Real-Field Formulation of Norm-Related Problems in the Frequency Domain

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Abstract – In this paper the isomorphism existing between the fields of real and complex numbers is explored in the context of some typical norm-related problems defined in the frequency domain. It is shown that certain properties of complex-valued linear equations are preserved, and some interpretations are clarified, by a transformation such that only real-valued matrices and vectors are involved. Useful results concerning singular values and typical measures of matrix size, such as norms and the determinant, are obtained. An application to the modeling of uncertainty based on identification data is included. By use of the results derived in this paper, the optimization task in the original problem formulation is transformed to a convex optimization problem based on linear matrix inequalities (LMIs).

I. Introduction

Many modeling, analysis and design techniques for multi-variable systems involve vector and matrix norms. When the problem is defined in the time domain, based on a state-space description of the system, the signals and the matrices are real. However, when the problem is defined in the frequency domain, using a transfer-function description, the signals and the transfer functions are complex. Since the frequency domain theory is well developed, this does not, in principle, make the formulation of the problem more difficult. Furthermore, calculations with complex numbers can easily be done using readily available software.

A good understanding of the problem is, of course, important for its formulation and solution. It is often quite helpful if certain aspects of the problem can be visualized, if not graphically, at least mentally.

Consider the equation \( y = Ax \), where \( x \) is a real-valued vector and \( A \) is a real-valued matrix, and assume that \( x \) is norm bounded, e.g., \( \| x \| \leq 1 \), where \( \| \cdot \| \) denotes the Euclidean vector norm. It is well known that the vector \( y \) covers a dim(\( y \))-dimensional ellipsoidal region when \( x \) varies over its range of admissible values. The size (volume) of this ellipsoid is proportional to \( \det(A) \) [1]. However, if \( A \) and \( x \) are complex-valued, it may appear more difficult to visualize what kind of region the complex-valued vector \( y \) covers, and what size it is. We shall show that this and other norm-related problems can be formulated and studied in the field of real numbers.

This also means that calculations can be done and various optimization problems can be formulated and solved using only real-valued variables and parameters. Norm restrictions, for example, can often be stated as linear matrix inequalities (LMIs) with the proposed technique. The formulation is also useful when it is desired to optimize an objective function with respect to the parameters of a transfer function over a range of frequencies. An application of the formulation of LMIs and the calculation of optimal filters in an uncertainty model is presented.

The isomorphism existing between the fields of real and complex numbers is, of course, a known property [2], but it is rarely mentioned in textbooks and other publications on linear algebra. Except for a reformulation of complex-valued LMIs [3], we have not seen applications of the concept in the literature.

II. Main Results

In this section we shall study some norm-related expressions typical for problems defined in the frequency domain by utilizing the isomorphism between the fields of real and complex numbers. We shall start with the relationship \( y = Ax \), after which we consider singular values and some measures related with these.

A. The Linear Equation \( y = Ax \)

Consider the linear equation

\[
y = Ax
\]

where \( x \in \mathbb{C}^m \), \( y \in \mathbb{C}^n \), \( A \in \mathbb{C}^{n \times m} \). The complex-valued entities can be expressed as

\[
x = x_R + x_I j, \quad y = y_R + y_I j, \quad A = A_R + A_I j
\]

where \( j = \sqrt{-1} \) is the imaginary unit. Combination of (1) and (2) gives

\[
y_R + y_I j = A_R x_R - A_I x_I + (A_R x_I + A_I x_R) j
\]
Hence, the real and imaginary parts of \( y \) are given by
\[
\begin{bmatrix}
y_R \\
y_I
\end{bmatrix} =
\begin{bmatrix}
A_R - A_I \\
A_I & A_R
\end{bmatrix}
\begin{bmatrix}
x_R \\
x_I
\end{bmatrix}
\]  
(4)

With the definitions
\[
x_{RI} \equiv \begin{bmatrix} x_R \\ x_I \end{bmatrix}, \ y_{RI} \equiv \begin{bmatrix} y_R \\ y_I \end{bmatrix}, \ R \equiv \begin{bmatrix} A_R - A_I \\ A_I & A_R \end{bmatrix}
\]  
(5)

we thus have
\[
y_{RI} = Rx_{RI}
\]  
(6)

where \( x_{RI} \in \mathbb{R}^{2m}, \ y_{RI} \in \mathbb{R}^{2n} \) and \( R \in \mathbb{R}^{2nx2m} \).

Assume that \( \|y_{RI}\| \leq 1 \), where \( \| \cdot \| \) denotes the Euclidean vector norm. Then \( y_{RI} \) covers a 2\( n \)-dimensional ellipsoid when \( x_{RI} \) varies over its admissible range of values. Since
\[
\|x\| = x^H x = x_R^H x_R + x_I^H x_I = \|y_{RI}\|, \quad \|y\| = \|y_{RI}\|
\]  
(7)

where \( ^T \) denotes transpose and \( ^* \) denotes complex conjugate transpose, \( y \) covers a region of the same size as \( y_{RI} \) when \( x \), \( \|y\| = \|y_{RI}\| \), varies over its admissible range of values in (1).

The volume of this region is proportional to \( \det(R) \) \[1\].

### B. Singular Values

The complex-valued matrix \( A \) has the singular value decomposition (SVD)
\[
A = U \Sigma V^*
\]  
(8)

where \( U \in \mathbb{C}^{n \times n} \) and \( V \in \mathbb{C}^{m \times m} \) are unitary matrices, thus satisfying
\[
U^* U = I, \quad V^* V = I
\]  
(9)

and \( \Sigma \in \mathbb{R}^{n \times m} \) is a diagonal matrix of singular values such that
\[
\Sigma = \text{diag}(\sigma_1, \ldots, \sigma_p), \quad \sigma_j \geq 0, \quad p = \min(m,n)
\]  
(10)

As before, we can write
\[
U = U_R + U_I j, \quad V = V_R + V_I j
\]  
(11)

According to (9) we then have
\[
U_R^T U_R + U_I^T U_I + (U_R^T U_I - U_I^T U_R) j = I
\]  
(12)

where the imaginary parts must be equal to zero. Likewise, (8) gives
\[
A_R + A_I j = U_R \Sigma V_R^T + U_I \Sigma V_I^T + (U_R \Sigma V_R^T - U_I \Sigma V_I^T) j
\]  
(13)

where the real and imaginary parts have to match. Based on this, we obtain
\[
R = \begin{bmatrix}
A_R - A_I \\
A_I & A_R
\end{bmatrix} = \begin{bmatrix}
U_R - U_I \\
U_I & U_R
\end{bmatrix} \Sigma \begin{bmatrix}
\Sigma & 0 \\
0 & \Sigma
\end{bmatrix} \begin{bmatrix}
V_R^T \\
V_I^T
\end{bmatrix}
\]  
(14)

It can be verified that the first and last matrix on the right hand side of (14) are orthogonal matrices by means of the relationships implicit in (12). Since the middle matrix is a real-valued diagonal matrix (after rearrangement of rows or columns if \( \Sigma \) is non-square) with all elements non-negative, (14) is a SVD of \( R \). This means that the singular values of \( A \) are duplicated twice in \( R \), and we have
\[
\sigma_{2i-1}(R) = \sigma_{2i}(R) = \sigma_i(A), \quad i = 1, \ldots, p
\]  
(15)

Obviously there is a distinct relationship between \( R \) and \( A \) in terms of measures and expressions involving singular values. Quite often this relationship is an identity, as illustrated in the next section.

### C. Measures of Matrix Size

A fundamental matrix measure used, e.g., in robust control is the maximum singular value \( \bar{\sigma} \). The minimum singular value, \( \underline{\sigma} \), is an important measure used, e.g., in controllability considerations. Of course, the condition number, \( \kappa \), is also a measure often encountered. From (15) it follows that
\[
\bar{\sigma}(R) = \bar{\sigma}(A), \quad \underline{\sigma}(R) = \underline{\sigma}(A), \quad \kappa(R) = \kappa(A)
\]  
(16)

where \( R \) is defined as in (5).

The induced Euclidean norm of a matrix is identical with the maximum singular value, i.e.,
\[
\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \bar{\sigma}(A)
\]  
(17)

Thus,
\[
\|R\| = \|A\|
\]  
(18)

If \( A \) is invertible, we also have
\[
\|R^{-1}\| = \|A^{-1}\|
\]  
(19)

The Frobenius norm \( \|\cdot\|_F \) is another often used matrix norm.

It can be shown that [4]
\[
\|A\|_F = \sqrt{\text{tr}(A^* A)} = \sqrt{\sum_i \sigma_i^2(A)}
\]  
(20)

From this and (15) it follows that
\[
\|R\|_F = \sqrt{\bar{\sigma}(A)}
\]  
(21)

As discussed previously in this paper, the determinant of a real-valued square matrix is proportional to the volume of the ellipsoid produced when the matrix operates on a norm-bounded vector. However, the determinant of a complex matrix \( A \) is generally complex-valued. This would not be difficult to handle, but in relevant problems the matrix usually has the quadratic form \( A^* A \), which has a real-valued positive determinant.

Use of the SVD in (8) for this matrix gives
\[
\det(A^* A) = \det(V \Sigma^T U^* U \Sigma V^*) = \det(\Sigma^T \Sigma) = \prod_{i=1}^n \sigma_i^2
\]  
(22)

where \( U \) and \( V \) are eliminated by use of (9) and the property
\[
\det(UY) = \det(U) \det(Y) = \det(YU)
\]  
(23)

for square matrices \( X \) and \( Y \). A similar derivation for the real matrix \( R \) in (14) gives
\[
\det(R^T R) = \det(\Sigma^T \Sigma) \det(\Sigma^T \Sigma) = \prod_{i=1}^n \sigma_i^4
\]  
(24)

From (23) and (24) it then follows that
\[
\det(R^T R) = \det(A^* A)^2
\]  
(25)

It is not necessary that the matrices are square, but in order for the expression to be meaningful, \( A \) (and thus \( \Sigma \) and \( R \)) should not have more columns than rows, i.e., \( n \geq m \).
If the matrices are square, (25) can be simplified to
\[ \det(R) = \det(A^*A) \] (26)
because \( \det(R) \) for \( R \) defined in (14) is always non-negative. This can be shown by means of Schur’s formula for the determinant of a block-partitioned matrix. For a matrix having the same structure as the orthogonal matrices in (14), this formula gives
\[ \det \begin{bmatrix} X & -Y \\ Y & X \end{bmatrix} = \det(X)\det(X + YX^{-1}Y) \] (27)
Applied to the orthogonal matrix in (14) with \( X = U_R \) and \( Y = U_1 \) this gives
\[ \det \begin{bmatrix} U_R^T & -U_1^T \\ U_R & U_1 \end{bmatrix} = \det(U_R)\det(U_R + U_1U_R^{-1}U_1) \]
\[ = \det(U_R^TU_R + U_1^TU_1U_R^{-1}U_1) = \det(U_R^TU_R + U_1^TU_1U_R^{-1}U_1) \]
\[ = \det(U_R^TU_R + U_1^TU_1) = \det(I) = 1 \] (28)
where we have made use of (23) and (12) and the fact that \( \det(X) = \det(X^*) \). Since the same result is obtained for the other orthogonal matrix (by including \( \det(U_R^T) \) from the right in the second step), it follows that
\[ \det(R) = \det(\Sigma)^2 = \sum_{i=1}^n \sigma_i^2 \] (29)

III. Application to Uncertainty Modeling

A technique for constructing norm-bounded uncertainty models based on system identification was developed and applied to a two-product pilot-scale distillation column in [5,6,7]. The problem was defined and solved in the frequency domain. We shall here show that the problem can be reformulated and solved as a convex optimization problem using relationship derived in the previous section.

A. Problem Formulation

A norm-bounded uncertainty model is derived by matching the input-output behavior of the model to known data. Ideally, these data should include all relevant dynamics, but no noise [7,8]. In order to comply with this request, a number of models \( G_k \), \( k = 1, \ldots, N \), is determined via identification [9], and the outputs \( y_k \) from these models, together with the corresponding inputs \( u_k \), are used as noise-free input-output data. In addition, a nominal model \( G_0 \) is determined for the uncertainty description.

The fact that different inputs give different models implies that each model \( G_k \) is valid only for the input \( u_k \) used to generate identification data for \( G_k \). This strongly suggests data matching as the leading principle for the uncertainty modeling, not model matching, which would give a more conservative model not justified by data [6].

Various types of uncertainty models were considered in [5,6,7,10]. In the application to distillation, it was clearly found that the data could be described with the least amount of conservatism by an output-multiplicative uncertainty model. A simple model of the form
\[ \tilde{G}(s) = (I + W(s)\Delta(s))G_0(s) \]
was sufficient. Here \( \Delta \) is an unstructured norm-bounded uncertainty, and \( W \) is a matrix of filters acting as uncertainty weights, which need to be determined.

The uncertainty manifests itself as a deviation from the nominal model output. Expressed in the frequency domain, (30) can for an arbitrary input \( u(j\omega) \) produce any deviation \( e(j\omega) \) satisfying
\[ e = Ww, \quad w = \Delta G_0 u, \quad \|\Delta\|_\infty \leq 1 \] (31)
for some admissible \( \Delta \). Since \( \Delta \) is capable of generating any \( w \), \( \|w\| \leq \|G_0u\| \), the possible deviations cover an ellipsoidal region. Because the volume of the ellipsoid is proportional to \( \det(W^*W) \), this, or equivalently \( \|\det(W)\|^2 \) if \( W \) is square, is a suitable criterion to minimize subject to appropriate data-matching constraints [5,7].

For experiment \( k \) with the input \( u_k \) and the noise-free output \( y_k \), the deviation \( e_k \) is given by
\[ e_k = y_k - G_0u_k \] (32)
The uncertainty model (30) can reproduce this deviation if, in analogy with (31),
\[ e_k = Ww_k, \quad w_k = \Delta_k G_0 u_k, \quad \|\Delta_k\|_\infty \leq 1 \] (33)
for some admissible \( \Delta = \Delta_k \). In the case of a square weight matrix, the norm-bounded \( \Delta_k \) gives the data-matching constraints [5,6,7]
\[ \|W^{-1}e_k\| \leq \|G_0u_k\|, \quad \forall k, \forall \omega \in \Omega \] (34)
where \( \Omega \) is a set of frequency points covering the range of relevant frequencies. This restriction is sufficient and necessary for data matching.

The derived data-matching constraints apply for an unstructured \( \Delta \) block. However, nothing prevents the inclusion of structural (or other) restrictions on \( W \). If \( WW \) denotes the set of admissible weights, the problem can be formulated as
\[ \min_{W} J(W,\omega) = \det(W^*W), \quad \forall \omega \in \Omega \] (35)
subject to (34).

As indicated, the minimization has to be performed at every relevant frequency. This gives \( W(j\omega), \forall \omega \in \Omega \), to which transfer function filters can be fitted [5,10].

However, such a 2-step procedure is obviously suboptimal. An alternative is to solve the problem by directly minimizing a (weighted) sum of \( J(W,\omega) \) over \( \Omega \) with respect to the filter parameters [10]. Another obvious modification is to replace \( W^{-1} \) in (34) by the matrix \( M = W^{-1} \) and to maximize \( \det(M^*M) \). In this way, the inversion of the weight matrix is avoided during the optimization.

B. Solution by Convex Optimization and LMIs

It is advantageous if an optimization problem can be formulated as a convex optimization since a found optimum is
then guaranteed to be global. Because a linear matrix inequality (LMI) is a convex constraint [3,11,12], optimization constraints are often expressed as LMIs, if possible.

We shall here show that the data-matching constraint (34) can be expressed as a set of LMIs using only real-valued vectors and matrices. As suggested above, we introduce

\[
M = W^{-1}
\]

(36)

In accordance with (2), we can write the complex-valued matrix \( M \) and data vector \( e_k \) as

\[
M = M_R + M_{11} \quad , \quad e_k = e_{kR} + e_{k1j} \quad \text{for each } \omega \in \Omega
\]

(37)

According to the main results in Section II.A we then have

\[
\|W^{-1}e_k\| = \|Me_k\| = \|RM_{e_k,R1}\|
\]

where

\[
RM = \begin{bmatrix} M_R & -M_{11} \\ M_{11} & M_R \end{bmatrix}, \quad e_{k,R} \equiv \begin{bmatrix} e_{kR} \\ e_{k1j} \end{bmatrix}
\]

(39)

Equation (34) can now be written

\[
\|Gh_k\|^{-1}e_{k,R}^T R_{M_{e_k,R1}} \geq 0, \quad \forall k, \forall \omega \in \Omega
\]

(40)

By means of the well-known Schur-complement lemma [11,12] this can be expressed as a set of LMIs

\[
\begin{bmatrix} I & R_{M_{e_k,R1}} \\ e_{k,R}^T R_{M} \end{bmatrix} \geq 0, \quad \forall k, \forall \omega \in \Omega
\]

(41)

where the block matrix is positive semidefinite. It is unnecessary to transform \( \|Gh_k\| \) since it is already a real-valued constant (but different for \( \forall k \) and \( \forall \omega \in \Omega \)) independent of \( R_M \).

From Section II.C it follows that the determinant in (35) can be written as

\[
\text{det}(W^*W) = \text{det}(M^*M)^{-1} = \text{det}(R_M)^{-1}
\]

(42)

which allows minimization with respect to \( R_M \). However, it is often advantageous to minimize the logarithm of a function instead of the function itself since the logarithm can introduce convexity [11,12]. The minimization problem is therefore rewritten as

\[
\min_{R_M} \log \text{det}(R_M)^{-1}, \quad \forall \omega \in \Omega
\]

subject to (41) and \( W \in \mathbb{W} \).

Note that \( R_M \) has a certain structure according to (39) and that \( M_R \) and \( M_{11} \) may be structurally constrained by \( W \in \mathbb{W} \).

Because of (22), the minimization of \( \log |W^*W| \) in the original formulation favors a solution with small singular values at the expense of some larger singular values \( \sigma_i(W) \). Therefore, it may be desirable to restrict the largest singular value, e.g., to enable a robustly stable control design.

Assume that we introduce the restriction

\[
\bar{\sigma}(W) = \|W\| \leq \gamma, \quad \forall \omega \in \Omega
\]

(44)

which may vary with the frequency. At low frequencies, some \( \gamma < 1 \) would be a typical choice. According to (18), the norm restriction can also be expressed as

\[
\|R_M\| \leq \gamma, \quad \forall \omega \in \Omega
\]

(45)

where

\[
R_M = \begin{bmatrix} W_R & -W_{1j} \\ W_{1j} & W_R \end{bmatrix}, \quad W = W_R + W_{1j}
\]

(46)

Equation (45) is equivalent with the matrix inequality

\[
R_M^T R_M \leq \gamma^2 I
\]

(47)

Since \( R_M R_M = I \), this can also be expressed as

\[
\gamma^2 - R_M^{-1} \geq 0
\]

(49)

Because \( R_M > 0 \), this is, according to the Schur-complement lemma, equivalent with the LMIs

\[
\begin{bmatrix} R_M & \gamma^{-1} I \\ \gamma^{-1} I & R_M^T \end{bmatrix} \geq 0, \quad \forall \omega \in \Omega
\]

(50)

This norm restriction may be included in the minimization problem (43).

C. Fitting Filter Transfer Functions

When \( R_M \) has been determined, \( R_M \), and thus \( W_R \) and \( W_{1j} \), can be calculated for \( \forall \omega \in \Omega \). The next stage is to fit stable transfer function filters to \( W_R \) and \( W_{1j} \).

Assume, for simplicity, that first-order filters are used, i.e., element \((i,j)\) of \( W \) has the form

\[
W_{ij}(s) = \frac{b_{ij}s + b_{ij0}}{a_{ij}s + 1}, \quad a_{ij} > 0
\]

(51)

The real and imaginary parts of the frequency response are then

\[
W_{R,ij} = \frac{b_{ij1} + a_{ij}^2 b_{ij2}}{1 + a_{ij}^2 \omega^2}, \quad W_{1j,ij} = \frac{b_{ij0} - a_{ij} b_{ij3}}{1 + a_{ij}^2 \omega^2}
\]

(52)

The right-hand side of these expressions can now be fitted to the calculated \( W_{R,ij}(\omega) \) and \( W_{1j,ij}(\omega) \). Since this could result in filters that violate the data-matching conditions, the fitting should be made subject to (34) with the filter expressions substituted into \( W \).

If higher-order filters are needed, it may be advantageous to define them in state-space form since stability issues can then easily be handled. If the complete set of filters is defined by the state-space description \((A,B,C,D)\), the frequency response of the corresponding transfer function matrix is given by

\[
W(j\omega) = C(j\omega I - A)^{-1} B + D
\]

(53)

Since

\[
(j\omega I - A)^{-1} = -(A^2 + \omega^2 I)^{-1}(j\omega I + A)
\]

(54)

the real and imaginary parts are

\[
W_R = -(C^2 + \omega^2 I)^{-1} AB D, \quad W_{1j} = -(C^2 + \omega^2 I)^{-1} BD \omega
\]

(55)

which can be fitted to the calculated values.

As mentioned previously, it tends to be suboptimal to fit the filter transfer functions separately. The fitting of filter transfer functions can be included in the optimization as formulated in the previous section simply by the calculation

\[
R_M(\omega) = R_M(\omega), \quad \forall \omega \in \Omega
\]

(56)

A modified objective function defined, e.g., as a (weighted) sum of the objective function in (43) over the whole frequency
range of interest, can then be directly minimized with respect to the filter parameters. A drawback of this approach is that the optimization problem, including all constraints, is no longer convex.

For this reason, it may be preferable to keep the calculation of optimal weights and the fitting of filter transfer functions separated. This preserves flexibility in the choice of filter orders and the suboptimality issue is probably of minor importance in practice. This is, after all, an attempt to model uncertainty.

**D. Case Study**

The distillation column of this study is a pilot-scale two-product column modeled as a $2 \times 2$ system [9]. It has been identified by applying a series of step changes to its high- and low-gain input directions, which can be estimated with good accuracy from certain flow gains. From these experiments, a nominal model as well as six additional models were determined as transfer matrix models composed of second-order transfer functions with dead time.

Before starting the task of optimizing uncertainty filters, it is useful to check the minimum norm of the weight matrix allowed by the available data. From (33) one can derive

$$\min \| P \| \geq \max_k \| v_k \|, \quad v_k = \frac{e_k}{G_k u_k}, \quad \forall \omega \in \Omega \quad (57)$$

according to which the minimum norm can readily be determined for every frequency. However, it is also instructive to consider the location of the end point of every vector $v_k$, $k = 1, \ldots, N$, in the appropriate space.

Figure 1 shows these points in a 2-dimensional space for the available steady-state data of the distillation column. The point farthest away from the origin gives $\max_k \| v_k \| = 0.46$, which means that a robustly stable control design is possible at least at steady state. The figure shows that the experimental points can be enclosed by an ellipse with significantly smaller area than the area of the smallest circle. This means that the minimization of $\det(W^*W)$ gives a less conservative uncertainty model that the minimization of $\|W\|$.

Figure 1 also reveals that the limiting data points for the ellipse are very close to the coordinate axes. This means that a diagonal weight $W$ is optimal, at least at steady state. In practice, this property tends to carry over to other frequencies.

Figure 2 shows the frequency response of a diagonal $W$ when the minimization is done frequency by frequency as well as when first-order transfer functions are directly fitted [10]. All frequencies are weighted equally in the latter case.

Robust $\mu$-optimal controllers have been designed using uncertainty models determined by this procedure. They have also been successfully tested on the distillation column [5].

**IV. Conclusions**

Multivariable problems defined in the frequency domain usually involve various measures of matrix size. Typical measures are singular values, norms and the determinant. Using these measures as they apply to complex-valued matrices may not be the best way of forming and solving a problem.

We have shown that the above mentioned measures can be expressed by equivalent expressions using only real-valued matrices and vectors. As illustrated in an application to uncertainty modeling, this may make it possible to express various optimization constraints as sets of LMIs, which enable the use of powerful convex optimization techniques.

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