The Sampled-Data $H_\infty$ Problem: A Unified Framework for Discretization Based Methods and Riccati Equation Solution*

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Abstract

A new discretization based solution to the sampled-data $H_\infty$ control problem is given. In contrast to previous solution procedures, the method is not based on the lifting technique. Instead, an equivalent finite-dimensional discrete problem representation is derived directly from a description of the sampled-data system. This is achieved via a closed-loop expression of the worst-case intersample disturbance and an associated variable transformation. In this way the solution is obtained completely in terms of classical linear-quadratic optimal control theory.

The discretization procedure described here is closely related to both the lifting-based technique and a two-Riccati equation solution of the sampled-data $H_\infty$ problem, in which the solution is obtained in terms of two coupled Riccati differential equations with jumps. In this way the method makes the connections between the various solution procedures transparent. In particular, it can be shown that the lifting approach and the two-Riccati equation solution lead to identical synthesis equations for the $H_\infty$-optimal sampled-data controller.

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1 Introduction


Various solution methods have been proposed for the sampled-data $H_\infty$ problem. A widely used approach is based on the lifting technique in which the sampled-data $H_\infty$ problem is transformed into an equivalent finite-dimensional discrete $H_\infty$ problem (Bamieh and Pearson 1992, Bamieh et al. 1991, Toivonen 1992a, Hayakawa et al. 1994, Chen and Francis 1995). In this method, the solution is obtained in two stages. In the first stage, the lifting transformation is applied. This gives a representation of the sampled-data system as a discrete system with a finite-dimensional state vector but with inputs and outputs which take values in the infinite-dimensional space $L_2[0, h)$ and which represent the continuous-time intersample input and output signals (Bamieh and Pearson 1992, Toivonen 1992a). In the second stage, a loop-shifting technique and an operator-valued version of Redheffer’s lemma is employed to transform the system into one with finite-dimensional inputs and outputs as well (Bamieh and Pearson 1992). The latter stage involves the computation of the system matrices of the equivalent discrete system. Various formulae for these have been proposed in the literature (Bamieh and Pearson 1992, Hara et al. 1992, Hayakawa et al. 1994, Chen and Francis 1996).

An alternative, more direct, solution method for the sampled-data $H_\infty$ problem is to apply $H_\infty$ control theory of time-varying systems (Ravi et al. 1991) to the sampled-data system. In this approach the sampled-data system is considered as a periodically time-varying hybrid discrete/continuous system and the solution of the $H_\infty$ problem is obtained in terms of two mixed discrete/continuous Riccati equations and an associated coupling condition (Başar and Bernhard 1991, Sun et al. 1992, 1993, Toivonen 1992b, 1995). The lifting method and the Riccati equation based methods both lead to problem representations in the form of finite-dimensional discrete $H_\infty$ problems but the procedures are derived in completely different ways and any connections between them are far from obvious. Although it has been claimed that the procedures lead to mathematically equivalent results (Sivashankar and Khargonekar 1994), the equivalence of the two solutions has not been explicitly established.
In this paper a new worst-case discretization procedure for the sampled-data $H_\infty$ problem is given. A particular feature of the method is that it does not employ the lifting transformation. Instead, the discretization is based on the continuous-time system equations and a closed-loop representation of the worst-case intersample disturbance followed by an associated variable transformation. In this way a characterization of the equivalent finite-dimensional discrete system can be determined by using standard linear optimal control theory.

A second contribution of this paper is to provide insight into the connections between various solution methods for the sampled-data $H_\infty$ problem. Although the method studied here has been derived without a lifting transformation, it is closely related to the lifting based solution. The connection can be interpreted to be due to the way in which the worst-case intersample disturbance is represented. Whereas the present approach employs a closed-loop representation of the worst-case intersample disturbance, the lifting-based solution can be interpreted to consist of a lifting transformation followed by an open-loop representation of the worst-case disturbance. From this connection it follows that the two methods give the same discrete problem representation although the expressions obtained for the equivalent discrete system are different. An advantage of the present procedure is that as it is developed using classical linear-quadratic optimal control theory, the derivation is particularly transparent and the resulting expressions for the equivalent discrete system are particularly simple.

The method studied here also clarifies the connections between the lifting technique and the two-Riccati equation based solution. In particular, it will be shown that the discretization equations which are employed in the lifting method are embedded as part of the Riccati equations with jumps which arise in the Riccati equation based method. It follows that the lifting approach and the two-Riccati equation approach lead to identical solutions and that their apparent differences are solely due to different ways of writing the equations involved. A consequence of the analysis is that for design purposes it does not matter which solution procedure is adopted because both methods can be shown to lead to the same set of synthesis equations and the same sampled-data $H_\infty$-optimal controller.

This paper is partly based on preliminary results reported in the conference contribution (Toivonen 1993) and the book chapter (Toivonen 1995) where the worst-case discretization procedure employed here was first described.

The paper is organized as follows. In Section 2 the sampled-data $H_\infty$-optimal problem is defined. The new characterization of the worst-case discretization step is introduced in Section 3 and its connections with the lifting-based method and the two-Riccati equation solution are developed in Sections 4 and 5 respectively. In Section 6 a simple numerical example illustrates the solution methods and their connections.
2 Problem statement

Consider the finite-dimensional linear continuous-time plant

\[
\begin{align*}
    \dot{x}(t) &= Ax(t) + B_1 w_c(t) + B_2 u_c(t), \quad x(0) = 0 \\
    z_c(t) &= C_1 x(t) + D_{12} u_c(t) \\
    y_k &= C_2 x(kh) + D_{21} v_k, \quad k = 0, 1, \ldots
\end{align*}
\]

where \(x(t) \in \mathbb{R}^n\) is the state vector, \(w_c(t) \in \mathbb{R}^{m_1}\) and \(v_k \in \mathbb{R}^{m_3}\) are continuous and discrete disturbances, \(u_c(t) \in \mathbb{R}^{m_2}\) is the control signal, and \(z_c(t) \in \mathbb{R}^{p_1}\) and \(y_k \in \mathbb{R}^{p_2}\) are the controlled and measured outputs. The measured output \(y\) is available at the discrete sampling instants \(kh\).

The solution of the sampled-data \(H_\infty\)-optimal control problem will be subject to the following standard assumptions on the system:

(A1) The discrete system \((e^{Ah}, \int_0^h e^{At} B_2 dt)\) is stabilizable and the discrete system \((C_2, e^{Ah})\) is detectable.
(A2) The continuous system \((A, B_1)\) is stabilizable and the continuous system \((C_1, A)\) is detectable.
(A3) \(D_{12}^T D_{12} > 0\).
(A4) \(D_{21}^T D_{21} > 0\).

It is assumed that the control signal is generated by a zero-order hold device, i.e.,

\[u_c(t) = u_k, \quad t \in [kh, kh + h).\]

Discrete-time controllers of the form

\[u = K y\]

are considered where \(K : l_2 \to l_2\) is a discrete causal linear transfer function.

A worst-case performance measure is defined as the induced norm of the closed-loop system with respect to disturbances \(w_c\) and \(v\) in the spaces \(L_2\) and \(l_2\) respectively, i.e.,

\[
J(K) := \sup_{(w_c, v) \neq 0} \left\{ \frac{\|z_c\|_{L_2}}{\|w_c\|_{L_2}^2 + \|v\|_{l_2}^2} \right\}^{1/2}.
\]

In the sampled-data \(H_\infty\) problem the objective is to find a stabilizing discrete controller \(K\) which achieves a specified attenuation level \(\gamma > 0\), i.e., such that \(J(K) < \gamma\) for equivalently

\[
L(w_c, v, u) := \|z_c\|_{L_2}^2 - \gamma^2 \|w_c\|_{L_2}^2 + \|v\|_{l_2}^2 < 0, \quad \text{all } (w_c, v) \neq 0.
\]
The procedures studied in this paper are based on a characterization of the sampled-data system $(1)\Gamma(2)$ as a mixed continuous/discrete system which can be represented as a linear system with jumps \( \dot{x}_e(t) = A_c x_e(t) + B_{c1} w_e(t), \quad t \neq k h \)
\( x_e(kh) = A_d x_e(kh^-) + B_{d2} u_k \)
\( z_e(t) = C_{c1} x_e(t) \)
\( y_k = C_{d2} x_e(kh^-) + D_{d1} v_k \)
where \( x_e(t) := [x'(t), u'(t)]^T \) and
\[
A_c := \begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix}, \quad B_{c1} := \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \quad C_{c1} := \begin{bmatrix} C_1 & D_{12} \end{bmatrix}
\]
\( A_d := \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad B_{d2} := \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad C_{d2} := \begin{bmatrix} C_2 & 0 \end{bmatrix}. \)

We will make use of the following definition of stability of hybrid discrete/continuous systems cf. Sun et al. (1993) Ravi et al. (1991).

**Definition 2.1** Consider the linear system with jumps described by
\[
\dot{x}(t + kh) = F_e(t)x(t + kh), \quad t \in [0, h) \\
x(kh) = F_{ex}(kh^-),
\]
and define the state transition matrix \( \Phi(t_2, t_1) \), so that \( x(t_2) = \Phi(t_2, t_1)x(t_1), \quad t_2 \geq t_1. \)

The system is exponentially stable if there exist \( c_1, c_2 > 0 \) such that
\[
\|\Phi(t_2, t_1)\| \leq c_1 e^{c_2(t_2-t_1)}, \quad \text{all } t_2 \geq t_1.
\]

The following auxiliary result will be used in the sequel (cf. Chen and Francis 1991).

**Lemma 2.1** Consider the linear system with jumps in equation (9). The system is exponentially stable if and only if the matrix \( \Phi(h^-, 0)F_d \) has all eigenvalues in the open unit disc.
3 A discretization based solution

In this section a discretization based method for the $H_{\infty}$ sampled-data problem is developed directly from the system representation (6) without using a lifting transformation.

Introduce the contribution from the sampling interval $[kh, kh+h]$ to the quadratic cost (5) $\Gamma_k(w, c) := \int_{kh}^{kh+h} (z'_c z_c - \gamma^2 w'_c w_c) d\tau. \tag{10}$

A necessary condition for the inequality (5) to hold is that (10) has a bounded supremum in $w_c$. In this case the maximum of (10) is achieved by a unique element $w^*_c$ in $L_2[0, h]$ which can be determined by standard linear optimal control methods.

Introduce the symmetric positive semidefinite matrix

$$P(t) := \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}(t) \tag{11}$$

which satisfies the following Riccati differential equation associated with the continuous-time part of (6) and the quadratic cost (10) $\Gamma_k$

$$- \dot{P}(t) = A'_c P(t) + P(t) A_c + \gamma^{-2} P(t) B_{c1} B'_{c1} P(t) + C'_{c1} C_{c1},$$

$$P(0) = 0, \quad t \in [0, h]. \tag{12}$$

Theorem 3.1 Consider the signal $z_c$ defined by equation (6) and the associated cost (10). The cost $L_k(w_c, x(kh), u_k)$ has a bounded supremum in $w_c$ for all $x(kh)$ and $u_k$ if and only if the Riccati differential equation (12) has a bounded solution on $[0, h]$. In this case, the maximum of $L_k(w_c, x(kh), u_k)$ is achieved by the optimal closed-loop strategy

$$w^*_c(t + kh) := \gamma^{-2} B'_{c1} P(t) x_c(t + kh), \quad t \in [0, h]. \tag{13}$$

Moreover, the cost (10) can be expressed as

$$L_k(w_c, x(kh), u_k) = \begin{bmatrix} x'(kh) \ u_k' \end{bmatrix} P(0) \begin{bmatrix} x(kh) \\ u_k \end{bmatrix} - \gamma^2 \int_{kh}^{kh+h} \tilde{w}'(\tau) \tilde{w}(\tau) d\tau \tag{14}$$

where $\tilde{w}(\cdot)$ is the pointwise deviation from the optimal closed-loop strategy,

$$\tilde{w}(t + kh) := w_c(t + kh) - \gamma^{-2} B'_{c1} P(t) x_c(t + kh), \quad t \in [0, h]. \tag{15}$$

Proof: For the first part of the theorem which relates the existence of a bounded supremum of the cost (10) to the solution of the Riccati equation (12) see for example Başar and Bernhard (1991; chapter 8) or Brockett (1970; section 24). The
optimal strategy (13) and the expansion (14) of the cost are standard in linear-quadratic optimal control theory (Brockett 1970).

Introducing the variable transformation implied by equation (14) takes the sampled-data $H_\infty$ problem to a finite-dimensional discrete $H_\infty$ problem as follows.

Theorem 3.2 Consider the sampled-data system described by (6), and suppose that the conditions in Theorem 3.1 hold. Define the finite-dimensional discrete system

\[
\begin{align*}
x_{k+1} &= Ax_k + B_1 \bar{v}_k + B_2 u_k \\
\bar{z}_k &= \bar{C}_1 x_k + \bar{D}_{12} u_k \\
y_k &= \bar{C}_2 x_k + \bar{D}_{21} v_k
\end{align*}
\]  

(16)

where

\[
\bar{A} := \Phi(h, 0),
\]

\[
\bar{B}_1 \bar{B}_1' := \int_0^h \Phi(h, \lambda) B_1 B_1' \Phi(h, \lambda) d\lambda
\]

\[
\bar{B}_2 := \int_0^h \Phi(h, \lambda) [B_2 + \gamma^{-2} B_1 B_1' P_{11}(\lambda)] d\lambda
\]  

(17)

\[
\bar{C}_2 := C_2
\]

\[
\bar{D}_{21} := D_{21}
\]

\[
\begin{bmatrix}
\bar{C}_1' \\
\bar{D}_{12}'
\end{bmatrix}
\begin{bmatrix}
\bar{C}_1 \\
\bar{D}_{12}
\end{bmatrix} := P(0)
\]

and $\Phi(\cdot, \cdot)$ is the state transition matrix associated with $A + \gamma^{-2} B_1 B_1' P_{11}(t)$. Then for any controller $u = Ky$ the following statements are equivalent:

(i) $K$ stabilizes the sampled-data system (6) and achieves the induced norm bound (5),

(ii) $K$ stabilizes the discrete system (16) and achieves the $H_\infty$ norm bound $\|\bar{z}\|^2_2 - \gamma^2 \|\bar{v}\|^2_2 + \|v\|^2_2 < 0$, all $(\bar{v}, v) \neq 0$.

Proof: Assume first that $K$ stabilizes both (6) and (16). From Theorem 3.1 introduce the variable $\bar{w}$ defined by (15) and the variable $\bar{z}_k$ defined in (16). Then the sampled-data system is described by

\[
\begin{align*}
\hat{x}(t + kh) &= [A + \gamma^{-2} B_1 B_1' P_{11}(t)]x(t + kh) + B_1 \bar{w}(t + kh) \\
&\quad + [B_2 + \gamma^{-2} B_1 B_1' P_{12}(t)] u_k, \ t \in [0, h) \\
\bar{z}_k &= \bar{C}_1 x(kh) + \bar{D}_{12} u_k.
\end{align*}
\]  

(18)
Theorem 3.1 implies \( \| \tilde{z} \|_2^2 - \gamma^2 \| \tilde{w} \|_2^2 = \| \tilde{z} \|_2^2 - \gamma^2 \| \tilde{w} \|_2^2 \). The linear transformation on \( L_2[kh, kh+h] \) in (15) which takes \( w_c(t+kh) \) to \( \tilde{w}(t+kh) \Gamma \) is one-to-one. Hence the induced norm bound (5) holds for (6) if and only if the norm bound

\[
\| \tilde{z} \|_2^2 - \gamma^2 \| \tilde{w} \|_2^2 + \| v \|_2^2 < 0, \quad \text{all} \quad (\tilde{w}, v) \neq 0,
\]

(19) holds for (18). The equivalence of the norm bound (19) and the \( H_\infty \) norm bound stated in part (ii) of the theorem can be established by defining the variable \( \tilde{v}_k \) as follows (cf. also Toivonen (1992b) and Hayakawa et al. (1994)). Integrating (18) over the sampling interval gives

\[
x(kh + h) = \tilde{A}x(kh) + \tilde{B}_2u_k + \tilde{B}_k \tilde{w} \\
\tilde{z}_k = \tilde{C}_1x(kh) + \tilde{D}_{12}u_k
\]

(20)

where \( \tilde{B}_k : L_2[kh, kh+h) \rightarrow R^n \) is defined as

\[
\tilde{B}_k \tilde{w} := \int_0^h \Phi(h, t) B_1 \tilde{w}(t+kh)dt.
\]

(21)

For any \( \tilde{w} \Gamma \) introduce \( \eta_k \in L_2[kh, kh+h) \Gamma \) such that \( \tilde{B}_k \eta_k = \tilde{B}_k \tilde{w} \) and which has minimum norm \( \Gamma \), i.e., \( \Gamma \)

\[
\eta_k = \arg \min_w \left\{ \int_{kh}^{kh+h} w'w\, dt : \tilde{B}_k w = \tilde{B}_k \tilde{w} \right\}.
\]

(22)

The problem of finding \( \eta_k \) is a minimum energy control problem whose solution is known from linear-quadratic control theory (cf. for example Brockett (1970; section 22)) \( \Gamma \) and is given by

\[
\eta_k(t + kh) := B'_k \Phi'(h, t) \xi, \quad t \in [0, h],
\]

where \( \xi \) satisfies the equation

\[
\tilde{B}_1 \tilde{B}'_k \xi = \tilde{B}_k \tilde{w}
\]

where \( \tilde{B}_k \) is defined according to (17). Moreover, defining \( \tilde{v}_k := \tilde{B}_k \xi \Gamma \) we have \( \tilde{B}_k \tilde{v}_k = \tilde{B}_k \tilde{w} \) and \( \tilde{v}'_k \tilde{v}_k = \| \eta_k \|_2^2 \). The norm result of the theorem then follows from the construction of \( \tilde{v}_k \).

In order to prove the stability part of the theorem \( \Gamma \) assume first that \( \mathcal{K} \) internally stabilizes (6) and achieves the bound (5). The controller \( \mathcal{K} \) stabilizes (16) \( \Gamma \) if and only if it stabilizes the system (6) when the input is given by \( w_c^e \) in (13). Therefore, consider the system (6) with the input \( w_c = w_c^e \). Let \( \tilde{x}(t) \) denote the combined state of the system (6) and the controller. Then (5) implies the existence of a real number \( M_1 \) such that for any initial time \( k_0h \) and initial state \( \tilde{x}(k_0h) \Gamma \)

\[
\| \tilde{z}_e \|_2^2 \| \tilde{w}_e \|_2^2 \leq M_1^2 \| \tilde{z}(k_0h) \|_2^2.
\]

(23)
By standard properties of Riccati differential equations the symmetric positive semidefinite solution of (12) satisfies $P(t) \leq P(0)$ on $[0, h]$. As $w_c = w_c^0$ implies $\dot{w} = 0$ it follows from equations (13) and (14) that there exists a real number $M_2$ such that
\[ \|w_c^0\|^2_{L_2[L_2(0, h, \infty)]} \leq M_2^2 \left[ \|z_c\|^2_{L_2[L_2(0, h, \infty)]} - \gamma^2 \|w_c^0\|^2_{L_2[L_2(0, h, \infty)]} \right]. \] (24)

Since $K$ stabilizes the system (6) it follows that there exists $\delta > 0$ such that
\[ \|\tilde{x}\|_{L_2[L_2(0, h, \infty)]} \leq \delta \left[ \|w_c\|_{L_2[L_2(0, h, \infty)]} + \|\tilde{x}(k_0 h)\| \right]. \] (25)

Hence
\[ \|\tilde{x}\|_{L_2[L_2(0, h, \infty)]} \leq \delta (M_1 M_2 + 1) \|\tilde{x}(k_0 h)\| \] (26)
which implies that the closed-loop system (6) with input $w_c^0$ is stable or equivalently that $K$ stabilizes the system (16). The reverse result can be shown analogously by considering the system (16) for equivalently (18) with the input $\tilde{w} = w_c^0$.

The discrete characterization of Theorem 3.2 relates the discretization step to a standard linear-quadratic optimal control problem. In this way standard formulae of linear optimal control theory can be applied to find explicit expression for the system matrices of the discrete problem representation.

In order to derive explicit formulas for the system matrices in (17) introduce the $2(n + m_2) \times 2(n + m_2)$ matrix
\[ \Pi(t) := \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{bmatrix}(t) := \exp \left( - \begin{bmatrix} A_c & -B_{c1} B'_{c1} \\ -C'_{c1} C_{c1} & -A'_{c} \end{bmatrix} t \right) \] (27)
where the matrices $A_c, B_{c1}$ and $C_{c1}$ are defined by (7). Observe that from the structure of the matrices $A_c$ and $B_{c1}$ it follows that $\Pi_{11}$ and $\Pi_{12}$ can be partitioned as
\[ \Pi_{11} = \begin{bmatrix} \Pi_{1111} & \Pi_{1112} \\ 0 & I \end{bmatrix}, \quad \Pi_{12} = \begin{bmatrix} \Pi_{1211} & 0 \\ 0 & 0 \end{bmatrix}. \] (28)

**Theorem 3.3** The system matrices in (17) are given by
\[ \tilde{A} = \Pi_{1111}(h)^{-1}, \quad \tilde{B}_1 \tilde{B}'_1 = -\gamma^2 \tilde{A} \Pi_{211}(h), \quad \tilde{B}_2 = -\tilde{A} \Pi_{112}(h) \] (29)

\[ \begin{bmatrix} \tilde{C}'_1 & \tilde{D}'_{12} \end{bmatrix} \begin{bmatrix} \tilde{C} & \tilde{D}_{12} \end{bmatrix} = \Pi_{211}(h) \Pi_{111}(h)^{-1}. \]
Proof: First note that the matrix \( P(t) \) of equation (12) can be expressed in terms of the matrix \( \Pi(t) \) as
\[
P(t) = \Pi_{21}(h - t)\Pi_{11}(h - t)^{-1}, \quad t \in [0, h]. \tag{30}
\]
The matrix \( \Pi(h) \) is the transition matrix from time \( h \) to time \( 0 \) of the system
\[
\begin{bmatrix}
P_1(t) \\
P_2(t)
\end{bmatrix} =
\begin{bmatrix}
A_c & \gamma^{-2} B_c B'_c \\
-C_c' C_c & -A'_c
\end{bmatrix}
\begin{bmatrix}
p_1(t) \\
p_2(t)
\end{bmatrix}.
\tag{31}
\]
Introduce the variable transformation
\[
\begin{bmatrix}
r(t) \\
s(t)
\end{bmatrix} :=
\begin{bmatrix}
I & 0 \\
-P(t) & I
\end{bmatrix}
\begin{bmatrix}
p_1(t) \\
p_2(t)
\end{bmatrix}.
\tag{32}
\]
Then we have from (31) and (12) \( \Gamma \)
\[
\begin{bmatrix}
\dot{r}(t) \\
\dot{s}(t)
\end{bmatrix} =
\begin{bmatrix}
A_c + \gamma^{-2} B_c B'_c P(t) \\
0
\end{bmatrix}
\begin{bmatrix}
\gamma^{-2} B_c B'_c \\
-(A_c + \gamma^{-2} B_c B'_c P(t))'
\end{bmatrix}
\begin{bmatrix}
r(t) \\
s(t)
\end{bmatrix}.
\tag{33}
\]
Using the upper triangular structure gives
\[
\begin{bmatrix}
r(0) \\
s(0)
\end{bmatrix} =
\begin{bmatrix}
\Phi_c(0, h) & -\Phi_c(0, h) W_c \\
0 & \Phi'_c(h, 0)
\end{bmatrix}
\begin{bmatrix}
r(h) \\
s(h)
\end{bmatrix}
\tag{34}
\]
where \( \Phi_c(\cdot, \cdot) \) is the transition matrix associated with \( A_c + \gamma^{-2} B_c B'_c P(t) \) and \( W_c \) is the Gramian
\[
W_c := \gamma^{-2} \int_0^h \Phi_c(h, t) B_c B'_c \Phi'_c(h, t) dt.
\tag{35}
\]
On the other hand \( \Gamma (31) \) and (32) give
\[
\begin{bmatrix}
r(0) \\
s(0)
\end{bmatrix} =
\begin{bmatrix}
I & 0 \\
-P(0) & I
\end{bmatrix}
\begin{bmatrix}
\Pi_{11}(h) & 0 \\
\Pi_{11}(h) P(h) & \Pi_{11}(h)
\end{bmatrix}
\begin{bmatrix}
r(h) \\
s(h)
\end{bmatrix} =
\begin{bmatrix}
\Pi_{11} & \Pi_{12} \\
0 & \Pi_{22} - \Pi_{21} \Pi_{11}^{-1} \Pi_{12}
\end{bmatrix}(h)
\begin{bmatrix}
r(h) \\
s(h)
\end{bmatrix}.
\tag{36}
\]
By (34) and (36) \( \Gamma \)
\[
\Phi_c(h, 0) = \Pi_{11}(h)^{-1}, \quad W_c = -\Phi_c(h, 0) \Pi_{12}(h).
\tag{37}
\]
From the structure of the matrices \( A_c \) and \( B_c \) we have
\[
A_c + \gamma^{-2} B_c B'_c P(t) =
\begin{bmatrix}
A + \gamma^{-2} B_1 B'_1 P_{11}(t) & B_2 + \gamma^{-2} B_1 B'_1 P_{12}(t) \\
0 & 0
\end{bmatrix}.
\tag{38}
\]
and it follows that the associated state transition matrix \( \Phi_c(h, 0) \) and Gramian matrix \( W_c \) can be expressed in terms of the matrices \( \tilde{A} \Gamma \tilde{B}_1 \) and \( \tilde{B}_2 \) defined in Theorem 3.2 as
\[
\Phi_c(h, 0) =
\begin{bmatrix}
\tilde{A} & \tilde{B}_2 \\
0 & I
\end{bmatrix}, \quad W_c =
\begin{bmatrix}
\tilde{B}_1 & 0 \\
0 & 0
\end{bmatrix}.
\tag{39}
\]
The formulas to be proved then follow from equations (39)\( \Gamma(37)\Gamma(28)\) and (30). □
4 Connection with lifting based solution

The discrete problem characterization of Theorem 3.2 was derived without making use of the lifting technique. In this section it is shown that the resulting discrete representation is closely related to the one obtained via the lifting technique.

In the lifting-based procedure a discrete characterization of the sampled-data system (1) of (2) is introduced by observing that the evolution of the state of the system (1) at the sampling instants $kh$ is described by a discrete system with inputs and outputs taking values in the infinite-dimensional space $L_2[0, h)$ (Bamieh and Pearson 1992) (Toivonen 1992a). Formally this is achieved by introducing the isometric lifting operator $T_h : L_2[0, \infty) \to L_2^{[0,h]} \Gamma$ which takes the signals $w_i$ and $z_i$ in $L_2[0, \infty)$ to discrete signals $\hat{w}$ and $\hat{z}$ in the space $L_2^{[0,h]} \Gamma$ of sequences with elements in $L_2[0, h)$ (Bamieh and Pearson 1992) (Toivonen 1992a). The sampled-data system can then be described as

$$x_{k+1} = \hat{A} x_k + \hat{B}_1 \hat{w}_k + \hat{B}_2 u_k, \quad x_0 = 0$$
$$\hat{z}_k = \hat{C}_1 x_k + \hat{D}_{11} \hat{w}_k + \hat{D}_{12} u_k$$
$$y_k = \hat{C}_2 x_k + \hat{D}_{21} v_k$$

where $x_k := x(kh) \in R^n$ is the state at the sampling instants $\hat{w}_k$ and $\hat{z}_k$ in $L_2[0, h)$ represent the input and output of the system during the sampling interval $[t, t+kh)$.

$$\hat{w}_k(t) = w_k(t + kh), \quad \hat{z}_k(t) = z_k(t + kh), \quad t \in [0, h),$$

$$\hat{A} = e^{Ah} \Gamma \hat{B}_2 = \int_0^h e^{A(kh-\lambda)} B_2 d\lambda \Gamma \hat{C}_2 = C_2 \Gamma \hat{D}_{21} = D_{21} \Gamma$$
and the operators $\hat{B}_1 : L_2[0, h) \to R^n \Gamma \hat{C}_1 : R^n \to L_2[0, h) \Gamma \hat{D}_{11} : L_2[0, h) \to L_2[0, h)$ and $\hat{D}_{12} : R^n \to L_2[0, h)$ are defined as (Bamieh and Pearson 1992) (Toivonen 1992a)

$$\hat{B}_1 \hat{w} := \int_0^h e^{A(kh-\lambda)} B_1 \hat{w}(\lambda) d\lambda$$
$$(\hat{C}_1 x)(\tau) := C_1 e^{A\tau} x$$
$$(\hat{D}_{11} \hat{w})(\tau) := \int_0^\tau C_1 e^{A(\tau-\lambda)} B_1 \hat{w}(\lambda) d\lambda$$
$$(\hat{D}_{12} u_k)(\tau) := \int_0^\tau C_1 e^{A(\tau-\lambda)} B_2 d\lambda + D_{12} u_k.$$  

By construction $\|\hat{w}\|_2 = \|w\|_2$ and $\|\hat{z}\|_2 = \|z\|_2$ and the $L_2^{[0,h]} \Gamma$-induced norm of the discrete system (40) is equal to $\mathcal{J}(\mathcal{K})$ for any stabilizing controller $\mathcal{K}$

$$\mathcal{J}(\mathcal{K}) = \sup_{(\hat{w}, v) \neq 0} \left\{ \frac{\|\hat{z}\|_2}{\|\hat{w}\|_2 + \|v\|_2^{1/2}} \right\}.$$
The characterization (40) and (43) was used in (Toivonen 1992a) to compute the performance measure (4) using a basis function expansion of the intersample signals. This approach is complicated by the fact that the system has infinite-dimensional inputs and outputs. The problem can however be transformed into an equivalent finite-dimensional discrete problem (Bamieh and Pearson 1992; Hayakawa et al. 1994) by introducing a variable transformation which takes the problem into an equivalent problem with $\hat{D}_{11} = 0$. Since all the remaining operators in (40) have finite ranks this reduces the problem to a finite-dimensional one.

In order to see how the discretization procedure of Section 3 and the lifting-based solutions are connected, note that the intersample cost (10) can be expressed in terms of the signals of the lifted system (40) as

$$L_k(w_c, x(kh), u_k) = L_k(\hat{w}_k, x_k, u_k) := \|\hat{z}_k\|^2_2 - \gamma^2 \|\hat{w}_k\|^2_2.$$  (44)

In analogy with the standard finite-dimensional discrete $H_{\infty}$ problem a necessary condition for $J_k < \gamma$ to hold is that $\|\hat{D}_{11}\| < \gamma$. Equivalently, it is required that for any bounded $x_k\Gamma u_k$ (44) has a bounded supremum in $\hat{w}_k$. The following theorem is the counterpart to Theorem 3.1 when the worst-case disturbance is expressed in open-loop form.

**Theorem 4.1** Consider the signal $\hat{z}_k$ defined in (40) and the associated cost (44). The cost $L_k(\hat{w}_k, x_k, u_k)$ has a bounded supremum in $\hat{w}_k$ if and only if $\|\hat{D}_{11}\| < \gamma$. In this case, the maximum of $L_k(\hat{w}_k, x_k, u_k)$ is achieved by the optimal open-loop strategy

$$\hat{w}_k^\ast := \gamma^{-2}(I - \gamma^{-2}\hat{D}_{11}^*\hat{D}_{11})^{-1}\hat{D}_{11}^*[\hat{C}_1 x_k + \hat{D}_{12}u_k]$$  (45)

where $\hat{D}_{11}^*$ denotes the adjoint operator. Moreover, the cost (44) can be expressed as

$$L_k(\hat{w}_k, x_k, u_k) = [x_k' \ u_k'] [\hat{D}_{12}^*] (I - \gamma^{-2}\hat{D}_{11}\hat{D}_{11}^*)^{-1} [\hat{C}_1 \ \hat{D}_{12}] [x_k' \ u_k']$$

$$-\gamma^2 < \hat{w}_k - \hat{w}_k^\ast, (I - \gamma^{-2}\hat{D}_{11}\hat{D}_{11}) (\hat{w}_k - \hat{w}_k^\ast) >$$  (46)

where $<\cdot,\cdot>$ denotes the inner product on $L_2[0, h]$.

**Proof:** The result follows from the definition of $\hat{z}_k$ and a straightforward completion of squares argument.  

Introducing the variable transformation implied by (46) into the system equation (40) gives an equivalent finite-dimensional discrete characterization of the sampled-data $H_{\infty}$ problem.
Theorem 4.2 Consider the sampled-data system (40), and suppose that \( \| \hat{D}_{11} \| < \gamma \).
Define the finite-dimensional discrete system

\[
\begin{align*}
    x_{k+1} &= \hat{A}x_k + \hat{B}_1\hat{v}_k + \hat{B}_2u_k \\
    \dot{z}_k &= \hat{C}_1x_k + \hat{D}_{12}u_k \\
    y_k &= \hat{C}_2x_k + \hat{D}_{21}v_k
\end{align*}
\]

where

\[
    \begin{align*}
    \hat{A} &:= \tilde{A} + \gamma^{-2}\hat{B}_1\hat{D}_{11}^* (I - \gamma^{-2}\hat{D}_{11}\hat{D}_{11}^*)^{-1}\hat{C}_1 \\
    \hat{B}_1\hat{B}_1' &:= \tilde{B}_1(I - \gamma^{-2}\hat{D}_{11}^*\hat{D}_{11})^{-1}\hat{B}_1' \\
    \hat{B}_2 &:= \tilde{B}_2 + \gamma^2\hat{B}_1\hat{D}_{11}^*(I - \gamma^{-2}\hat{D}_{11}\hat{D}_{11}^*)^{-1}\hat{D}_{12}
    \end{align*}
\]

Then for any controller \( K \) the following statements are equivalent:

(i) \( K \) stabilizes the system (40) and achieves the performance bound \( J(K) < \gamma \),

(ii) \( K \) stabilizes the system (47) and achieves the \( H_\infty \) norm bound \( \| \dot{z} \|^2 + \gamma^2\| \hat{v} \|^2 + \| v \|^2 < 0 \), all \((\hat{v}, v) \neq 0\).

Proof: The result can be shown in a completely analogous way to Theorem 3.2 by introducing the variable transformation implied by (46) into the system (40).

Note that the discrete characterization in Theorem 4.2 is identical to the characterization given in the literature (Bamieh and Pearson 1992, Hayakawa et al. 1994) as it has been derived via a loop-shifting technique and Redheffer's lemma. The present treatment shows that the characterizations in Theorems 3.2 and 4.2 are closely related: the characterization in Theorem 3.2 is based on the closed-loop representation of the worst-case intersample disturbance (Theorem 3.1) while the characterization in Theorem 4.2 is based on the open-loop representation in Theorem 4.1. From the construction of the discrete systems it follows that they are mathematically equivalent as stated in the following theorem.

Theorem 4.3 The system matrices in equations (17) and (48) are equal,

\[
\begin{align*}
    \tilde{A} &= \hat{A} \\
    \tilde{B}_1\tilde{B}_1' &= \hat{B}_1\hat{B}_1' \\
    \tilde{B}_2 &= \hat{B}_2 \\
    \begin{bmatrix} \hat{C}_1' & \hat{D}_{12}' \end{bmatrix} [\hat{C}_1 & \hat{D}_{12}] &= \begin{bmatrix} \hat{C}_1' & \hat{D}_{12}' \end{bmatrix} [\hat{C}_1 & \hat{D}_{12}]
    \end{align*}
\]

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Proof: The result follows from the observation that the worst-case inputs in equations (13) and (45) are equivalent and from the construction of the discrete systems in Theorems 3.2 and 4.2.

It is worthwhile to observe that the finite-dimensional discrete systems in Theorems 3.2 and 4.2 depend on the attenuation level $\gamma$. In particular, as the system matrices $\bar{A}$ and $\bar{B}_2$ differ from the corresponding matrices in (40) the set of stabilizing controllers of the discrete system (16) or (47) is in general not the same as the set of controllers which stabilize the sampled-data system (1) or (2). This possible inconvenience can however be avoided by defining a norm-preserving variable transformation which takes the system in (16) or (47) to a finite-dimensional discrete system with the system matrices $\bar{A}$ and $\bar{B}_2$ of the representation (40) cf. Hayakawa et al. (1994).

5 Equivalence of discretization method and Riccati equation based solution

The sampled-data system can be characterized as a periodically time-varying mixed continuous/discrete system described by (6) and the sampled-data $H_\infty$ problem can be solved by adapting the $H_\infty$ control theory of time-varying systems (Ravi et al. 1991) to (6). This results in a solution of the sampled-data $H_\infty$ control problem in terms of coupled Riccati equations with jumps (Sun et al. 1992).

In this section it will be shown that the discretization based methods and the two-Riccati equation solution lead to the same synthesis equations for the $H_\infty$-optimal controller. This is achieved by showing that the Riccati equations with jumps and the associated coupling condition can be represented in terms of equivalent discrete Riccati equations and a discrete coupling condition which are precisely the equations that are obtained when a standard discrete two-Riccati equation solution is applied to the discrete $H_\infty$ problem defined for the system of Theorem 3.2. The discrete representations presented here are expressed in terms of the system matrices defined in Theorems 3.2 and 3.3 and they differ substantially from the discrete representations given by Sun et al. (1992).

This section is structured as follows. The solution of the sampled-data $H_\infty$ problem in terms of two coupled periodic Riccati equations with jumps due to Sun et al. (1992) is first stated in Theorem 5.1. The equivalent discrete representations of the Riccati equations and the associated coupling condition which relates the solution to the discretization based solution are given in Theorems 5.2, 5.3 and 5.4 respectively.

Theorem 5.1 Consider the mixed discrete/continuous system (6). Suppose that assumptions (A1)–(A4) hold. There exists a stabilizing controller $u = Ky$ which
achieves the induced norm bound $\mathcal{J}(K) < \gamma$ if and only if the following conditions are satisfied:

(a) There exists a bounded symmetric positive semidefinite matrix function

$$ S(t) = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}(t) $$

on $[0, h)$ which satisfies the equations

$$ -\dot{S}(t) = A' S(t) + S(t) A + \gamma^{-2} S(t) B_c B'_c S(t) + C'_c C_c, \quad 0 \leq t < h, \quad S(h^-) = S $$

$$ S = A'_0 S(0) A_d - A'_d S(0) B_d [B'_d S(0) B_d]^{-1} B'_d S(0) A_d $$

such that

$$ \dot{x}_e(t + kh) = (A_e + B_e K_1(t)) x_e(t + kh), \quad t \in [0, h) $$

$$ x_e(kh) = (A_d + B_d K_2) x_e(kh^-) $$

is exponentially stable, where

$$ K_1(t) := \gamma^{-2} B'_c S(t), \quad t \in [0, h) $$

$$ K_2 := -[B'_d S(0) B_d]^{-1} B'_d S(0) A_d. $$

(b) There exists a bounded symmetric positive semidefinite matrix function $Q(t)$ on $[0, h)$ which satisfies the equation

$$ \dot{Q}(t) = A Q(t) + Q(t) A' + \gamma^{-2} Q(t) C'_C C_c Q(t) + B_1 B'_1, \quad 0 \leq t < h $$

$$ Q(0) = Q - QC'_2 (D_{21} D'_{21} + C_2 Q C'_2)^{-1} C_2 Q, \quad Q = Q(h^-) $$

such that

$$ \dot{x}(t + kh) = (A + L_1(t) C_1) x(t + kh), \quad t \in [0, h) $$

$$ x(kh) = (I + L_2 C_2) x(kh^-) $$

is exponentially stable, where

$$ L_1(t) := \gamma^{-2} Q(t) C'_1, \quad t \in [0, h) $$

$$ L_2 := -Q C'_2 (D_{21} D'_{21} + C_2 Q C'_2)^{-1}. $$

(c) The matrices $S_{11}(t)$ and $Q(t)$ satisfy the spectral radius inequality $\rho(S_{11}(t) Q(t)) < \gamma^2$ for all $t \in [0, h)$.
Moreover, when these conditions are satisfied, a controller which achieves the induced norm performance bound is given by

\[
\begin{align*}
\dot{x}(t + kh) &= \bar{A}(t)\dot{x}(t + kh) + \bar{B}_2(t)u_k, \quad t \in [0, h) \\
\dot{x}(kh) &= \dot{x}(kh^-) + NC_2'(D_{2_1}D_{2_{11}} + C_2NC_2')^{-1}[y_k - C_2\dot{x}(kh^-)] \\
u_k &= -S_{22}^{-1}(0)S_{12}(0)\dot{x}(kh)
\end{align*}
\]

where

\[
N := Q(I - \gamma^{-2}S_{11}^1)^{-1}
\]

and

\[
\begin{align*}
\bar{A}(t) &= A + \gamma^{-2}B_1B_1'S_{11}(t) \\
\bar{B}_2(t) &= B_2 + \gamma^{-2}B_1B_1'S_{12}(t).
\end{align*}
\]

The Riccati equation with jumps of Theorem 5.1(a) can be represented in terms of a discrete Riccati equation assoicated with the discrete system of Theorem 3.2 as follows.

**Theorem 5.2** There exists a bounded symmetric positive semidefinite matrix function \(S(t)\) on \([0, h]\) which satisfies the Riccati equation with jumps (51) and the associated stability condition if and only if the matrix \(\Pi_{11}(h)\) in (27) is nonsingular and there exists a symmetric positive semidefinite matrix \(X\) which satisfies the algebraic Riccati equation

\[
X = \bar{A}'X\bar{A} - \begin{pmatrix} \bar{B}_1'X\bar{A} \\ \bar{B}_2'X\bar{A} + D_{12}'\bar{C}_1 \end{pmatrix}'G(X)^{-1}\begin{pmatrix} \bar{B}_1'X\bar{A} \\ \bar{B}_2'X\bar{A} + D_{12}'\bar{C}_1 \end{pmatrix} + \bar{C}_1'\bar{C}_1
\]

where

\[
G(X) := \begin{pmatrix} -\gamma^2I & 0 \\ 0 & D_{12}'D_{12} \end{pmatrix} + \begin{pmatrix} \bar{B}_1' \\ \bar{B}_2' \end{pmatrix}X[\bar{B}_1 \quad \bar{B}_2]
\]

such that

\[
\gamma^2I - \bar{B}_1'XB_1 > 0
\]

and

\[
\bar{A}_X := \bar{A} - [\bar{B}_1 \quad \bar{B}_2]G(X)^{-1}\begin{pmatrix} \bar{B}_1'X\bar{A} \\ \bar{B}_2'X\bar{A} + D_{12}'\bar{C}_1 \end{pmatrix}
\]

has all eigenvalues in the open unit disc. Moreover, \(S(\cdot)\) satisfies (51) and the associated stability condition if and only if \(X := S_{11} = S_{11}(h^-)\) is a positive semidefinite solution of (61)-(64).
Proof: First observe that the existence of a bounded positive semidefinite solution to (51) implies the existence of a bounded positive semidefinite matrix $P(t)$ which satisfies equation (12). Then $\Pi_{11}(t) = \Phi_{s}(h-t, h)$ where $\Phi_{s}(\cdot, \cdot)$ denotes the state transition matrix associated with the system matrix $A_{s} + \gamma^{-2}B_{s2}B'_{s1}P(t)\Gamma$ of the proof of Theorem 3.3. Hence $\Pi_{11}(t)\Gamma$ and $\Pi_{111}(t)\Gamma$ are nonsingular on $[0, h]$.

The matrix $S(t)$ in equation (51)$\Gamma$ when it exists is given by

$$S(h-t) = [\Pi_{21}(t) + \Pi_{22}(t)S][\Pi_{11}(t) + \Pi_{12}(t)S]^{-1},$$

and conversely when the matrix in (65) exists it satisfies (51). In this case following the arguments of the proof of Theorem 3.3, the arguments of the proof of Theorem 3.4, and conversely when the matrix in (66) exists, it satisfies (51) if and only if the matrix in (66) is nonsingular.

Equation (65) can be written in the form

$$S(h-t) = [\Pi_{22}(t) - \Pi_{21}(t)\Pi_{11}(t)^{-1}\Pi_{12}(t)][I + S\Pi_{11}(t)^{-1}\Pi_{12}(t)]^{-1}S\Pi_{11}(t)^{-1} + \Pi_{21}(t)\Pi_{11}(t)^{-1}.$$  

(67)

Using the fact that $(\Pi_{22} - \Pi_{21}\Pi_{11}^{-1}\Pi_{12})' = \Pi_{11}^{-1}\Gamma$ which follows from the fact that $\Pi_{11}(\cdot)$ is symplectic, invoking the structure of $\Pi_{11}(\cdot)$ of equation (28)$\Gamma$ and introducing the matrices in Theorem 3.3 gives

$$S(0) = \begin{bmatrix} \tilde{A}' \\ \tilde{B}'_2 \end{bmatrix} (I - \gamma^{-2}S_{11}\tilde{B}_1\tilde{B}'_1)^{-1}S_{11} \begin{bmatrix} \tilde{A} \\ \tilde{B}_2 \end{bmatrix} + \begin{bmatrix} \tilde{C}_1' \\ \tilde{D}_{12}' \end{bmatrix} \begin{bmatrix} \tilde{C}_1 \\ \tilde{D}_{12} \end{bmatrix}$$

(68)

where $S_{11} = S_{11}(h^{-})$ has been introduced. Combining (68) and (51) some lengthy but straightforward manipulations give

$$S_{11} = \tilde{A}'S_{11}\tilde{A} - \left[\tilde{B}'_2S_{11}\tilde{A} + \tilde{D}_{12}'\tilde{C}_1 \right]'G(S_{11})^{-1} \left[\tilde{B}'_2S_{11}\tilde{A} + \tilde{D}_{12}'\tilde{C}_1 \right] + \tilde{C}_1'\tilde{C}_1.$$  

(69)

Thus $S_{11}$ and $X$ satisfy the same discrete Riccati equation.

Next it will be shown that the existence of a bounded positive semidefinite matrix function $S(t)$ on $[0, h]$ is equivalent to the positive definiteness condition in (63) with $X = S_{11}$. The existence of $S(t)$ is equivalent to the nonsingularity of (66) on $[0, h]$. From the structures of the matrices $\Pi_{11}$ and $\Pi_{12}$ (cf. (28)) it follows that the matrix in (66) is nonsingular if and only if the matrix

$$\Pi_{111}(t) + \Pi_{121}(t)S_{11}$$

(70)
is nonsingular, which holds if and only if the matrix
\[ \gamma^2 I + \gamma^2 \Pi_{1111}(t)^{-1} \Pi_{1211}(t) S_{11} \]
is nonsingular. From the proof of Theorem 3.3 it follows that (cf. (35) Γ(37) and the structure of \( \Phi_s(\cdot, \cdot) \))
\[ - \gamma^2 \Pi_{1111}(t)^{-1} \Pi_{1211}(t) = \int_{h^{-}}^{h} \Phi(h, \lambda) B_i B'_i S_{11}(t) d\lambda \]
where \( \Phi(\cdot, \cdot) \) denotes the state transition matrix associated with the system matrix \( A + \gamma^{-2} B_i B'_i P_{11}(t) \). The matrix in (72) is positive semidefinite and nonincreasing in \( t \). It follows that the matrix (71) is nonsingular for all \( t \in [0, h] \) if and only if it is nonsingular for \( t = h \). Introducing the definition of \( \bar{B}_i \) in Theorem 3.3 the nonsingularity of (71) for \( t = h \) is equivalent to
\[ \gamma^2 I - \bar{B}_i^t S_{11} \bar{B}_i > 0 \]
which is the condition corresponding to (63).

Finally, consider the exponential stability condition associated with the closed-loop system. The closed-loop linear system with jumps in Theorem 5.1 (a) can be written in the form
\[
\begin{align*}
\dot{x}_s(t + kh) &= [A_s + \gamma^{-2} B_{i1} B'_{i1} S(t)] x_s(t + kh), \quad t \in [0, h) \\
x_s(kh) &= \begin{bmatrix} I & 0 \\ -S_{22}^{-1}(0) S'_{12}(0) & 0 \end{bmatrix} x_s(kh^-).
\end{align*}
\]
The state transition matrix from \( kh \) to \( kh + h^- \) is (cf. (66))
\[ \Phi_S(h^-, 0) = [\Pi_{11}(h) + \Pi_{12}(h) S]^{-1}. \]
Introducing the structures of \( \Pi_{11}(\cdot) \) and \( \Pi_{12}(\cdot) \) in (28) and the matrices defined in Theorem 3.3 gives
\[ \Phi_S(h^-, 0) = \begin{bmatrix} \bar{A}_S & \bar{B}_{S2} \\ 0 & I \end{bmatrix} \]
where
\[
\begin{align*}
\bar{A}_S &= (I - \gamma^{-2} B_i B'_i S_{11})^{-1} A \\
\bar{B}_{S2} &= (I - \gamma^{-2} B_i B'_i S_{11})^{-1} B_2.
\end{align*}
\]
By Lemma 2.11 we have that the closed-loop system (74) is exponentially stable if and only if the matrix
\[
\bar{A}_S - \bar{B}_{S2} S_{22}^{-1}(0) S'_{12}(0)
\]
has all eigenvalues in the open unit disc. Here it can be shown from (68) that

\[ S_{22}^{-1}(0)S_{12}^t(0) = (\tilde{B}'_2S_{11}^{-1}\tilde{B}_2 + \tilde{D}'_{12}\tilde{D}_{12})^{-1}(\tilde{B}'_2S_{11}^{-1}\tilde{A} + \tilde{D}'_{12}\tilde{C}_1) \]  

(80)

where

\[ S_{11}^{-1} := S_{11}(I - \gamma^{-2}\tilde{B}_1\tilde{B}'_1S_{11})^{-1}. \]

Combining these expressions give after some lengthy manipulations for (79) with \( S_{11} = X \Gamma \)

\[ \tilde{A} = -B_2S_{22}^{-1}(0)S_{12}^t(0) = \tilde{A}_X. \]  

(81)

Hence the stability condition in Theorem 5.1 (a) is equivalent to the stability of the matrix in (64). \( \square \)

Theorem 5.2 relates the Riccati differential equation with jumps (51) to the discrete Riccati equation (61) associated with the discrete system of Theorem 3.2. A similar result relating the Riccati differential equation with jumps (54) and a corresponding discrete Riccati equation is given by the following theorem.

**Theorem 5.3** There exists a bounded symmetric positive semidefinite matrix function \( Q(t) \) on \([0, h)\) which satisfies the Riccati equation with jumps (54) and the associated stability condition if and only if the matrix \( \Pi_{11}(h) \) in (27) is nonsingular and there exists a symmetric positive semidefinite matrix \( Y \) which satisfies the algebraic Riccati equation

\[ Y = \tilde{A}Y\tilde{A}' - \tilde{A}Y[\tilde{C}_1', \tilde{C}_2']E(Y)^{-1}\begin{bmatrix} \tilde{C}_1' \\ \tilde{C}_2' \end{bmatrix}Y\tilde{A}' + B_1\tilde{B}'_1 \]  

(82)

where

\[ E(Y) := \begin{bmatrix} -\gamma^2I & 0 \\ 0 & \tilde{D}_{21}\tilde{D}'_{21} \end{bmatrix} + \begin{bmatrix} \tilde{C}_1' \\ \tilde{C}_2' \end{bmatrix}Y[\tilde{C}_1' \tilde{C}_2']. \]  

(83)

such that

\[ I + \tilde{C}_2'(\tilde{D}_{21}\tilde{D}'_{21})^{-1}\tilde{C}_2Y - \gamma^{-2}\tilde{C}_1'\tilde{C}_1Y \]  

has only positive eigenvalues, and the matrix

\[ \tilde{A} = \tilde{A}Y[\tilde{C}_1' \tilde{C}_2']YE(Y)^{-1}\begin{bmatrix} \tilde{C}_1' \\ \tilde{C}_2' \end{bmatrix} \]  

(85)

has all eigenvalues in the open unit disc. Moreover, \( Q(\cdot) \) satisfies (54) and the associated stability condition if and only if \( Y := Q = Q(h^\cdot) \) is a positive semidefinite solution of (82)-(85).
Proof: Define the matrix exponential

$$
\Gamma(t) := \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix} (t) := \exp \left( \begin{bmatrix} A & \gamma^{-2}B_1B_1' \\ -C_1C_1' & -A' \end{bmatrix} t \right).
$$

(86)

The first part of the proof is similar to Theorem 5.2\Gamma and follows in the same way using (86) and the connections between the matrix exponentials \(P(t)\) and \(\Gamma(t)\) (cf. (90) and (91) below). Observe that the existence of a bounded symmetric positive semidefinite solution to (54) implies that \(\Gamma_{22}(t)\) is nonsingular on \([0, h]\) cf. the proof of Theorem 5.2. The matrix \(Q(t)\) in equation (54)\Gamma when it exists is given by

$$
Q(t) = [\gamma^2 \Gamma_{12}(t) + \Gamma_{11}(t)Q(0)][\Gamma_{22}(t) + \gamma^{-2}\Gamma_{21}(t)Q(0)]^{-1}
$$

(87)

and it is bounded and positive semidefinite if assuming that \(Q(0)\) is\(\Gamma\)if and only if the matrix

$$
\Gamma_{22}(t) + \gamma^{-2}\Gamma_{21}(t)Q(0)
$$

(88)

is nonsingular\(\Gamma\) cf. the proof of Theorem 5.2. Equation (87) can be written in the form

$$
Q(t) = [\Gamma_{11}(t) - \Gamma_{12}(t)\Gamma_{22}(t)^{-1}\Gamma_{21}(t)][I + \gamma^{-2}Q(0)\Gamma_{22}(t)^{-1}\Gamma_{21}(t)]^{-1}Q(0)\Gamma_{22}(t)^{-1} + \gamma^2\Gamma_{12}(t)\Gamma_{22}(t)^{-1}.
$$

(89)

In order to express the solution in terms of the matrices in (29)\Gamma define the matrix

$$
\Lambda(t) := \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix} (t) := \exp \left( - \begin{bmatrix} A & \gamma^{-2}B_1B_1' \\ -C_1C_1' & -A' \end{bmatrix} t \right) = \Gamma(t)^{-1}.
$$

(90)

Observe that from the definition and structure of the matrix of \(\Pi(\cdot)\) in (27) we have

$$
\begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix} (t) = \begin{bmatrix} \Pi_{1111} & \Pi_{1211} \\ \Pi_{2111} & \Pi_{2211} \end{bmatrix} (t).
$$

(91)

From (90) and (91)\Gamma the fact that \(\Gamma_{22}^{-1} = (\Gamma_{11} - \Gamma_{12}\Gamma_{22}^{-1}\Gamma_{21})\Gamma\) which is implied by the symplectic nature of the matrix \(\Gamma(\cdot)\) the definitions in (29)\Gamma and (89) it follows that

$$
Q(h^-) = \tilde{A}[I - \gamma^{-2}Q(0)\tilde{C}_1\tilde{C}_1']^{-1}Q(0)\tilde{A}' + \tilde{B}_1\tilde{B}_1'.
$$

(92)

Combining equations (92) and (54) gives for \(Q := Q(h^-)\Gamma

$$
Q = \tilde{A}Q\tilde{A}' - \tilde{A}Q[\tilde{C}_1' \tilde{C}_2'] E(Q)^{-1} \left[\begin{array}{c} \tilde{C}_1' \\ \tilde{C}_2' \end{array} \right] Q\tilde{A}' + \tilde{B}_1\tilde{B}_1'.
$$

(93)

Hence\(\Gamma Q\) and \(Y\) satisfy the same discrete Riccati equation.
Next it is shown that there exists a bounded symmetric positive semidefinite matrix function \( Q(t) \) on \([0, h]\) which satisfies (54) if and only if the matrix in (84) \( \Gamma \) with \( Y = \bar{Q}\Gamma \) has only positive eigenvalues. The existence of \( Q(t) \) is equivalent to the nonsingularity of the matrix in (88). From (54) we have

\[
\Gamma_{22}(t) + \gamma^{-2} \Gamma_{21}(t)Q(0) = \Gamma_{22}(t)^{-1}[I + \bar{C}_2'\bar{D}_{21}^{-1}\bar{C}_2Q + \gamma^{-2} \Gamma_{22}(t)^{-1}\Gamma_{21}(t)Q][I + \bar{C}_2'\bar{D}_{21}^{-1}\bar{C}_2Q]^{-1}. \tag{94}
\]

Equation (90) implies that \( \Gamma_{22}(t)^{-1}\Gamma_{21}(t) = -\Lambda_{21}(t) \Lambda_{11}(t)^{-1} \). Here \( \Lambda \)

\[
\Lambda_{21}(t)\Lambda_{11}(t)^{-1} = P_{11}(h - t), \tag{95}
\]

where \( P_{11}(\cdot) \) is the positive semidefinite matrix in equation (11) \( \Gamma \) and satisfies the equation

\[-\dot{P}_{11}(\tau) = A'P_{11}(\tau) + P_{11}(\tau)A + \gamma^{-2}P_{11}(\tau)B_1B_1'P_{11}(\tau) + C_1'C_1, \quad P_{11}(h) = 0.\]

By standard properties of the Riccati differential equation \( P_{11}(\cdot) \) is nonincreasing on \([0, h]\) and from (95) \( \Gamma(91) \) and (29) we have \( P_{11}(0) = \bar{C}_1'C_1 \). Hence it follows that the nonsingularity of (88) on \([0, h]\) is equivalent to the condition that the matrix

\[
I + \bar{C}_2'\bar{D}_{21}^{-1}\bar{C}_2Q - \gamma^{-2}\bar{C}_1'C_1Q \tag{96}
\]

has only positive eigenvalues \( \Gamma \) which is the condition in (84) with \( Y = Q \). The stability part of the theorem can be shown in the same way as in Theorem 5.2. \( \square \)

Next the coupling condition in Theorem 5.1 (c) will be expressed in terms of the solutions of the discrete Riccati equations (61) and (82).

**Theorem 5.4** Assume that there exist bounded positive semidefinite matrix functions \( S(t) \) and \( Q(t) \) on \([0, h]\) which satisfy the conditions in Theorem 5.1 (a) and (b). Then \( \rho(S_{11}(t)Q(t)) < \gamma^2 \) is satisfied for all \( t \in [0, h] \) if and only if \( \rho(XY) < \gamma^2 \), where \( X \) and \( Y \) are symmetric positive semidefinite solutions of the discrete Riccati equation of Theorems 5.2 and 5.3, respectively.

**Proof:** By Theorems 5.2 and 5.3 we have \( S_{11} = X \) and \( Q = Y \). Observe that \( S_{11}(t) \) can be expressed in terms of \( S_{11} = S_{11}(h^-) \) as

\[
S_{11}(h - t) = [\Lambda_{21}(t) + \Lambda_{22}(t)S_{11}][\Lambda_{11}(t) + \Lambda_{12}(t)S_{11}]^{-1} \tag{97}
\]

where \( \Lambda(t) \) is defined by equation (90). Similarly \( \Gamma Q(t) \) can be expressed in terms of \( Q = Q(h^-) \) as

\[
Q(h - t) = [\gamma^{2}\Lambda_{12}(t) + \Lambda_{11}(t)Q][\Lambda_{22}(t) + \gamma^{-2}\Lambda_{21}(t)Q]^{-1}. \tag{98}
\]
Using the fact that \( \Lambda(\cdot) \) is a symplectic matrix so that
\[
\begin{bmatrix}
\Lambda_{11} & \Lambda_{12} \\
\Lambda_{21} & \Lambda_{22}
\end{bmatrix}^T
\begin{bmatrix}
0 & -I \\
I & 0
\end{bmatrix}
\begin{bmatrix}
\Lambda_{11} & \Lambda_{12} \\
\Lambda_{21} & \Lambda_{22}
\end{bmatrix} =
\begin{bmatrix}
0 & -I \\
I & 0
\end{bmatrix},
\] (99)
and the relations (97) and (98) we have
\[
\gamma^2 I - QS_{11} =
\begin{bmatrix}
0 & -I \\
I & 0
\end{bmatrix}
\begin{bmatrix}
I \\
S_{11}
\end{bmatrix}
= \begin{bmatrix} \Lambda_{22}(t) + \gamma^2 \Lambda_{21}(t)Q \end{bmatrix} \gamma^2 I - Q(h - t)S_{11}(h - t)] [\Lambda_{11}(t) + \Lambda_{12}(t)S_{11}].
\] (100)

Here the first and third factors are nonsingular by the assumption that there exist bounded solutions \( Q(t) \) and \( S_{11}(t) \) on \([0, h]\). As \( Q(t) \) and \( S_{11}(t) \) are continuous functions of \( t \) it follows that \( \gamma^2 - \rho(S_{11}(t)Q(t)) \) does not change sign on \( t \in [0, h] \).

\[ \Box \]

**Remark 5.1:** Note that equation (100) implies that it is sufficient to check the coupling condition \( \rho(S_{11}(t)Q(t)) < \gamma^2 \) in Theorem 5.1 (c) at a single point \( t \in [0, h] \). Moreover when the value of \( \gamma \) is such that equality holds \( \rho(S_{11}(t)Q(t)) = \gamma^2 \) will hold for all \( t \in [0, h] \).

It is interesting to compare the discrete Riccati equations of Theorems 5.2 and 5.3 with the discrete representations given by Sun et al. (1992). Besides the fact that the equations have been expressed in a different form the most important difference is that in Sun et al. (1992) the discrete representation of (51) was expressed in terms of \( S(0) \) whereas the representation (61) is given in terms of \( S_{11} := S(h^-) \). Also the discrete representation of the coupling condition stated in Theorem 5.4 is new.

Combining the above results we have the following theorem which establishes the equivalence between the sampled-data \( H_\infty \)-optimal controller in Theorem 5.1 and the \( H_\infty \)-optimal controller which can be constructed for the discrete system (16).

**Theorem 5.5** The conditions in Theorem 5.1 are satisfied if and only if the following conditions hold:
(a) There exists a positive semidefinite matrix \( X \) which satisfies the equations (61)-(64), in which case \( S_{11} = X \).
(b) There exists a positive semidefinite matrix \( Y \) which satisfies the equations (82)-(85), in which case \( Q = Y \).
(c) The matrices \( X \) and \( Y \) satisfy the spectral radius inequality \( \rho(XY) < \gamma^2 \).
Moreover, when the above conditions are satisfied, the controller (57) can be written in the form

\[
\begin{align*}
\hat{x}_{k+1} &= \tilde{A} \hat{x}_k \\
\hat{x}_k &= \hat{x}_{k-1} + ZC_2^t(\check{D}_{21}\check{D}_{21}^t + \check{C}_2ZC_2^t)^{-1}(y_k - \check{C}_2\hat{x}_{k-1}) \quad (101) \\
u_k &= -(\check{B}_2^tX^-\check{B}_2 + \check{D}_{12}\check{D}_{12}^t)^{-1}((\check{B}_2^tX^-\check{A} + \check{D}_{12}\check{C}_1)\hat{x}_k)
\end{align*}
\]

where

\[
X^- := X(I - \gamma^{-2}\check{B}_1\check{B}_1^tX)^{-1} \quad (102)
\]

and

\[
Z := Y(I - \gamma^{-2}XY)^{-1}. \quad (103)
\]

Proof: The first part of the theorem follows from Theorems 5.2, 5.3 and 5.4. The equivalence of the controllers (57) and (101) can be shown by integrating the continuous-time part of (57). Following the proof of Theorem 5.2 gives

\[
\hat{z}(kh + h^-) = [\check{A}_S - \check{B}_{S2}S_{22}^{-1}(0)\check{S}_{12}'(0)]\hat{z}(kh).
\]

The result then follows from (81) and (80).

Note that the Riccati equations of Theorems 5.2 and 5.3 and the controller (101) are the discrete Riccati equations and the corresponding \(H_\infty\)-optimal controller which are obtained for the discrete system in equation (16) cf. Stoorvogel (1992). In view of Theorems 4.2 and 4.3 it follows that the two-Riccati equation approach and the lifting technique give rise to the same synthesis equations and the same \(H_\infty\)-optimal controller.

6 Numerical example

The following simple numerical example illustrates some of the results of this paper. Consider the scalar system

\[
\begin{align*}
\dot{x}(t) &= -0.8x(t) + w_z(t) + 2u_z(t), \quad x(0) = 0 \\
z_z(t) &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_z(t) \quad (104) \\
y_k &= x(kh) + 0.3v_k, \quad k = 0, 1, \ldots
\end{align*}
\]

When the system is controlled by a sampled-data controller (2)\(\Gamma\)(3) with the sampling period \(h = 1\) the minimum achievable attenuation level is \(\gamma_{inf} = 0.670\). Figure 1 shows the solutions of the Riccati equations with jumps in Theorem 5.1 for three values of \(\gamma\). Note that \(X := S_{11}(h^-)\) and \(Y := Q(h^-)\) solve the discrete Riccati
equations (61) and (82) and that the quantity $\gamma^2 - \rho(S_{11}(t)Q(t))$ does not change sign in the interval $[0, h]$ cf. Theorem 5.4 and Remark 5.1. Table 1 shows the parameters of the equivalent discrete system representations. It is interesting to note that for the values of $\gamma$ shown in the example the discrete systems are unstable although the continuous-time system (104) is stable. This phenomenon could be avoided by applying a second variable transformation as described by Hayakawa et al. (1994).

7 Conclusion

A worst-case discretization procedure which is not based on the lifting transformation has been studied for the sampled-data $H_\infty$ problem. The method employs results of standard linear-quadratic theory to derive the discrete problem characterization. The procedure results in simple expressions for the equivalent discrete system. The expressions are different from previously obtained expressions and require only a single matrix exponential of order $2(n + m_2)$ where $n$ and $m_2$ denote the dimensions of the state vector and control signal respectively.

The discretization procedure has been used to show that the lifting technique and a two-Riccati equation approach give the same synthesis equations for the sampled-data $H_\infty$ problem. As far as the synthesis of $H_\infty$-optimal controllers is concerned the two approaches can therefore be considered equivalent. Note that the result is not restricted to the synthesis problem but it can be generalized to the problem of checking whether the $H_\infty$ norm bound holds for the closed loop for any linear sampled-data controller.

It is, however, important to note that despite the equivalences shown here the characterizations in Theorems 5.1 and 3.2 are complementary and cannot be interchanged in general. The lifting technique is theoretically appealing and provides an equivalent discrete characterization of sampled-data control systems without explicitly referring to a Riccati equation based characterization. The two-Riccati equation solution of Theorem 5.1 can again be modified to construct $H_\infty$-optimal controllers for dual-rate systems including asynchronous dual-rate systems to which the lifting technique is not applicable cf. Säfors and Toivonen (1996).

ACKNOWLEDGEMENT

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REFERENCES


Fig. 1. The solutions $S_{11}(t)$ and $Q(t)$ in Theorem 5.1 and $S_{11}(t)Q(t)$ for three levels of $\gamma$ (left: $\gamma = 0.680$; middle: $\gamma = \gamma_{inf} = 0.670$; right: $\gamma = 0.660$).
<table>
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<th>$\gamma = \gamma_{in.f} = 0.670$</th>
<th>$\gamma = 0.660$</th>
</tr>
</thead>
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<td>$\bar{A}$</td>
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<td>1.100</td>
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<td>0.901</td>
<td>0.911</td>
<td>0.922</td>
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<td>2.370</td>
<td>2.431</td>
<td>2.494</td>
</tr>
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<td>$\bar{C}_1$</td>
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<td>[0.911 0]</td>
<td>[0.922 0]</td>
</tr>
<tr>
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<td>[1.055 1.182]</td>
<td>[1.075 1.183]</td>
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<td>$\bar{X}$</td>
<td>0.469</td>
<td>0.473</td>
<td>0.478</td>
</tr>
<tr>
<td>$\bar{Y}$</td>
<td>0.919</td>
<td>0.948</td>
<td>0.978</td>
</tr>
</tbody>
</table>

**Table 1.** Parameters of the equivalent discrete system representation (16) and the solutions to the discrete Riccati equations (61) and (82) for three values of $\gamma$. 