gamma^2 I - \bar{D}_1\bar{D}'_1) - \bar{B}_1\bar{D}'_1]' + \bar{B}_1[I + \bar{D}'_1(\gamma^2 I - \bar{D}_1\bar{D}'_1)^{-1}\bar{D}_1]\bar{B}'_1
\]

with \( \Delta = H \). By the chain rule, we have
\[
\left( \frac{\partial J(\tilde{G}, K_1, K_2)}{\partial K_i} \right) = \left( \frac{\partial J(\tilde{G}, K_1, K_2)}{\partial K_i} \right)_{\Delta = H}
+ \sum_{k,l} \left( \frac{\partial J(\tilde{G}, K_1, K_2)}{\partial \Delta_{k,l}} \right)_{\Delta = H} \left( \frac{\partial \Delta_{k,l}}{\partial K_i} \right), \quad i = 1, 2.
\]

It is straightforward to show that
\[
\left( \frac{\partial J(\tilde{G}, K_1, K_2)}{\partial \Delta} \right)_{\Delta = H} = 0.
\]

The result then follows from standard gradient expressions [7] and some lengthy but straightforward matrix manipulations. \( \Box \)

The gradient expression in Theorem 3.1 together with the optimality conditions
\[
\frac{\partial}{\partial K_1}J(\tilde{G}, K_1, K_2) = 0, \quad \frac{\partial}{\partial K_2}J(\tilde{G}, K_1, K_2) = 0
\]
provide a set of synthesis equations for the optimal controller which are the discrete-time counterparts of the synthesis equations which have been given for the continuous-time problem in [1, 8, 3]. A crucial question in connection with the optimality conditions concerns their necessity and sufficiency. A partial answer to this question is provided by the following theorem.
Theorem 3.2 (a) Suppose that the infimum of $J(\hat{G}, K_1, K_2)$ is achieved for some $K_1^*, K_2^*$ such that the closed-loop system (14) is asymptotically stable and has $H_\infty$ norm from $\hat{w}$ to $\hat{z}_1$ less than $\gamma$. Then the optimality conditions (26) are satisfied for $K_1^*, K_2^*$.

(b) Suppose that there exist $K_1^*, K_2^*$ such that the closed-loop system (14) is asymptotically stable and has $H_\infty$ norm from $\hat{w}$ to $\hat{z}_1$ less than $\gamma$, and that

\[ Y > 0 \]
\[ \bar{B}_1' P (\bar{A} + H \bar{C}_1) + \bar{D}_{02}' (\bar{C}_0 + \bar{D}_{02} K_1) = 0 \]  
\[ (\bar{B}_2 + H \bar{D}_{12})' P (\bar{B}_1 + H \bar{D}_1) + \bar{D}_{02}' (\bar{D}_{01} + \bar{D}_{02} K_2) = 0 \]  

hold at $K_1^*, K_2^*$. Then

\[ J(\hat{G}, K_1^*, K_2^*) = \inf_{K_1, K_2} J(\hat{G}, K_1, K_2) \]  

Proof: Part (a) follows from the fact that the cost is continuously differentiable in the interior of the region where the $H_\infty$ norm bound is satisfied (cf. Theorem 3.1). The proof of part (b) is given in the Appendix.

4 Conclusions

Synthesis equations for the discrete mixed $H_2/H_\infty$ control problem have been derived. The equations are the discrete versions of the continuous-time equations given in References [1, 8, 3]. Necessary and sufficient conditions for the optimality of a controller which satisfies the equations have been obtained.

An important question associated with the controller synthesis problem concerns the numerical solution of the synthesis equations. The equations consist of a set of coupled nonlinear matrix equations which is quite hard to solve directly. In the continuous-time context, homotopy methods have been suggested for their solution. An alternative approach is to apply a convergent gradient-based descent algorithm [7] for the minimization problem in (12), which achieves convergence to a stationary point $(K_1^*, K_2^*)$ such that (26) holds, provided such a point exists. As minimization problems can in general be solved more efficiently than systems of nonlinear equations, a convergent minimization based procedure can be expected to provide a numerically powerful method for solving the synthesis equations as well.

5 Appendix

The proof of Theorem 3.2 will be based on the convex problem characterization due to Kaminer et al. [5], and it is summarized in the following lemmas.
**Lemma A.1** Consider the system (14), and suppose that the matrix $\tilde{A}$ has all eigenvalues in the open unit disc. Then the following statements are equivalent:

(a) There exists a symmetric positive semidefinite matrix $Y$ which satisfies equations (15) and the associated positive definiteness condition (16) such that the matrix $A_Y$ in (17) is asymptotically stable.

(b) There exists a symmetric $Z > 0$ such that

$$R(Z) := \tilde{A}Z\tilde{A}' - Z + (\tilde{A}Z\tilde{C}'_1 + \tilde{B}_1'\tilde{D}_1')M(Z)^{-1}(\tilde{A}Z\tilde{C}'_1 + \tilde{B}_1'\tilde{D}_1')' + \tilde{B}_1\tilde{B}_1' < 0$$

$$M(Z) := \gamma^2I - \tilde{D}_1'\tilde{D}_1 - \tilde{C}_1Z\tilde{C}'_1 > 0$$

Moreover, the conditions in (b) are equivalent to

$$Q(Z, K_1, K_2) := [\begin{array}{cc} \tilde{A} & \tilde{C}'_1 \\ \tilde{C}_1 & \tilde{A}' \end{array}] Z [\begin{array}{cc} \tilde{A}' & \tilde{C}'_1 \\ \tilde{C}_1 & \tilde{A} \end{array}] + [\begin{array}{cc} \tilde{B}_1' \\ \tilde{D}_1' \end{array}] [\begin{array}{cc} \tilde{B}_1 \\ \tilde{D}_1 \end{array}] - [\begin{array}{cc} Z & 0 \\ 0 & \gamma^2I \end{array}] < 0$$

For convenience, introduce the notation $g(K_1, K_2)$ for $J(\tilde{G}, K_1, K_2)$,

$$g(K_1, K_2) := \text{tr}[\tilde{C}_0 + \tilde{D}_{20}K_1]Y(\tilde{C}_0 + \tilde{D}_{20}K_1)' + (\tilde{D}_{01} + \tilde{D}_{02}K_2)(\tilde{D}_{01} + \tilde{D}_{02}K_2)'$$

and define the function

$$f(Z, K_1, K_2) := \text{tr}[(\tilde{C}_0 + \tilde{D}_{20}K_1)Z(\tilde{C}_0 + \tilde{D}_{20}K_1)' + (\tilde{D}_{01} + \tilde{D}_{02}K_2)(\tilde{D}_{01} + \tilde{D}_{02}K_2)']$$

**Lemma A.2** Assume that $K_1, K_2$ are such that the closed-loop system (14) is stable and achieves the $H_\infty$ norm bound of Theorem 3.1. Then the cost $g(K_1, K_2)$ can be characterized as

$$g(K_1, K_2) = \inf\{f(Z, K_1, K_2) : Z = Z', R(Z) < 0, \text{ and } M(Z) > 0\}.$$  \hspace{1cm} \text{(31)}

or, equivalently,

$$g(K_1, K_2) = \inf\{f(Z, K_1, K_2) : Z = Z', Q(Z, K_1, K_2) < 0\}.$$ \hspace{1cm} \text{(32)}

Following Kaminer et al. [5], introduce the variable $W := K_1 Z$, and define $f_W(Z, W, K_2) := f(Z, WZ^{-1}, K_2)$,

$$f_W(Z, W, K_2) := \text{tr}[(\tilde{C}_0Z + \tilde{D}_{20}W)Z^{-1}(\tilde{C}_0Z + \tilde{D}_{20}W)' + (\tilde{D}_{01} + \tilde{D}_{02}K_2)(\tilde{D}_{01} + \tilde{D}_{02}K_2)']$$

$$+(\tilde{D}_{01} + \tilde{D}_{02}K_2)(\tilde{D}_{01} + \tilde{D}_{02}K_2)'$$

$$+(\tilde{D}_{01} + \tilde{D}_{02}K_2)(\tilde{D}_{01} + \tilde{D}_{02}K_2)'.$$  \hspace{1cm} \text{(33)}
and $Q_W(Z, W, K_2) := Q(Z, WZ^{-1}, K_2)$,

$$
Q_W(Z, W, K_2) := \begin{bmatrix} \tilde{A}Z + \tilde{B}_2W \\ \tilde{C}_1Z + \tilde{D}_{12}W \end{bmatrix} \begin{bmatrix} \tilde{A}Z + \tilde{B}_2W' \\ \tilde{C}_1Z + \tilde{D}_{12}W' \end{bmatrix}' + \begin{bmatrix} \tilde{B}_1 + \tilde{B}_2K_2 \\ \tilde{D}_{11} + \tilde{D}_{12}K_2 \end{bmatrix} \begin{bmatrix} \tilde{B}_1 + \tilde{B}_2K_2' \\ \tilde{D}_{11} + \tilde{D}_{12}K_2' \end{bmatrix}' - \begin{bmatrix} Z & 0 \\ 0 & \gamma^2I \end{bmatrix},
$$

where the definitions of $\tilde{A}$, $\tilde{C}_1$, $\tilde{B}_1$ and $\tilde{D}_1$ have been used. Introduce the sets

$$
\Omega := \{(Z, W, K_2) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times p} \times \mathbb{R}^{m \times p} : Z = Z' > 0\} \tag{34}
$$

$$
\Phi := \{(Z, W, K_2) : (Z, W, K_2) \in \Omega : Q_W(Z, W, K_2) < 0\}. \tag{35}
$$

Then the minimum cost can be characterized as follows.

**Lemma A.3** Consider the cost in (29). Then

$$
\inf_{K_1, K_2} g(K_1, K_2) = \inf\{f_W(Z, W, K_2) : (Z, W, K_2) \in \Phi\} \tag{36}
$$

where the infimum with respect to $K_1, K_2$ is taken over matrices which give a stable closed loop and which achieve the $H_\infty$ norm bound.

The significance of the above result is that it expresses the minimization problem as a convex optimization problem.

**Lemma A.4** The mapping $f_W : \Omega \to \mathbb{R}^+$ is a convex function on $\Omega$, and the mapping $Q_W : \Omega \to \mathbb{R}^{n \times n}$ is convex.

We can now complete the proof of Theorem 3.2.

**Proof of Theorem 3.2 (b):** The proof is be based on the convex characterization in Lemma A.3 and Lemma A.4. Consider the equations (27), and define $Z^* := Y > 0$ and $W^* := K_1^*Y$. Consider any $(Z, W, K_2) \in \Phi$, and define

$$
Z(\alpha) := Z^* + \alpha(Z - Z^*)
$$

$$
W(\alpha) := W^* + \alpha(W - W^*)
$$

$$
K_2(\alpha) := K_2^* + \alpha(K_2 - K_2^*) \tag{37}
$$

By convexity (Lemma A.4), $(Z(\alpha), W(\alpha), K_2(\alpha)) \in \Phi$ on $\alpha \in (0, 1]$, and

$$
f_W(Z(\alpha), W(\alpha), K_2(\alpha)) \leq f_W(Z^*, W^*, K_2^*) + \alpha[f_W(Z, W, K_2) - f_W(Z^*, W^*, K_2^*)]. \tag{38}
$$

Note that $(Z^*, W^*, K_2^*) \notin \Phi$, but by continuity, (38) can be extended to $\alpha = 0$. On $\alpha \in [0, 1]$, define $\tilde{K}_1(\alpha) := W(\alpha)Z^{-1}(\alpha)$. Invertibility of $Z(\alpha)$ follows from the positive definiteness of $Z^*$ and $Z$. By construction, the inequalities in Lemma A.1 (b) hold on $\alpha \in (0, 1]$. The inequality $R(Z(\alpha)) < 0$ implies that the matrix $\tilde{A}$
is asymptotically stable. It follows that the pair \((K_1(\alpha), K_2(\alpha))\) is stabilizing and satisfies the \(H_\infty\) norm bound on \(\alpha \in [0,1]\). Let \(Y(\alpha)\) denote the solution of the Riccati equation for \(Y\) with \(K_1 = K_1(\alpha), K_2 = K_2(\alpha)\). Then, \(R(Z(\alpha)) < 0\) and \(R(Y(\alpha)) = 0\), and from standard monotonicity properties of the Riccati equation it follows that \(Z(\alpha) \geq Y(\alpha)\). Hence,

\[
g(K_1(\alpha), K_2(\alpha)) \leq f(Z(\alpha), K_1(\alpha), K_2(\alpha)) = f_W(Z(\alpha), W(\alpha), K_2(\alpha)), \tag{39}
\]

and, introducing \(f_W(Z^*, W^*, K_2^*) = g(K_1^*, L K 2^*)\), which follows from the definitions of \(Z^*\) and \(W^*\), it then follows from (38) that

\[
g(K_1(\alpha), K_2(\alpha)) \leq g(K_1^*, K_2^*) + \alpha[f_W(\ddot{Z}, \ddot{W}, \ddot{K}_2) - g(K_1^*, K_2^*)]. \tag{40}
\]

The inequality (40) implies

\[
\left. \frac{d}{d\alpha} g(K_1(\alpha), K_2(\alpha)) \right|_{\alpha = 0} \leq f_W(\ddot{Z}, \ddot{W}, \ddot{K}_2) - g(K_1^*, K_2^*) \tag{41}
\]

provided the derivative exists. On the other hand, we have

\[
\left. \frac{d}{d\alpha} g(K_1, K_2) \right|_{\alpha = 0} = \text{tr} \left[ \left( \frac{\partial g(K_1(\alpha), K_2(\alpha))}{\partial K_1} \right)' \left( \frac{\partial K_1(\alpha)}{\partial \alpha} \right) \right] + \text{tr} \left[ \left( \frac{\partial g(K_1, K_2)}{\partial K_2} \right)' \left( \frac{\partial K_2(\alpha)}{\partial \alpha} \right) \right]. \tag{42}
\]

Here, (27) imply that \(\partial g/\partial K_1 = 0\) and \(\partial g/\partial K_2 = 0\) at \(K_1^*, K_2^*\), and as

\[
\left. \left( \frac{\partial K_1(\alpha)}{\partial \alpha} \right) \right|_{\alpha = 0} = [\ddot{W} - K_1^* \ddot{Z}](Z^*)^{-1} \tag{43}
\]

\[
\left. \left( \frac{\partial K_2(\alpha)}{\partial \alpha} \right) \right|_{\alpha = 0} = \ddot{K}_2 - K_2^* \tag{44}
\]

are bounded, it follows that

\[
\left. \frac{d}{d\alpha} g(K_1(\alpha), K_2(\alpha)) \right|_{\alpha = 0} = 0. \tag{45}
\]

From (41) and (45) it then follows that

\[
f_W(\ddot{Z}, \ddot{W}, \ddot{K}_2) - g(K_1^*, K_2^*) \geq 0. \tag{46}
\]

As \((\ddot{Z}, \ddot{W}, \ddot{K}_2) \in \Phi\) was arbitrary, it follows in view of Lemma A.3 that \(K_1^*, K_2^*\) is a minimizer, which was the result to be proven. \(\square\)
References


