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A Kreĭn space coordinate free version of the de Branges complementary space

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Abstract

Let $\mathcal{Z}$ be a maximal nonnegative subspace of a Kreĭn space $\mathcal{X}$, and let $\mathcal{X}/\mathcal{Z}$ be the quotient of $\mathcal{X}$ modulo $\mathcal{Z}$. Define

$$\mathcal{H}(\mathcal{Z}) = \bigl\{h \in \mathcal{X}/\mathcal{Z} \bigm| \sup\bigl\{-[x,x]_{\mathcal{X}} \bigm| x \in h\bigr\} < \infty\bigr\}.$$

It is proved that $\sup\{-[x,x]_{\mathcal{X}} \mid x \in h\} \geqslant 0$ for $h \in \mathcal{H}(\mathcal{Z})$, and that $\mathcal{H}(\mathcal{Z})$ is a Hilbert space with norm

$$\|h\|_{\mathcal{H}(\mathcal{Z})} = \bigl(\sup\bigl\{-[x,x]_{\mathcal{X}} \bigm| x \in h\bigr\}\bigr)^{1/2},$$

which is continuously contained in $\mathcal{X}/\mathcal{Z}$, and the properties of this space are studied. Given any fundamental decomposition $\mathcal{X} = -\mathcal{Y}\,[\dot{+}]\,\mathcal{U}$ of $\mathcal{X}$, the subspace $\mathcal{Z}$ can be written as the graph of a contraction $A : \mathcal{U} \to \mathcal{Y}$. There is a natural isomorphism between $\mathcal{X}/\mathcal{Z}$ and $\mathcal{Y}$, and under this isomorphism the space $\mathcal{H}(\mathcal{Z})$ is mapped isometrically onto the complementary space $\mathcal{H}(A)$ of the range space of A studied by de Branges and Rovnyak. The space $\mathcal{H}(\mathcal{Z})$ is used as state space in a construction of a canonical passive state/signal shift realization of a linear observable and backward conservative discrete time invariant state/signal system with a given passive future behavior, equal to a given maximal nonnegative right-shift invariant subspace $\mathcal{Z}$ of the Kreĭn space $\mathcal{X} = k_+^2(\mathcal{W})$ of all ℓ^2-sequences on $\mathbb{Z}^+$ with values in the Kreĭn

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signal space $\mathcal{W}$. This state/signal realization is related to the de Branges–Rovnyak model of a linear observable and backward conservative scattering input/state/output system whose scattering matrix is a given Schur class function in the same way as $\mathcal{H}(\mathcal{Z})$ is related to $\mathcal{H}(A)$.

Keywords: Kreĭn space; Hilbert space; Quotient space; Maximal nonnegative subspace; Contractive operator; Passive system; State space; Signal space; Behavior

1. Introduction

The main result of this article concerns the geometry of Kreĭn spaces, and it describes the relationship between the orthogonal companion $\mathcal{Z}^{[\perp]}$ of a maximal nonnegative subspace $\mathcal{Z}$ of a Kreĭn space $\mathcal{X}$ and a certain Hilbert space $\mathcal{H}(\mathcal{Z})$ that is continuously (but not necessarily densely) contained in the quotient space $\mathcal{X}/\mathcal{Z}$. This result was discovered as a byproduct of our continuing research on passive linear discrete time invariant s/s (state/signal) systems, which so far has resulted in the publications [2–5]. The subspace $\mathcal{H}(\mathcal{Z})$ can be interpreted as a coordinate free version of the de Branges complement of the range space of a contractive operator A between two Hilbert spaces $\mathcal{U}$ and $\mathcal{Y}$, as will be explained in more detail in Section 3.

Let $\mathcal{Z}$ be a maximal nonnegative subspace of a Kreĭn space $\mathcal{X}$, and let $\mathcal{X}/\mathcal{Z}$ be the quotient of $\mathcal{X}$ modulo $\mathcal{Z}$. Define

$$\mathcal{H}(\mathcal{Z}) = \bigl\{h \in \mathcal{X}/\mathcal{Z} \bigm| \sup\bigl\{-[x,x]_{\mathcal{X}} \bigm| x \in h\bigr\} < \infty\bigr\}. \tag{1.1}$$

As we shall prove in Theorem 2.3, $\sup\{-[x,x]_{\mathcal{X}} \mid x \in h\} \geqslant 0$ for $h \in \mathcal{H}(\mathcal{Z})$, and $\mathcal{H}(\mathcal{Z})$ is a Hilbert space with norm

$$\|h\|_{\mathcal{H}(\mathcal{Z})} = \bigl(\sup\bigl\{-[x,x]_{\mathcal{X}} \bigm| x \in h\bigr\}\bigr)^{1/2}, \quad h \in \mathcal{H}(\mathcal{Z}), \tag{1.2}$$

which is continuously contained in $\mathcal{X}/\mathcal{Z}$. In Lemma 2.4 we prove that the following "Schwarz type" inequality

$$\bigl|[x,z]_{\mathcal{X}}\bigr|^2 \leqslant [z,z]_{\mathcal{X}}\bigl([x,x]_{\mathcal{X}} + \|h\|^2_{\mathcal{H}(\mathcal{Z})}\bigr), \quad h \in \mathcal{H}(\mathcal{Z}),\ x \in h,\ z \in \mathcal{Z}, \tag{1.3}$$

holds, and that it collapses to an equality if and only if either $[z,z]_{\mathcal{X}} = 0$ or $[z,z]_{\mathcal{X}} \neq 0$ and $x = ([x,z]_{\mathcal{X}}/[z,z]_{\mathcal{X}})z + z^\dagger$ for some $z^\dagger \in \mathcal{Z}^{[\perp]}$, where $\mathcal{Z}^{[\perp]}$ is the orthogonal companion of $\mathcal{Z}$ in $\mathcal{X}$, i.e.,

$$\mathcal{Z}^{[\perp]} = \bigl\{x \in \mathcal{X} \bigm| [x,z]_{\mathcal{X}} = 0 \text{ for all } z \in \mathcal{Z}\bigr\}. \tag{1.4}$$

In Lemma 2.4 and Theorem 2.5 we prove a number of additional results about the space $\mathcal{H}(\mathcal{Z})$, such as the following. Define the subspace $\mathcal{H}^0(\mathcal{Z})$ of $\mathcal{X}/\mathcal{Z}$ by

$$\mathcal{H}^0(\mathcal{Z}) := \bigl\{z^\dagger + \mathcal{Z} \bigm| z^\dagger \in \mathcal{Z}^{[\perp]}\bigr\}, \tag{1.5}$$

where $z^\dagger + \mathcal{Z}$ stands for the equivalence class in $\mathcal{X}/\mathcal{Z}$ which contains $z^\dagger$. Then $\mathcal{H}^0(\mathcal{Z}) \subset \mathcal{H}(\mathcal{Z})$, and the supremum in (1.1) is achieved if and only if $h \in \mathcal{H}^0(\mathcal{Z})$. The space $\mathcal{H}^0(\mathcal{Z})$ has a natural positive inner product induced by $\mathcal{X}$, namely

$$\left(z_1^\dagger + \mathcal{Z}, z_2^\dagger + \mathcal{Z}\right)_{\mathcal{H}^0(\mathcal{Z})} = -\left[z_1^\dagger, z_2^\dagger\right]_{\mathcal{X}}, \quad z_1^\dagger,\ z_1^\dagger \in \mathcal{Z}^{[\perp]}, \tag{1.6}$$

and this inner product coincides with the inner product inherited from $\mathcal{H}(\mathcal{Z})$. Moreover, $\mathcal{H}^0(\mathcal{Z})$ is dense in $\mathcal{H}(\mathcal{Z})$. This means that the Hilbert space $\mathcal{H}(\mathcal{Z})$ is the completion of $\mathcal{H}^0(\mathcal{Z})$. Furthermore, if we define $\mathcal{Z}_0 = \mathcal{Z} \cap \mathcal{Z}^{[\perp]}$, then $\mathcal{Z}^{[\perp]}/\mathcal{Z}_0$ can be identified in a natural way with $\mathcal{H}^0(\mathcal{Z})$ with the positive inner product inherited from $-\mathcal{X}$, so that we may also regard $\mathcal{H}(\mathcal{Z})$ as the completion of $\mathcal{Z}^{[\perp]}/\mathcal{Z}_0$ with respect to this inner product.

Since $\mathcal{Z}^{[\perp]}$ can be interpreted as a maximal nonnegative subspace of the anti-space $-\mathcal{X}$ of the Kreĭn space $\mathcal{X}$, it follows that there is a dual version of the space $\mathcal{H}(\mathcal{Z})$ that we denote by $\mathcal{H}(\mathcal{Z}^{[\perp]})$. Connections between the spaces $\mathcal{H}(\mathcal{Z})$ and $\mathcal{H}(\mathcal{Z}^{[\perp]})$ are studied in Theorem 2.12.

All our results on the Hilbert spaces $\mathcal{H}(\mathcal{Z})$ and $\mathcal{H}(\mathcal{Z}^{[\perp]})$, including those mentioned above, are formulated and proved in Section 2. As we show in Section 3, there is a close connection between the Hilbert space $\mathcal{H}(\mathcal{Z})$ and the de Branges complementary space $\mathcal{H}(A)$ induced by the contraction A that appear in the graph representation of $\mathcal{Z}$ with respect to some fundamental decomposition of the Kreĭn space $\mathcal{X}$. All the proofs that we give in Section 2 are "coordinate free" in the sense that they make no use of such a graph representation. They are also self-contained in the sense that they require no a priori knowledge whatsoever of the de Branges complementary space $\mathcal{H}(A)$. The connection between $\mathcal{H}(\mathcal{Z})$ and the space $\mathcal{H}(A)$ is described in detail in Section 3. In this section we have also included some alternative proofs, which are based on the above graph representation, of some of the results in Section 2. The main reason for including these proofs is that they illustrate the connection between the space $\mathcal{H}(\mathcal{Z})$ and the space $\mathcal{H}(A)$. Our coordinate free proofs in Section 2 are written in the spirit of the original proof by de Branges and Rovnyak of Theorem 7 in [10], given on pp. 24–26 of that book, whereas the alternative proofs in Section 3 are written in the spirit of more recent proofs of the same result.

More precisely, to each space $\mathcal{H}(\mathcal{Z})$ there corresponds not only one space $\mathcal{H}(A)$, but a whole family of spaces $\mathcal{H}(A)$. As is well known, if $\mathcal{X} = -\mathcal{Y}\,[\dot{+}]\,\mathcal{U}$ is a fundamental decomposition of the Kreĭn space $\mathcal{X}$, then each maximal nonnegative subspace $\mathcal{Z}$ of $\mathcal{X}$ is the graph of a linear contraction $A : \mathcal{U} \to \mathcal{Y}$, and conversely, the graph of every linear contraction $A : \mathcal{U} \to \mathcal{Y}$ between the Hilbert spaces $\mathcal{U}$ and $\mathcal{Y}$ is a maximal nonnegative subspace of the Kreĭn space $\mathcal{X} = -\mathcal{Y}[\dot{+}]\mathcal{U}$. However, the correspondence between $\mathcal{Z}$ and the contraction A is far from one-to-one, since A obviously does not depend only on $\mathcal{Z}$ but also on the choice of the fundamental decomposition $\mathcal{X} = -\mathcal{Y}\,[\dot{+}]\,\mathcal{U}$. It is easy to see that $\mathcal{X}$ is the direct sum of $\mathcal{Z}$ and $-\mathcal{Y}$, and this implies that there is a natural isomorphism $T : \mathcal{X}/\mathcal{Z} \to \mathcal{Y}$. It turns out that the restriction of T to $\mathcal{H}(\mathcal{Z})$ is a unitary map of $\mathcal{H}(\mathcal{Z})$ onto the de Branges complement $\mathcal{H}(A)$ of the range space $\mathcal{M}(A)$ of the operator A. As we mentioned above, this makes it is possible to prove some of the results in Section 2 by appealing to known results about $\mathcal{H}(A)$ and $\mathcal{M}(A)$ due to de Branges and Rovnyak. However, it is also possible to proceed in the opposite direction, and to prove results about the spaces $\mathcal{H}(A)$ by appealing to the results about the Hilbert space $\mathcal{H}(\mathcal{Z})$ given in Section 2.

As we mentioned at the beginning, the present article is an outgrowth of our research on passive linear discrete time invariant s/s (state/signal) systems. In Section 4 we describe how the space $\mathcal{H}(\mathcal{Z})$ is used in passive s/s systems theory. There we take $\mathcal{X} = k_+^2(\mathcal{W})$, the set of all ℓ^2-sequences on $\mathbb{Z}^+$ with values in the Kreĭn signal space $\mathcal{W}$, with the indefinite inner product in $k_+^2(\mathcal{W})$ induced by the inner product in $\mathcal{W}$. The subspace $\mathcal{Z}$ is a maximal nonnegative right-shift invariant subspace of $\mathcal{X}$, or in the state/system terminology, $\mathcal{Z}$ is a passive future behavior. The space $\mathcal{H}(\mathcal{Z})$ is used as the state space of the canonical shift realization of a linear observable and backward conservative discrete time invariant s/s system with the given passive future behavior $\mathcal{Z}$ that we construct in Section 4. This s/s realization is related to the de Branges–Rovnyak model of a linear observable and backward conservative scattering input/state/output system whose scattering matrix is a given Schur class function in the same way as $\mathcal{H}(\mathcal{Z})$ is related to $\mathcal{H}(A)$. The latter model has been studied in [9,10], and the more recently in [1], with a different terminology: there the given Schur class function is realized as the "characteristic function" of a "coisometric and closely outer connected colligation." The idea of using a quotient of two vector valued sequence spaces as the state space of a (not necessarily passive) finite-dimensional input/state/output realization of a given rational transfer function with finite-dimensional state, input, and output spaces goes back to Kalman; see, e.g., the last part of [11].

Notations and conventions. The space of bounded linear operators from one Kreĭn space $\mathcal{X}$ to another Kreĭn space $\mathcal{Y}$ is denoted by $\mathcal{B}(\mathcal{X};\mathcal{Y})$. The domain, range, and kernel of a linear operator A are denoted by $\mathcal{D}(A)$, $\mathcal{R}(A)$, and $\mathcal{N}(A)$, respectively. The restriction of A to some subspace $\mathcal{Z} \subset \mathcal{D}(A)$ is denoted by $A|_{\mathcal{Z}}$. The identity operator on $\mathcal{X}$ is denoted by $1_{\mathcal{X}}$. The projection onto a closed subspace $\mathcal{Y}$ of a space $\mathcal{X}$ along some complementary subspace $\mathcal{U}$ is denoted by $P_{\mathcal{Y}}^{\mathcal{U}}$, or by $P_{\mathcal{Y}}$ if $\mathcal{Y}$ and $\mathcal{U}$ are orthogonal with respect to a Hilbert or Kreĭn space inner product in $\mathcal{X}$.

We denote the ordered product of the two locally convex topological vector spaces $\mathcal{Y}$ and $\mathcal{U}$ by $\left[\begin{smallmatrix}\mathcal{Y}\\ \mathcal{U}\end{smallmatrix}\right]$, and sometimes write $\mathcal{Y} \dotplus \mathcal{U}$ for $\left[\begin{smallmatrix}\mathcal{Y}\\ \mathcal{U}\end{smallmatrix}\right]$ (interpreting $\mathcal{Y} \dotplus \mathcal{U}$ as an ordered sum), identifying vectors $\left[\begin{smallmatrix}y\\ 0\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}0\\ u\end{smallmatrix}\right]$ with y and u for $y \in \mathcal{Y}$ and $u \in \mathcal{U}$.

The inner product in a Hilbert space $\mathcal{X}$ is denoted by $(\cdot,\cdot)_{\mathcal{X}}$, and by $[\cdot,\cdot]_{\mathcal{X}}$ in the case of a Kreĭn space $\mathcal{X}$. The orthogonal sum of two Hilbert spaces $\mathcal{Y}$ and $\mathcal{U}$ is denoted by $\mathcal{Y} \oplus \mathcal{U}$, and the orthogonal sum of two Kreĭn spaces $\mathcal{Y}$ and $\mathcal{U}$ is denoted by $\mathcal{Y} [\dotplus] \mathcal{U}$. We identify $\mathcal{U}$ and $\mathcal{Y}$ with the appropriate subspaces of these sums.

A Hilbert space $\mathcal{Y}$ is *continuously contained* in a topological vector space $\mathcal{X}$ if $\mathcal{Y}$ is a subspace of $\mathcal{X}$, and the inclusion map of $\mathcal{Y} \hookrightarrow \mathcal{X}$ is continuous.

If $\mathcal{X}$ is a Kreĭn space with inner product $[\cdot,\cdot]_{\mathcal{X}}$, then the Kreĭn space $-\mathcal{X}$ is the same vector space with the inner product $-[\cdot,\cdot]_{\mathcal{X}}$. We call $-\mathcal{X}$ the anti-space of $\mathcal{X}$.

2. The Hilbert space $\mathcal{H}(\mathcal{Z})$

2.1. Preliminaries on Kreĭn spaces

We assume the reader to be familiar with basic notions and results in Kreĭn space theory. A short introduction to this theory can be found in, e.g. [1], and more detailed treatments in [6] or [7]. Nevertheless, we include here a short summary of Kreĭn space theory in order to establish the notations.

Let $\mathcal{X}$ be a Kreĭn space. This means that $\mathcal{X}$ is a vector space with an indefinite inner product $[\cdot,\cdot]_{\mathcal{X}}$, and that $\mathcal{X}$ has a fundamental decomposition $\mathcal{X} = -\mathcal{Y} [\dotplus] \mathcal{U}$, where $\mathcal{Y}$ and $\mathcal{U}$ are Hilbert spaces. The topology of $\mathcal{X}$ is the one induced by the Hilbert space norm $\|x\|^2 = -[y,y]_{\mathcal{X}} + [u,u]_{\mathcal{X}}$, where $x = u + y$ with $y \in \mathcal{Y}$ and $u \in \mathcal{U}$ (different fundamental decompositions give different but equivalent norms). Such a norm is called an *admissible norm*.

A subspace $\mathcal{Z}$ of $\mathcal{X}$ is nonnegative if $[x,x]_{\mathcal{X}} \geqslant 0$ for every $x \in \mathcal{Z}$. It is maximal nonnegative if it is not properly contained in any other nonnegative subspace of $\mathcal{X}$. Nonpositive and maximal nonpositive subspaces are defined analogously. The orthogonal companion $\mathcal{Z}^{[\perp]}$ of a subspace $\mathcal{Z}$ is defined by (1.4). It is well known that $(\mathcal{Z}^{[\perp]})^{[\perp]} = \mathcal{Z}$ if and only if $\mathcal{Z}$ is closed. The subspace $\mathcal{Z}$ is neutral if $[z,z]_{\mathcal{X}} = 0$ for all $z \in \mathcal{Z}$, or equivalently, if $\mathcal{Z} \subset \mathcal{Z}^{[\perp]}$. It is called Lagrangian if $\mathcal{Z}^{[\perp]} = \mathcal{Z}$.

A nonnegative subspace $\mathcal{Z}$ is uniformly positive if it is a Hilbert space with respect to the inner product inherited from $\mathcal{X}$, and it is uniformly negative if it is a Hilbert space with respect to the inner product norm inherited from $-\mathcal{X}$. If $\mathcal{Z}$ is both uniformly positive and maximal nonnegative, and only in this case, $\mathcal{Z}$ induces a fundamental decomposition

$$\mathcal{X} = \mathcal{Z}^{[\perp]} \, [\dot{+}] \, \mathcal{Z}, \tag{2.1}$$

and $\mathcal{Z}^{[\perp]}$ uniformly negative and maximal nonpositive. In general the intersection

$$\mathcal{Z}_0 := \mathcal{Z} \cap \mathcal{Z}^{[\perp]} \tag{2.2}$$

can be different from $\{0\}$. It is the maximal neutral subspace contained in $\mathcal{Z}$, and at the same time the maximal neutral subspace contained in $\mathcal{Z}^{[\perp]}$.

In the proof of Theorem 2.3 below we shall need the following lemma.

Lemma 2.1. *If $\mathcal{Z}$ is a maximal nonnegative subspace of a Kreĭn space $\mathcal{X}$, then*

$$\mathcal{X} = \mathcal{Z} \dot{+} \mathcal{Y} \tag{2.3}$$

for every uniformly negative and maximal nonpositive subspace $\mathcal{Y}$ of $\mathcal{X}$.

Proof. If $x \in \mathcal{Z} \cap \mathcal{Y}$, then, on one hand, $[x, x]_{\mathcal{X}} \geqslant 0$ since $x \in \mathcal{Z}$, and on the other hand $[x, x]_{\mathcal{X}} \leqslant 0$ since $x \in \mathcal{Y}$. The uniform negativity of $\mathcal{Y}$ implies that $x = 0$. Thus, $\mathcal{Z} \cap \mathcal{Y} = \{0\}$.

We next show that $\mathcal{Z} + \mathcal{Y}$ is dense in $\mathcal{X}$, or equivalently, that $x = 0$ whenever $x \in (\mathcal{Z} + \mathcal{Y})^{[\perp]}$. The condition $x \in (\mathcal{Z} + \mathcal{Y})^{[\perp]}$ is equivalent to $x \in \mathcal{Z}^{[\perp]} \cap \mathcal{Y}^{[\perp]}$. Since $\mathcal{Z}$ is nonpositive and $\mathcal{Y}^{[\perp]}$ is uniformly positive, this implies that $x = 0$ (by an argument analogous to the one above).

Finally, we show that $\mathcal{Z} + \mathcal{Y}$ is closed in $\mathcal{X}$. Let $x_n = z_n + y_n \to x$ in $\mathcal{X}$, with $z_n \in \mathcal{Z}$ and $y_n \in \mathcal{Y}$. Let $P_{\mathcal{Y}}$ and $P_{\mathcal{Y}^{[\perp]}}$ be the complementary orthogonal projections onto $\mathcal{Y}$ and $\mathcal{Y}^{[\perp]}$, respectively. Then, for each n and m,

$$\begin{aligned} 0 &\leqslant [z_n - z_m, z_n - z_m]_{\mathcal{X}} \\ &= \big[P_{\mathcal{Y}}(z_n - z_m), P_{\mathcal{Y}}(z_n - z_m)\big]_{\mathcal{X}} + \big[P_{\mathcal{Y}^{[\perp]}}(z_n - z_m), P_{\mathcal{Y}^{[\perp]}}(z_n - z_m)\big]_{\mathcal{X}}, \end{aligned}$$

and hence

$$\begin{aligned} 0 &\leqslant -\big[P_{\mathcal{Y}}(z_n - z_m), P_{\mathcal{Y}}(z_n - z_m)\big]_{\mathcal{X}} \\ &\leqslant \big[P_{\mathcal{Y}^{[\perp]}}(z_n - z_m), P_{\mathcal{Y}^{[\perp]}}(z_n - z_m)\big]_{\mathcal{X}} \\ &= \big[P_{\mathcal{Y}^{[\perp]}}(x_n - x_m), P_{\mathcal{Y}^{[\perp]}}(x_n - x_m)\big]_{\mathcal{X}}. \end{aligned}$$

Here the final expression tends to zero as $n, m \to \infty$, hence so do the other two. As both $-\mathcal{Y}$ and $\mathcal{Y}^{[\perp]}$ are uniformly positive, this implies that both $P_{\mathcal{Y}}(z_n - z_m)$ and $P_{\mathcal{Y}^{[\perp]}}(z_n - z_m)$ tend to zero in $\mathcal{X}$ as $n, m \to \infty$, and consequently $z_n - z_m \to 0$ in $\mathcal{X}$ as $n, m \to \infty$. By the completeness of $\mathcal{X}$, the limit $\lim_{n\to\infty} z_n := z$ exists, and hence also $\lim_{n\to\infty} y_n := y = x - z$ exists. Both $\mathcal{Z}$ and $\mathcal{Y}$ are closed, so $z \in \mathcal{Z}$, $y \in \mathcal{Y}$, and $x = z + y \in \mathcal{Z} + \mathcal{Y}$. □

The partial converse of Lemma 2.1 is also true: if $\mathcal{Z}$ is nonnegative and (2.3) holds for some uniformly negative and maximal nonpositive subspace, then $\mathcal{Z}$ is maximal nonnegative.

2.2. *Preliminaries on quotient spaces*

Let $\mathcal{X}$ be a topological vector space, and let $\mathcal{Z}$ be a closed subspace of $\mathcal{X}$. The *quotient space* of $\mathcal{X}$ modulo $\mathcal{Z}$ (or over $\mathcal{Z}$) is denoted by $\mathcal{X}/\mathcal{Z}$. Each element in $\mathcal{X}/\mathcal{Z}$ is an equivalence class of vectors in $\mathcal{X}$, where x_1 and $x_2 \in \mathcal{X}$ are considered to be equivalent if $x_1 - x_2 \in \mathcal{Z}$. Thus, each equivalence class is a closed affine subset of $\mathcal{X}$. The equivalence class in $\mathcal{X}/\mathcal{Z}$ which contains a particular $x \in \mathcal{X}$ is denoted by $x + \mathcal{Z}$. The quotient $\mathcal{X}/\mathcal{Z}$ is a vector space with addition and scalar multiplication defined by $(x_1 + \mathcal{Z}) + (x_2 + \mathcal{Z}) = (x_1 + x_2) + \mathcal{Z}$ and $\lambda(x + \mathcal{Z}) = (\lambda x) + \mathcal{Z}$. The quotient map $x \mapsto x + \mathcal{Z}$ is denoted by $\pi_{\mathcal{Z}}$, or shortly by π. The *quotient topology* in $\mathcal{X}/\mathcal{Z}$ is the one inherited from $\mathcal{X}$ through the quotient map π, i.e., $\Omega \subset \mathcal{X}/\mathcal{Z}$ is open in $\mathcal{X}/\mathcal{Z}$ if and only if its inverse image $\pi^{-1}(\Omega)$ of Ω is open in $\mathcal{X}$. The *quotient map* π is obviously linear, and it is both continuous and open with respect to the quotient topology in $\mathcal{X}/\mathcal{Z}$.

If the topology in $\mathcal{X}$ is induced by a Hilbert space norm $\|\cdot\|_{\mathcal{X}}$, then the topology in $\mathcal{X}/\mathcal{Z}$ is induced by the Hilbert space *quotient norm*

$$\|h\|_{\mathcal{X}/\mathcal{Z}} = \min\{\|x\|_{\mathcal{X}} \mid x \in h\}. \tag{2.4}$$

In particular, this is true if $\mathcal{X}$ is a closed subspace of a Kreĭn space (since the topology of a Kreĭn space is induced by a Hilbert space norm). In both these cases the quotient map π has a bounded right-inverse, since π is surjective, and since the topologies in $\mathcal{X}/\mathcal{Z}$ and $\mathcal{X}$ are induced by Hilbert space norms. If $\mathcal{Z}$ is a maximal nonnegative subspace of a Kreĭn space $\mathcal{X}$, then we can say more:

Lemma 2.2. *If $\mathcal{Z}$ is a maximal nonnegative subspace of a Kreĭn space $\mathcal{X}$ and $\mathcal{Y}$ is an arbitrary uniformly negative and maximal nonpositive subspace of $\mathcal{X}$, then the quotient map $\pi : \mathcal{X} \to \mathcal{X}/\mathcal{Z}$ has a unique bounded right-inverse T with range $\mathcal{Y}$.*

Proof. By Lemma 2.1, $\mathcal{W} = \mathcal{Z} \dotplus \mathcal{Y}$. This implies that the restriction of the quotient map $\pi_{\mathcal{Z}}$ to $\mathcal{Y}$ is a continuous linear bijection from $\mathcal{Y}$ to $\mathcal{X}/\mathcal{Z}$. Since the topology in $\mathcal{X}/\mathcal{Z}$ is induced by a Hilbert space norm, this implies that the inverse T of this map is continuous. This map T is the unique right-inverse of $\pi_{\mathcal{Z}}$ with $\mathcal{R}(T) = \mathcal{Y}$. □

2.3. *The space $\mathcal{H}(\mathcal{Z})$*

As an introduction to our first main result, presented in Theorem 2.3, we first consider the case where $\mathcal{Z}$ is a uniformly positive maximal nonnegative subspace of $\mathcal{X}$. Then (2.1) is a fundamental decomposition of $\mathcal{X}$, and every $x \in \mathcal{X}$ has a unique decomposition

$$x = z_x^\dagger + z_x \quad \text{with } z_x^\dagger \in \mathcal{Z}^{[\perp]} \text{ and } z_x \in \mathcal{Z};$$

here $z_x^\dagger = P_{\mathcal{Z}^{[\perp]}} x$ and $z_x = P_{\mathcal{Z}} x$. When x is decomposed in this way we get, for every $z \in \mathcal{Z}$,

$$\begin{aligned}[x - z, x - z]_{\mathcal{X}} &= \big[z_x^\dagger + (z_x - z), z_x^\dagger + (z_x - z)\big]_{\mathcal{X}} \\ &= \big[z_x^\dagger, z_x^\dagger\big]_{\mathcal{X}} + [z_x - z, z_x - z]_{\mathcal{X}},\end{aligned}$$

where $z_x - z \in \mathcal{Z}$. Hence, since $\mathcal{Z}$ is nonnegative,

$$\sup_{z \in \mathcal{Z}} \bigl(-[x-z, x-z]_{\mathcal{X}}\bigr) = -\bigl[z_x^\dagger, z_x^\dagger\bigr]_{\mathcal{X}}, \tag{2.5}$$

and z_x is the unique vector in $\mathcal{Z}$ for which the supremum is achieved. The right-hand side is the square of the norm of $z_x^\dagger$ in the Hilbert space $-\mathcal{Z}^{[\perp]}$, and the left-hand side can be interpreted as the square of a Hilbert norm in the quotient space $\mathcal{X}/\mathcal{Z}$. We denote $\mathcal{X}/\mathcal{Z}$ equipped with the above norm by $\mathcal{H}(\mathcal{Z})$, and denote the norm of $h \in \mathcal{H}(\mathcal{Z})$ by (1.2). With this notation (2.5) becomes

$$\|x + \mathcal{Z}\|^2_{\mathcal{H}(\mathcal{Z})} = -\bigl[z_x^\dagger, z_x^\dagger\bigr]_{\mathcal{X}}, \tag{2.6}$$

where $x + \mathcal{Z}$ stands for the equivalence class in $\mathcal{X}/\mathcal{Z}$ to which x belongs. The mapping $x \mapsto z_x^\dagger := P_{\mathcal{Z}^{[\perp]}} x$ is a unitary map of $\mathcal{H}(\mathcal{Z})$ onto $-\mathcal{Z}^{[\perp]}$, whose inverse $z^\dagger \mapsto z^\dagger + \mathcal{Z}$ is the restriction of the quotient map π to $\mathcal{Z}^{[\perp]}$.

We now proceed to discuss the general case where $\mathcal{Z}$ is maximal nonnegative but not necessarily uniformly positive. In this case the supremum in (1.1) can be infinite for some equivalence classes $h \in \mathcal{X}/\mathcal{Z}$, and, if finite, it need not always be achieved for some $z \in \mathcal{Z}$. Nonetheless, we *define* $\mathcal{H}(\mathcal{Z})$ to be the subset of $\mathcal{X}/\mathcal{Z}$ for which the supremum in (1.2) is finite. As we show in the following theorem, it is still true that $\mathcal{H}(\mathcal{Z})$ is a Hilbert space which is continuously contained in $\mathcal{X}/\mathcal{Z}$.

Theorem 2.3. *Let $\mathcal{Z}$ be a maximal nonnegative subspace of a Kreĭn space $\mathcal{X}$. Define $\mathcal{H}(\mathcal{Z})$ by* (1.1)*, and define $\|\cdot\|_{\mathcal{H}(\mathcal{Z})}$ by* (1.2)*. Then $\mathcal{H}(\mathcal{Z})$ is a Hilbert space with the norm $\|\cdot\|_{\mathcal{H}(\mathcal{Z})}$ which is continuously contained in $\mathcal{X}/\mathcal{Z}$.*

Proof.

Step 1 *(The supremum in (1.1) in nonnegative).* If $h = \mathcal{Z}$, then the supremum in (1.1) is zero since $\mathcal{Z}$ is nonnegative. We claim that the supremum is strictly positive if it is finite and $h \neq \mathcal{Z}$.

Suppose that $x_0 \in h$ but $x_0 \notin \mathcal{Z}$, and that $\sup\{-[x,x]_{\mathcal{X}} \mid x \in h\} \leqslant 0$, i.e., that

$$[x_0 + z, x_0 + z]_{\mathcal{X}} \geqslant 0, \quad z \in \mathcal{Z}. \tag{2.7}$$

Define $\mathcal{Z}' = \{\lambda x_0 + z \mid \lambda \in \mathbb{C},\ z \in \mathcal{Z}\}$. Then $\mathcal{Z}'$ is a subspace which strictly contains $\mathcal{Z}$. We claim that $\mathcal{Z}'$ is nonnegative. If $\lambda = 0$ then $[\lambda x_0 + z, \lambda x_0 + z]_{\mathcal{X}} \geqslant 0$ because of the nonnegativity of $\mathcal{Z}$, whereas if $\lambda \neq 0$, then

$$[\lambda x_0 + z, \lambda x_0 + z]_{\mathcal{X}} = |\lambda|^2 \biggl[x_0 + \frac{1}{\lambda} z, x_0 + \frac{1}{\lambda} z\biggr]_{\mathcal{X}} \geqslant 0 \tag{2.8}$$

because of (2.7). This proves that $\mathcal{Z}'$ is nonnegative. However, $\mathcal{Z}$ was assumed to be *maximal* nonnegative, so it cannot have a nontrivial nonnegative extension. This shows that (2.7) cannot hold, and consequently the supremum in (1.1) is strictly positive.

Step 2 *($\|\lambda h\|_{\mathcal{H}(\mathcal{Z})} = \|\lambda\|h\|_{\mathcal{H}(\mathcal{Z})}$ for all $h \in \mathcal{H}(\mathcal{Z})$).* If $\lambda = 0$, then both sides are equal to zero, and if $\lambda \neq 0$ this follows from the same identity that was used in (2.8).

Step 3 *($\|\cdot\|_{\mathcal{H}(\mathcal{Z})}$ satisfies the parallelogram law).* It is easy to verify that for all $x_1, x_2 \in \mathcal{X}$ and $z_1, z_2 \in \mathcal{Z}$ we have

$$\begin{aligned}
&[x_1 + x_2 + z_1, x_1 + x_2 + z_1]_{\mathcal{X}} + [x_1 - x_2 + z_2, x_1 - x_2 + z_2]_{\mathcal{X}} \\
&\quad = 2\left[x_1 + \frac{1}{2}(z_1 + z_2), x_1 + \frac{1}{2}(z_1 + z_2)\right]_{\mathcal{X}} \\
&\qquad + 2\left[x_2 + \frac{1}{2}(z_1 - z_2), x_2 + \frac{1}{2}(z_1 - z_2)\right]_{\mathcal{X}},
\end{aligned}$$

and hence

$$\begin{aligned}
&\|x_1 + x_2 + \mathcal{Z}\|^2_{\mathcal{H}(\mathcal{Z})} + \|x_1 - x_2 + \mathcal{Z}\|^2_{\mathcal{H}(\mathcal{Z})} \\
&\quad = \sup_{z_1, z_2 \in \mathcal{Z}} \big(-[x_1 + x_2 + z_1, x_1 + x_2 + z_1]_{\mathcal{X}} \\
&\qquad - [x_1 - x_2 + z_2, x_1 - x_2 + z_2]_{\mathcal{X}}\big) \\
&\quad = 2 \sup_{z_1, z_2 \in \mathcal{Z}} \left(-\left[x_1 + \frac{1}{2}(z_1 + z_2), x_1 + \frac{1}{2}(z_1 + z_2)\right]_{\mathcal{X}}\right. \\
&\qquad \left. - \left[x_2 + \frac{1}{2}(z_1 - z_2), x_2 + \frac{1}{2}(z_1 - z_2)\right]_{\mathcal{X}}\right) \\
&\quad = 2 \sup_{z_1', z_2' \in \mathcal{Z}} \big(-\big[x_1 + z_1', x_1 + z_1'\big]_{\mathcal{X}} - \big[x_2 + z_2', x_2 + z_2'\big]_{\mathcal{X}}\big) \\
&\quad = 2\|x_1 + \mathcal{Z}\|^2_{\mathcal{H}(\mathcal{Z})} + 2\|x_2 + \mathcal{Z}\|^2_{\mathcal{H}(\mathcal{Z})}.
\end{aligned}$$

This shows that $\|\cdot\|_{\mathcal{H}(\mathcal{Z})}$ satisfies the parallelogram law.

Step 4 *($\mathcal{H}(\mathcal{Z})$ is a subspace and $\|\cdot\|_{\mathcal{H}(\mathcal{Z})}$ is a norm in $\mathcal{H}(\mathcal{Z})$ induced by an inner product).* It follows from the homogeneity property proved in Step 2 and the parallelogram law proved in Step 3 that if both $\sup\{-[x,x]_{\mathcal{X}} \mid x \in h_1\} < \infty$ and $\sup\{-[x,x]_{\mathcal{X}} \mid x \in h_2\} < \infty$, then $\sup\{-[x,x]_{\mathcal{X}} \mid x \in \lambda_1 h_1 + \lambda_2 h_2\} < \infty$ for all $\lambda_1, \lambda_2 \in \mathbb{C}$. Thus $\mathcal{H}(\mathcal{Z})$ is a subspace of $\mathcal{X}/\mathcal{Z}$. Since $\|\cdot\|_{\mathcal{H}(\mathcal{Z})}$ is a strictly positive homogeneous function on $\mathcal{H}(\mathcal{Z})$ satisfying the parallelogram law, it is a norm on $\mathcal{H}(\mathcal{Z})$ induced by an inner product in $\mathcal{H}(\mathcal{Z})$, which can be defined in terms of $\|\cdot\|_{\mathcal{H}(\mathcal{Z})}$ via the standard polarisation identity

$$\begin{aligned}
4(x_1 + \mathcal{Z}, x_2 + \mathcal{Z})_{\mathcal{H}(\mathcal{Z})} &= \|x_1 + x_2 + \mathcal{Z}\|^2_{\mathcal{H}(\mathcal{Z})} - \|x_1 - x_2 + \mathcal{Z}\|^2_{\mathcal{H}(\mathcal{Z})} \\
&\quad + i\|x_1 + ix_2 + \mathcal{Z}\|^2_{\mathcal{H}(\mathcal{Z})} - i\|x_1 - ix_2 + \mathcal{Z}\|^2_{\mathcal{H}(\mathcal{Z})}.
\end{aligned}$$

Step 5 *(The inclusion map $\mathcal{H}(\mathcal{Z}) \hookrightarrow \mathcal{X}/\mathcal{Z}$ is continuous).* Let $h_n \to 0$ in $\mathcal{H}(\mathcal{Z})$, and let R be a bounded right-inverse to the quotient map π with a uniformly negative range (such a right-inverse exists by Lemma 2.2). Then $Rh_n \in h_n$, and hence $-\langle Rh_n, Rh_n\rangle_{\mathcal{X}} \leqslant \|h_n\|^2_{\mathcal{H}(\mathcal{Z})}$. This implies that

$\liminf_{n\to\infty}\langle Rh_n, Rh_n\rangle_{\mathcal{X}} \geqslant 0$. Together with the fact that the range of R is uniformly negative, this implies that $Rh_n \to 0$ in $\mathcal{X}$ as $n \to \infty$. Consequently, $h_n = \pi Rh_n \to 0$ in $\mathcal{X}/\mathcal{Z}$.

Step 6 *(The space $\mathcal{H}(\mathcal{Z})$ with the norm $\|\cdot\|_{\mathcal{H}(\mathcal{Z})}$ is complete).* Let $\{h_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $\mathcal{H}(\mathcal{Z})$. Since the inclusion map $\mathcal{H}(\mathcal{Z}) \hookrightarrow \mathcal{X}/\mathcal{Z}$ is continuous, $\{h_n\}_{n=1}^{\infty}$ is also a Cauchy sequence in $\mathcal{X}/\mathcal{Z}$, and since $\mathcal{X}/\mathcal{Z}$ is complete, h_n converges to some limit h in $\mathcal{X}/\mathcal{Z}$. We claim that $h \in \mathcal{H}(\mathcal{Z})$, and that h_n converges to h in $\mathcal{H}(\mathcal{Z})$.

Let $\epsilon > 0$, and choose N so large that $\|h_n - h_m\|_{\mathcal{H}(\mathcal{Z})} \leqslant \epsilon$ whenever both $n \geqslant N$ and $m \geqslant N$. Let R be a continuous right-inverse to the quotient map π (such a right-inverse exists by Lemma 2.2), and define $x_n = Rh_n$ and $x = Rh$. Then $h_n = x_n + \mathcal{Z}$, $h = x + \mathcal{Z}$, $x_m \to x$ in $\mathcal{X}$ as $m \to \infty$, and for every $z \in \mathcal{Z}$,

$$-[x - x_n + z, x - x_n + z]_{\mathcal{X}} = \lim_{m\to\infty}\bigl(-[x_m - x_n + z, x_m - x_n + z]_{\mathcal{X}}\bigr),$$

where

$$-[x_m - x_n + z, x_m - x_n + z]_{\mathcal{X}} \leqslant \|x_m - x_n + \mathcal{Z}\|^2_{\mathcal{H}(\mathcal{Z})} \leqslant \epsilon^2$$

whenever both $n \geqslant N$ and $m \geqslant N$. Hence

$$-[x - x_n + z, x - x_n + z]_{\mathcal{X}} \leqslant \epsilon^2$$

for all $z \in \mathcal{Z}$ when $n \geqslant N$. By the definition of $\|\cdot\|_{\mathcal{H}(\mathcal{Z})}$, $x - x_n + \mathcal{Z} \in \mathcal{H}(\mathcal{Z})$ and $\|x - x_n + \mathcal{Z}\|_{\mathcal{H}(\mathcal{Z})} \leqslant \epsilon$. Since $x_n + \mathcal{Z} \in \mathcal{H}(\mathcal{Z})$ also $x \in \mathcal{H}(\mathcal{Z})$, and $h_n = x_n + \mathcal{Z} \to x + \mathcal{Z} = h$ in $\mathcal{H}(\mathcal{Z})$ as $n \to \infty$. $\square$

2.4. Properties of the space $\mathcal{H}(\mathcal{Z})$

We continue to study some properties of the Hilbert space $\mathcal{H}(\mathcal{Z})$. In particular we will show that the subspace $\mathcal{H}^0(\mathcal{Z})$ defined in (1.5) is a dense subspace of $\mathcal{H}(\mathcal{Z})$. We begin with a preliminary lemma.

Since $\mathcal{Z}$ is an nonnegative subspace of $\mathcal{X}$, the Schwarz inequality says that

$$\bigl|[x, z]_{\mathcal{X}}\bigr|^2 \leqslant [z, z]_{\mathcal{X}}[x, x]_{\mathcal{X}}, \quad x, z \in \mathcal{Z}.$$

A generalisation of this inequality is presented in part (1) of the following lemma.

Lemma 2.4. *Let $\mathcal{Z}$ be a maximal nonnegative subspace of a Kreĭn space $\mathcal{X}$.*

(1) *The inequality* (1.3) *holds.*
(2) *The inequality* (1.3) *collapses to an equality if and only if either* $[z, z]_{\mathcal{X}} = 0$ *or* $[z, z]_{\mathcal{X}} \neq 0$ *and* $x = ([x, z]_{\mathcal{X}}/[z, z]_{\mathcal{X}})z + z^{\dagger}$ *where* $z^{\dagger} \in \mathcal{Z}^{[\perp]}$.
(3) *The supremum in* (1.1) *is achieved if and only if* $h = z^{\dagger} + \mathcal{Z}$ *for some* $z^{\dagger} \in \mathcal{Z}^{[\perp]}$, *and in this case the supremum is equal to*

$$\max\bigl\{-\bigl[z^{\dagger} + z, z^{\dagger} + z\bigr]_{\mathcal{X}} \bigm| z \in \mathcal{Z}\bigr\} = -\bigl[z^{\dagger}, z^{\dagger}\bigr]_{\mathcal{X}}. \tag{2.9}$$

(4) *If* $z^\dagger \in \mathcal{Z}^{[\perp]}$, $x \in \mathcal{X}$, *and* $x + \mathcal{Z} \in \mathcal{H}(\mathcal{Z})$, *then* $z^\dagger + \mathcal{Z} \in \mathcal{H}(\mathcal{Z})$ *and*

$$\bigl(z^\dagger + \mathcal{Z}, x + \mathcal{Z}\bigr)_{\mathcal{H}(\mathcal{Z})} = -\bigl[z^\dagger, x\bigr]_{\mathcal{X}}. \tag{2.10}$$

Proof.

Proof of claim (1). Let $h \in \mathcal{H}(\mathcal{Z})$, $x \in h$, $z \in \mathcal{Z}$, and $\lambda \in \mathbb{C}$. Then

$$-[x - \lambda z, x - \lambda z]_{\mathcal{X}} \leqslant \|h\|^2_{\mathcal{H}(\mathcal{Z})},$$

or equivalently,

$$|\lambda|^2[z, z]_{\mathcal{X}} - 2\Re[x, \lambda z]_{\mathcal{X}} + [x, x]_{\mathcal{X}} + \|h\|^2_{\mathcal{H}(\mathcal{Z})} \geqslant 0. \tag{2.11}$$

If $[z, z]_{\mathcal{X}} = 0$, then this implies that $[x, z]_{\mathcal{X}} = 0$ (since the inequality is true for all $\lambda \in \mathbb{C}$), so (1.3) holds in the trivial form $0 = 0$. If $[z, z]_{\mathcal{X}} \neq 0$, then we can take $\lambda = [x, z]_{\mathcal{X}}/[z, z]_{\mathcal{X}}$ in (2.11) and multiply the resulting formula by $[z, z]_{\mathcal{X}}$ to get (1.3).

Proof of claim (3). It $h = z^\dagger + \mathcal{Z}$ for some $z^\dagger \in \mathcal{Z}^{[\perp]}$, then

$$-\bigl[z^\dagger + z, z^\dagger + z\bigr]_{\mathcal{X}} = -\bigl[z^\dagger, z^\dagger\bigr]_{\mathcal{X}} - [z, z]_{\mathcal{X}} \leqslant -\bigl[z^\dagger, z^\dagger\bigr]_{\mathcal{X}}.$$

The supremum in (1.1) is achieved by taking $z = 0$, and it is equal to $-[z^\dagger, z^\dagger]_{\mathcal{X}}$.

Conversely, if the supremum in (1.1) is achieved at some point $x_0 \in h$, then it follows from (1.3) with x replaced by x_0 that $[x_0, z]_{\mathcal{X}} = 0$ for all $z \in \mathcal{Z}$. Consequently, $x_0 \in \mathcal{Z}^{[\perp]}$.

Proof of claim (2). If the inequality (1.3) holds in the form of an equality and $[z, z]_{\mathcal{X}} \neq 0$, then we get equality in (2.11) by taking $\lambda = [x, z]_{\mathcal{X}}/[z, z]_{\mathcal{X}}$. This implies that the supremum in (1.1) is achieved for the vector $x - ([x, z]_{\mathcal{X}}/[z, z]_{\mathcal{X}})$. By claim (3), $x - ([x, z]_{\mathcal{X}}/[z, z]_{\mathcal{X}}) \in \mathcal{Z}^{[\perp]}$.

Proof of claim (4). It follows from part (3) that $z^\dagger + \mathcal{Z} \in \mathcal{H}(\mathcal{Z})$. In order to prove (2.10) is suffices to prove that, for all $z^\dagger \in \mathcal{Z}^{[\perp]}$ and $x \in \mathcal{X}$ with $x + \mathcal{Z} \in \mathcal{H}(\mathcal{Z})$ we have

$$\bigl\|x + z^\dagger + \mathcal{Z}\bigr\|^2_{\mathcal{H}(\mathcal{Z})} = \|x + \mathcal{Z}\|^2_{\mathcal{H}(\mathcal{Z})} - \bigl[z^\dagger, x\bigr]_{\mathcal{X}} - \bigl[x, z^\dagger\bigr]_{\mathcal{X}} + \bigl\|z^\dagger + \mathcal{Z}\bigr\|^2_{\mathcal{H}(\mathcal{Z})}, \tag{2.12}$$

because (2.10) then follows from the polarisation identity. However, for all $z^\dagger \in \mathcal{Z}^{[\perp]}$, $x \in \mathcal{X}$ with $x + \mathcal{Z} \in \mathcal{H}(\mathcal{Z})$, and $z \in \mathcal{Z}$,

$$\begin{aligned}-\bigl[x + z^\dagger + z, x + z^\dagger + z\bigr]_{\mathcal{X}} &= -[x + z, x + z]_{\mathcal{X}} - \bigl[x, z^\dagger\bigr]_{\mathcal{X}} \\ &\quad - \bigl[z^\dagger, x\bigr]_{\mathcal{X}} - \bigl[z^\dagger, z^\dagger\bigr]_{\mathcal{X}}.\end{aligned} \tag{2.13}$$

After taking the supremum over all $z \in \mathcal{Z}$ and using the fact that $\|z^\dagger + \mathcal{Z}\|^2_{\mathcal{H}(\mathcal{Z})} = -[z^\dagger, z^\dagger]_{\mathcal{X}}$ we get (2.12). $\square$

The following theorem contains a geometrical interpretation of the Hilbert space $\mathcal{H}(\mathcal{Z})$ as the completion of the pre-Hilbert space $\mathcal{H}^0(\mathcal{Z})$ with the inner product given in (1.6).

Theorem 2.5. *Let $\mathcal{Z}$ be a maximal nonnegative subspace of a Kreĭn space $\mathcal{X}$. Then the subspace $\mathcal{H}^0(\mathcal{Z})$ of $\mathcal{X}/\mathcal{Z}$ defined in* (1.5) *is a dense subspace of the Hilbert space $\mathcal{H}(\mathcal{Z})$ defined in Theorem* 2.3*, and the inner product in $\mathcal{H}^0(\mathcal{Z})$ inherited from $\mathcal{H}(\mathcal{Z})$ is given by* (1.6)*. Thus, $\mathcal{H}(\mathcal{Z})$ is a Hilbert space completion of the space $\mathcal{H}^0(\mathcal{Z})$ with the inner product* (1.6).

Proof. It follows from part (3) of Lemma 2.4 that $\mathcal{H}^0(\mathcal{Z}) \subset \mathcal{H}(\mathcal{Z})$. That the inner product in $\mathcal{H}^0(\mathcal{Z})$ inherited from $\mathcal{H}(\mathcal{Z})$ is given by (1.6) follows from (2.10).

To show that the $\mathcal{H}^0(\mathcal{Z})$ is dense in $\mathcal{H}(\mathcal{Z})$ it suffices to show that $(\mathcal{H}^0(\mathcal{Z}))^\perp = \{0\}$. Let $x \in \mathcal{X}$ and $x + \mathcal{Z} \in \mathcal{H}(\mathcal{Z})$, and suppose that $(x + \mathcal{Z}, z^\dagger + \mathcal{Z})_{\mathcal{H}(\mathcal{Z})} = 0$ for all $z^\dagger \in \mathcal{Z}^{[\perp]}$. Then by (2.10), $[x, z^\dagger]_{\mathcal{X}} = 0$ for all $z^\dagger \in \mathcal{Z}^{[\perp]}$, and hence $x \in (\mathcal{Z}^{[\perp]})^{[\perp]} = \mathcal{Z}$ and $x + \mathcal{Z} = \mathcal{Z}$ is the zero vector in $\mathcal{X}/\mathcal{Z}$. □

2.5. *The spaces $\mathcal{Z}^{[\perp]}/\mathcal{Z}_0$ and $\mathcal{U}(\mathcal{Z})$*

Up to now we have concentrated our attention on the two subspaces $\mathcal{H}(\mathcal{Z})$ and $\mathcal{H}^0(\mathcal{Z})$ of $\mathcal{X}/\mathcal{Z}$. It turns out that the latter of these spaces is closely related to the space $\mathcal{Z}^{[\perp]}/\mathcal{Z}_0$, where as before $\mathcal{Z}_0 = \mathcal{Z} \cap \mathcal{Z}^{[\perp]}$. The space $\mathcal{Z}^{[\perp]}/\mathcal{Z}_0$ is defined in the standard way as the quotient of $\mathcal{Z}^{[\perp]}$ modulo its closed subspace $\mathcal{Z}_0$. Since the topology in $\mathcal{Z}^{[\perp]}$ inherited from $-\mathcal{X}$ is induced by a Hilbert space norm, it follows that also the standard quotient topology in $\mathcal{Z}^{[\perp]}/\mathcal{Z}_0$ is induced by a Hilbert space norm. In particular, $\mathcal{Z}^{[\perp]}/\mathcal{Z}_0$ is complete with respect to the quotient topology.

We define the space $\mathcal{U}(\mathcal{Z})$ to be the same vector space as $\mathcal{Z}^{[\perp]}/\mathcal{Z}_0$, but with a different topology induced by the positive inner product inherited from $-\mathcal{X}$, i.e.,

$$\big(z_1^\dagger + \mathcal{Z}_0, z_2^\dagger + \mathcal{Z}_0\big)_{\mathcal{U}(\mathcal{Z})} = -\big[z_1^\dagger, z_2^\dagger\big]_{\mathcal{X}}, \quad z_1^\dagger, z_2^\dagger \in \mathcal{Z}^{[\perp]}. \tag{2.14}$$

That this is, indeed, a positive inner product on the vector space $\mathcal{Z}^{[\perp]}/\mathcal{Z}_0$ follows from the fact that $\mathcal{Z}^{[\perp]}$ is a nonnegative subspace of $-\mathcal{X}$, and that $\mathcal{Z}_0$ is the maximal neutral subspace in $\mathcal{Z}^{[\perp]}$. The topology induced by this inner product is weaker than the standard quotient topology of $\mathcal{Z}^{[\perp]}/\mathcal{Z}_0$, so that the embedding of $\mathcal{Z}^{[\perp]}/\mathcal{Z}_0$ in $\mathcal{U}(\mathcal{Z})$ is continuous. However, the inverse of this embedding map need not be continuous, and $\mathcal{U}(\mathcal{Z})$ need not be complete. Thus, $\mathcal{U}(\mathcal{Z})$ is a unitary space (a pre-Hilbert space), but $\mathcal{U}(\mathcal{Z})$ need not be a Hilbert space. It is a Hilbert space if and only if $\mathcal{Z}^{[\perp]}$ is the direct sum of $\mathcal{Z}_0$ and a uniformly negative subspace in $\mathcal{X}$, or equivalently, if and only if $\mathcal{Z}$ is the direct sum of $\mathcal{Z}_0$ and a uniformly positive subspace in $\mathcal{X}$. In this case (and only in this case) the topologies of $\mathcal{Z}^{[\perp]}/\mathcal{Z}_0$ and $\mathcal{U}(\mathcal{Z})$ coincide.

Theorem 2.6. *Let $\mathcal{Z}$ be a maximal nonnegative subspace of a Kreĭn space $\mathcal{X}$. Define $\mathcal{Z}_0 = \mathcal{Z} \cap \mathcal{Z}^{[\perp]}$, and let $\mathcal{U}(\mathcal{Z})$, $\mathcal{H}^0(\mathcal{Z})$, and $\mathcal{H}(\mathcal{Z})$ be the spaces defined earlier in this section. Then the formula*

$$S\big(z^\dagger + \mathcal{Z}_0\big) = z^\dagger + \mathcal{Z}, \quad x^\dagger \in \mathcal{Z}^{[\perp]}, \tag{2.15}$$

defines an linear isometric map S from the unitary space $\mathcal{U}(\mathcal{Z})$ into the Hilbert space $\mathcal{H}(\mathcal{Z})$ with $\mathcal{R}(S) = \mathcal{H}^0(\mathcal{Z})$. In particular $\mathcal{R}(S)$ is dense in $\mathcal{H}(\mathcal{Z})$.

Proof. That (2.15) defines a linear isometric operator S from $\mathcal{U}(\mathcal{Z})$ onto $\mathcal{H}^0(\mathcal{Z})$ follows from the formulas (2.14) and (1.6) for the inner products in $\mathcal{U}(\mathcal{Z})$ and $\mathcal{H}^0(\mathcal{Z})$, respectively. That $\mathcal{H}^0(\mathcal{Z})$ is dense in $\mathcal{H}(\mathcal{Z})$ is part of the conclusion of Theorem 2.5. □

Remark 2.7. The linear bijection S defined in (2.15) is an isomorphism with respect to the two inner products in $\mathcal{U}(\mathcal{Z})$ and $\mathcal{H}^0(\mathcal{Z})$, and it is still continuous if we replace the topology in $\mathcal{U}(\mathcal{Z})$ by the quotient topology of $\mathcal{Z}^{[\perp]}/\mathcal{Z}_0$ or if we replace the topology in $\mathcal{H}^0(\mathcal{Z})$ by the quotient topology inherited from $\mathcal{X}/\mathcal{Z}$. This follows from the fact that the quotient topology in $\mathcal{Z}^{[\perp]}/\mathcal{Z}_0$ in stronger than the inner product topology of $\mathcal{U}(\mathcal{Z})$, and that the inner product topology of $\mathcal{H}^0(\mathcal{Z})$ is stronger than the quotient topology inherited from $\mathcal{X}/\mathcal{Z}$. However, the inverse S^{-1} need not be continuous with respect to the quotient topologies. It is continuous if and only if $\mathcal{Z}$ is the direct sum of $\mathcal{Z}_0$ and a uniformly positive subspace. In this case the spaces $\mathcal{U}(\mathcal{Z})$ and $\mathcal{H}^0(\mathcal{Z})$ are complete, and hence $\mathcal{H}^0(\mathcal{Z}) = \mathcal{H}(\mathcal{Z})$ and S is a unitary map of $\mathcal{U}(\mathcal{Z})$ onto $\mathcal{H}(\mathcal{Z})$. Theorems 2.14 and 3.7 list a number of other equivalent conditions for this case to occur.

2.6. *Further properties of the space $\mathcal{H}(\mathcal{Z})$*

Convergence of a sequence in $\mathcal{H}(\mathcal{Z})$ is related to convergence of the corresponding representatives in $\mathcal{X}$ as follows.

Lemma 2.8. *Let $\mathcal{Z}$ be a maximal nonnegative subspace of a Kreĭn space $\mathcal{X}$, and let $\mathcal{H}(\mathcal{Z})$ be the Hilbert space defined in Theorem* 2.3.

(1) *If $y_n + \mathcal{Z} \to x + \mathcal{Z}$ in $\mathcal{H}(\mathcal{Z})$ as $n \to \infty$, then there exists a sequence $x_n \in \mathcal{X}$ such that $x_n \to x$ in $\mathcal{X}$ and $x_n + \mathcal{Z} = y_n + \mathcal{Z} \to x + \mathcal{Z}$ in $\mathcal{H}(\mathcal{Z})$ as $n \to \infty$. The same claim remains true if we throughout replace the strong convergence by weak convergence.*
(2) *Given any $x + \mathcal{Z} \in \mathcal{H}(\mathcal{Z})$ there exists a sequence $x_n \in \mathcal{Z} + \mathcal{Z}^{[\perp]}$ such that $x_n \to x$ in $\mathcal{X}$ and $x_n + \mathcal{Z} \to x + \mathcal{Z}$ in $\mathcal{H}(\mathcal{Z})$ as $n \to \infty$.*
(3) *If $z_n^\dagger \in \mathcal{Z}^{[\perp]}$ and $\sup_{n \geqslant 0}(-[z_n^\dagger, z_n^\dagger]_{\mathcal{X}}) < \infty$, then there exist a vector $x \in \mathcal{X}$, a subsequence $z_{n_j}^\dagger$, and a sequence $z_{n_j} \in \mathcal{Z}$ such that $z_{n_j}^\dagger + z_{n_j} \to x$ weakly in $\mathcal{X}$ and $z_{n_j}^\dagger + \mathcal{Z} \to x + \mathcal{Z}$ weakly in $\mathcal{H}(\mathcal{Z})$ as $j \to \infty$.*

Proof.

Proof of claim (1). Since $\mathcal{H}(\mathcal{Z})$ is continuously contained in $\mathcal{X}/\mathcal{Z}$, the sequence $y_n + \mathcal{Z}$ converges to $x + \mathcal{Z}$ also in the topology of $\mathcal{X}/\mathcal{Z}$. The quotient map $\pi_{\mathcal{Z}}$ has a bounded right-inverse, and this implies that there exists a sequence $x_n' \in \mathcal{X}$ which tends to a limit $x' \in \mathcal{X}$ such that $x_n' + \mathcal{Z} = y_n + \mathcal{Z}$ for all n and $x' + \mathcal{Z} = x + \mathcal{Z}$. In particular, $x - x' \in \mathcal{Z}$. Define $x_n = x_n' + x - x'$. Then $x_n + \mathcal{Z} = y_n + \mathcal{Z}$ for all n, $x_n + \mathcal{Z} \to x + \mathcal{Z}$ in $\mathcal{H}(\mathcal{Z})$, and $x_n \to x$ in $\mathcal{X}$ as $n \to \infty$. The version where the strong convergence has been replaced by weak convergence is proved in the same way.

Proof of claim (2). Since $\mathcal{H}^0(\mathcal{Z})$ is dense in $\mathcal{H}(\mathcal{Z})$, for each $x + \mathcal{Z} \in \mathcal{H}(\mathcal{Z})$ there exists a sequence $y_n \in \mathcal{Z}^{[\perp]}$ such that $y_n + \mathcal{Z} \to x + \mathcal{Z}$ in $\mathcal{H}(\mathcal{Z})$. By applying claim (1) to this sequence we can find a sequence x_n satisfying the conclusion of claim (2), since the condition $x_n + \mathcal{Z} = y_n + \mathcal{Z}$ implies that $x_n \in \mathcal{Z} + \mathcal{Z}^{[\perp]}$.

Proof of claim (3). For each $z_n^\dagger$ we have $\|z_n^\dagger + \mathcal{Z}\|^2_{\mathcal{H}(\mathcal{Z})} = -[z_n^\dagger, z_n^\dagger]_{\mathcal{X}}$, so the given condition implies that the sequence $z_n^\dagger + \mathcal{Z}$ is bounded in $\mathcal{H}(\mathcal{Z})$. The unit ball in $\mathcal{H}(\mathcal{Z})$ is weakly sequentially compact, and hence some subsequence $z_{n_j} + \mathcal{Z}$ converges weakly to a limit $x + \mathcal{Z}$ in $\mathcal{H}(\mathcal{Z})$. The conclusion of claim (3) now follows from claim (1). $\square$

Proposition 2.9. *Let $\mathcal{Z}$ be a maximal nonnegative subspace of a Kreĭn space, and define $\mathcal{H}(\mathcal{Z})$ as in Theorem* 2.3. *Then $\|x+\mathcal{Z}\|_{\mathcal{H}(\mathcal{Z})}$ defined in* (1.2) (*finite or infinite*) *is equal to*

$$\|x+\mathcal{Z}\|_{\mathcal{H}(\mathcal{Z})} = \sup\bigl\{\bigl|[z^\dagger, x]_{\mathcal{X}}\bigr| \bigm| z^\dagger \in \mathcal{Z}^{[\perp]} \text{ and } -[z^\dagger, z^\dagger]_{\mathcal{X}} \leqslant 1\bigr\}. \tag{2.16}$$

Proof. If $x \in \mathcal{X}$ and $x+\mathcal{Z} \in \mathcal{H}(\mathcal{Z})$, then (2.16) follows from (2.10) and density of $\mathcal{H}^0(\mathcal{Z})$ in $\mathcal{H}(\mathcal{Z})$. Conversely, suppose that the supremum in (2.16) is finite. Then the linear functional $F(z^\dagger) := [z^\dagger, x]_{\mathcal{X}} : \mathcal{Z}^{[\perp]} \to \mathbb{C}$ is bounded on $\mathcal{Z}^{[\perp]}$ (with respect to the semi-norm inherited from $-\mathcal{X}$). However, every such functional can be interpreted as a bounded linear functional on $\mathcal{H}^0(\mathcal{Z})$ with respect to the norm inherited from $\mathcal{H}(\mathcal{Z})$. Since $\mathcal{H}^0(\mathcal{Z})$ is dense in $\mathcal{H}(\mathcal{Z})$ and $\mathcal{H}(\mathcal{Z})$ is a Hilbert space, there is some $y+\mathcal{Z} \in \mathcal{H}(\mathcal{Z})$ such that $F(z^\dagger) = [z^\dagger + \mathcal{Z}, y+\mathcal{Z}]_{\mathcal{H}(\mathcal{Z})}$ for every $z^\dagger \in \mathcal{Z}^{[\perp]}$. By (2.10), $[z^\dagger + \mathcal{Z}, y+\mathcal{Z}]_{\mathcal{H}(\mathcal{Z})} = [z^\dagger, y]_{\mathcal{X}}$. Thus, $F(z^\dagger) = [z^\dagger, x]_{\mathcal{X}} = [z^\dagger, y]_{\mathcal{X}}$ for all $z^\dagger \in \mathcal{Z}^{[\perp]}$. This implies that $x-y \in \mathcal{Z}$, and so $x+\mathcal{Z} = y+\mathcal{Z} \in \mathcal{H}(\mathcal{Z})$. As we observed above, this implies (2.16). □

Given a maximal nonnegative subspace $\mathcal{Z}$ of a Kreĭn space $\mathcal{X}$ we define $\mathcal{L}(\mathcal{Z})$ by

$$\mathcal{L}(\mathcal{Z}) = \bigl\{x+\mathcal{Z} \bigm| x \in \overline{\mathcal{Z}+\mathcal{Z}^{[\perp]}}\bigr\}. \tag{2.17}$$

Lemma 2.10. *The set $\mathcal{L}(\mathcal{Z})$ defined above is a closed subspace of $\mathcal{X}/\mathcal{Z}$.*

Proof. It is easy to see that $\mathcal{L}(\mathcal{Z})$ is a subspace. To see that it is closed we argue as follows. Let $h_n \in \mathcal{L}(\mathcal{Z})$, and let $h_n \to h$ in $\mathcal{X}/\mathcal{Z}$ as $n \to \infty$. Let R be a bounded right-inverse of the quotient map $\pi_{\mathcal{Z}}$, and define $x_n = Rh_n$ and $x = Rh$. Then $x_n \to x$ in $\mathcal{X}$ as $n \to \infty$, and $x_n + \mathcal{Z} = h_n \in \mathcal{L}(\mathcal{Z})$ for all n. This implies that $x_n - y_n \in \mathcal{Z}$ for some $y_n \in \overline{\mathcal{Z}+\mathcal{Z}^{[\perp]}}$, and consequently $x_n \in \overline{\mathcal{Z}+\mathcal{Z}^{[\perp]}}$. Therefore also $x = \lim_{n\to\infty} x_n \in \overline{\mathcal{Z}+\mathcal{Z}^{[\perp]}}$. Thus $h = x+\mathcal{Z} \in \mathcal{L}(\mathcal{Z})$. □

Proposition 2.11. *Let $\mathcal{Z}$ be a maximal nonnegative subspace of a Kreĭn space, and let $\mathcal{H}(\mathcal{Z})$ and $\mathcal{H}^0(\mathcal{Z})$ be the spaces defined earlier in this section. Then the closure in $\mathcal{X}/\mathcal{Z}$ of each of the spaces $\mathcal{H}^0(\mathcal{Z})$ and $\mathcal{H}(\mathcal{Z})$ in $\mathcal{X}/\mathcal{Z}$ is equal to $\mathcal{L}(\mathcal{Z})$ defined in* (2.17).

Proof. In view of claim (2) of Lemma 2.8 and the continuous inclusion of $\mathcal{H}(\mathcal{Z})$ in $\mathcal{X}/\mathcal{Z}$, $\mathcal{H}(\mathcal{Z}) \subset \mathcal{L}(\mathcal{Z})$. Consequently,

$$\overline{\mathcal{H}^0(\mathcal{Z})} \subset \overline{\mathcal{H}(\mathcal{Z})} \subset \overline{\mathcal{L}(\mathcal{Z})} = \mathcal{L}(\mathcal{Z}).$$

To complete the proof we still have to show that $\mathcal{L}(\mathcal{Z}) \subset \overline{\mathcal{H}^0(\mathcal{Z})}$. Take any $h \in \mathcal{L}(\mathcal{Z})$, and choose some $x \in \overline{\mathcal{Z}+\mathcal{Z}^{[\perp]}}$ such that $h = x+\mathcal{Z}$. Then there exists a sequence $x_n \in \mathcal{Z}+\mathcal{Z}^{[\perp]}$ such that $x_n \to x$ in $\mathcal{X}$ as $n \to \infty$. By the definition of $\mathcal{H}^0(\mathcal{Z})$, $x_n + \mathcal{Z} \in \mathcal{H}^0(\mathcal{Z})$ for all n. By the continuity of the quotient map $\pi_{\mathcal{Z}}$, $x_n+\mathcal{Z} \to x+\mathcal{Z}$ in $\mathcal{X}/\mathcal{Z}$, and so $h = x+\mathcal{Z} \in \overline{\mathcal{H}^0(\mathcal{Z})}$. □

2.7. *Relation between $\mathcal{H}(\mathcal{Z})$ and $\mathcal{H}(\mathcal{Z}^{[\perp]})$*

If $\mathcal{Z}$ is a maximal nonnegative subspace of a Kreĭn space $\mathcal{X}$, then $\mathcal{Z}^{[\perp]}$ can be interpreted as a maximal nonnegative subspace of the anti-space $-\mathcal{X}$ of $\mathcal{X}$. We can therefore repeat the same construction presented above with $\mathcal{X}$ replaced by $-\mathcal{X}$, and with $\mathcal{Z}$ replaced by $\mathcal{Z}^{[\perp]}$ to get the

Hilbert space $\mathcal{H}(\mathcal{Z}^{[\perp]})$ which is a Hilbert completion of the inner product space $\mathcal{H}^0(\mathcal{Z}^{[\perp]})$. The unitary space $\mathcal{U}(\mathcal{Z}^{[\perp]})$ is obtained in an analogous way, and it can be canonically embedded in the Hilbert space $\mathcal{H}(\mathcal{Z}^{[\perp]})$.

As the following theorem shows, there are certain connections between the two spaces $\mathcal{H}(\mathcal{Z})$ and $\mathcal{H}(\mathcal{Z}^{[\perp]})$. To explore this connection we investigate the following two subspaces of $\mathcal{X}$:

$$\mathcal{X}(\mathcal{Z}) = \{x \in \mathcal{X} \mid x + \mathcal{Z} \in \mathcal{H}(\mathcal{Z})\}, \tag{2.18}$$

$$\mathcal{X}(\mathcal{Z}^{[\perp]}) = \{x \in \mathcal{X} \mid x + \mathcal{Z}^{[\perp]} \in \mathcal{H}(\mathcal{Z}^{[\perp]})\}. \tag{2.19}$$

Indeed, they are subspaces, since $\mathcal{X}(\mathcal{Z})$ is the inverse image of $\mathcal{H}(\mathcal{Z})$ under the quotient map $\pi_{\mathcal{Z}}$, and $\mathcal{X}(\mathcal{Z}^{[\perp]})$ is the inverse image of $\mathcal{H}(\mathcal{Z}^{[\perp]})$ under the quotient map $\pi_{\mathcal{Z}^{[\perp]}}$.

Theorem 2.12. *Let $\mathcal{Z}$ be a maximal nonnegative subspace of a Kreĭn space $\mathcal{Z}$, let $\mathcal{Z}_0 = \mathcal{Z} \cap \mathcal{Z}^{[\perp]}$, and let $\mathcal{X}(\mathcal{Z})$, $\mathcal{X}(\mathcal{Z}^{[\perp]})$, $\mathcal{H}(\mathcal{Z})$, $\mathcal{H}^0(\mathcal{Z})$, $\mathcal{H}(\mathcal{Z}^{[\perp]})$, and $\mathcal{H}^0(\mathcal{Z}^{[\perp]})$ be the spaces defined above. Then the following claims are true.*

(1) $\mathcal{X}(\mathcal{Z}) = \mathcal{X}(\mathcal{Z}^{[\perp]})$.
(2) $\mathcal{Z} + \mathcal{Z}^{[\perp]} \subset \mathcal{X}(\mathcal{Z}) \subset \overline{\mathcal{Z} + \mathcal{Z}^{[\perp]}}$.
(3) *$\mathcal{Z}_0$ is the maximal subspace of $\mathcal{Z} + \mathcal{Z}^{[\perp]}$ which is orthogonal to $\mathcal{Z} + \mathcal{Z}^{[\perp]}$, and the same statement remains true if we replace $\mathcal{Z} + \mathcal{Z}^{[\perp]}$ by $\mathcal{X}(\mathcal{Z})$ or by $\overline{\mathcal{Z} + \mathcal{Z}^{[\perp]}}$.*
(4) *Let $\mathcal{F}$ be one of the spaces listed in part* (2). *Then the formula*

$$[y_1 + \mathcal{Z}_0, y_2 + \mathcal{Z}_0]_{\mathcal{F}/\mathcal{Z}_0} = [y_1, y_2]_{\mathcal{X}}, \quad y_1, y_2 \in \mathcal{F}, \tag{2.20}$$

defines a nondegenerate indefinite inner product in the quotient space $\mathcal{F}/\mathcal{Z}_0$.
(5) *The formula*

$$Q(x + \mathcal{Z}_0) = \begin{bmatrix} x + \mathcal{Z} \\ x + \mathcal{Z}^{[\perp]} \end{bmatrix}, \quad x \in \mathcal{X}(\mathcal{Z}), \tag{2.21}$$

defines a linear isometric map from the space $\mathcal{X}(\mathcal{Z})/\mathcal{Z}_0$ with the inner product defined in (2.20) *with $\mathcal{F} = \mathcal{X}(\mathcal{Z})$ into the Kreĭn space $-\mathcal{H}(\mathcal{Z})\,[\dot{+}]\,\mathcal{H}(\mathcal{Z}^{[\perp]})$. The image of $(\mathcal{Z} + \mathcal{Z}^{\perp})/\mathcal{Z}_0$ under Q is $-\mathcal{H}^0(\mathcal{Z})[\dot{+}]\mathcal{H}^0(\mathcal{Z}^{[\perp]})$. In particular, both $\mathcal{R}(Q|_{(\mathcal{Z}+\mathcal{Z}^{\perp})/\mathcal{Z}_0})$ and $\mathcal{R}(Q)$ are dense in $-\mathcal{H}(\mathcal{Z})\,[\dot{+}]\,\mathcal{H}(\mathcal{Z}^{[\perp]})$.*

Proof.

Proof of claim (1). Let $x \in \mathcal{X}$ and $x + \mathcal{Z} \in \mathcal{H}(\mathcal{Z})$. It follows from (1.6) and (2.12) that

$$\begin{aligned}[x + z^\dagger, x + z^\dagger]_{\mathcal{X}} &= [x, x]_{\mathcal{X}} + [z^\dagger, x]_{\mathcal{X}} + [x, z^\dagger]_{\mathcal{X}} + [z^\dagger, z^\dagger]_{\mathcal{X}} \\ &= \|x + \mathcal{Z}\|^2_{\mathcal{H}(\mathcal{Z})} + [x, x]_{\mathcal{X}} - \|x + z^\dagger + \mathcal{Z}\|^2_{\mathcal{H}(\mathcal{Z})}.\end{aligned}$$

Taking the supremum over all $z^\dagger \in \mathcal{Z}^{[\perp]}$ we find that $x + \mathcal{Z}^{[\perp]} \in \mathcal{H}(\mathcal{Z}^{[\perp]})$, and that

$$\|x + \mathcal{Z}^{[\perp]}\|^2_{\mathcal{H}(\mathcal{Z}^{[\perp]})} = \|x + \mathcal{Z}\|^2_{\mathcal{H}(\mathcal{Z})} + [x, x]_{\mathcal{X}} - \inf_{z^\dagger \in \mathcal{Z}^{[\perp]}} \|x + z^\dagger + \mathcal{Z}\|^2_{\mathcal{H}(\mathcal{Z})}.$$

Since $\mathcal{H}^0(\mathcal{Z})$ is dense in $\mathcal{H}(\mathcal{Z})$, the infimum is zero, and hence

$$[x,x]_{\mathcal{X}} = -\|x+\mathcal{Z}\|^2_{\mathcal{H}(\mathcal{Z})} + \|x+\mathcal{Z}^{[\perp]}\|^2_{\mathcal{H}(\mathcal{Z}^{[\perp]})}. \tag{2.22}$$

That $x+\mathcal{Z} \in \mathcal{H}(\mathcal{Z})$ whenever $x+\mathcal{Z}^{[\perp]} \in \mathcal{H}(\mathcal{Z}^{[\perp]})$ can be proved in the same way (or by replacing $\mathcal{X}$ by $-\mathcal{X}$ and $\mathcal{Z}$ by $\mathcal{Z}^{[\perp]}$).

Proof of claim (2). That $\mathcal{Z}+\mathcal{Z}^{[\perp]} \subset \mathcal{X}(\mathcal{Z})$ follows from part (3) of Lemma 2.4, and that $\mathcal{X}(\mathcal{Z}) \subset \overline{\mathcal{Z}+\mathcal{Z}^{[\perp]}}$ follows from Proposition 2.11.

Proof of claims (3), (4). The straightforward proofs of claims (3) and (4) are left to the reader.

Proof of claim (5). That Q is well defined it follows from the fact that $\mathcal{N}(\pi_{\mathcal{Z}}) \cap \mathcal{N}(\pi_{\mathcal{Z}^{[\perp]}}) = \mathcal{Z}_0$. That Q is an isometry follows from (2.20) and (2.22). The remaining claims are obvious. □

Remark 2.13. The operator Q in (2.21) is also a well-defined continuous linear map from $\overline{\mathcal{Z}+\mathcal{Z}^{[\perp]}}/\mathcal{Z}_0$ into $\left[\begin{smallmatrix}\mathcal{X}/\mathcal{Z} \\ \mathcal{X}/\mathcal{Z}^{[\perp]}\end{smallmatrix}\right]$, but the range of this extended map is not necessarily contained in $-\mathcal{H}(\mathcal{Z})\ [\dotplus]\ \mathcal{H}(\mathcal{Z}^{[\perp]})$, neither does it necessarily contain $-\mathcal{H}(\mathcal{Z})\ [\dotplus]\ \mathcal{H}(\mathcal{Z}^{[\perp]})$. In fact, the space $\mathcal{X}(\mathcal{Z})$ is the maximal subspace of $\mathcal{X}$ whose image under this extended map Q is contained in $-\mathcal{H}(\mathcal{Z})\,[\dotplus]\,\mathcal{H}(\mathcal{Z}^{[\perp]})$. However, the image of $\mathcal{X}(\mathcal{Z})$ need not be all of $-\mathcal{H}(\mathcal{Z})\,[\dotplus]\,\mathcal{H}(\mathcal{Z}^{[\perp]})$. To see this it suffices to observe that the intersection of the image with the closed subspace $-\mathcal{H}(\mathcal{Z})$ of $-\mathcal{H}(\mathcal{Z})\ [\dotplus]\ \mathcal{H}(\mathcal{Z}^{[\perp]})$ is equal to $-\mathcal{H}^0(\mathcal{Z})$. It is not difficult to show that the image is all of $-\mathcal{H}(\mathcal{Z})\ [\dotplus]\ \mathcal{H}(\mathcal{Z}^{[\perp]})$ if and only if $\mathcal{H}^0(\mathcal{Z}) = \mathcal{H}(\mathcal{Z})$.

Theorem 2.14. *Let $\mathcal{Z}$ be a maximal nonnegative subspace of a Kreĭn space $\mathcal{X}$, let $\mathcal{Z}_0 = \mathcal{Z} \cap \mathcal{Z}^{[\perp]}$, and let $\mathcal{H}(\mathcal{Z})$, $\mathcal{H}(\mathcal{Z}^{[\perp]})$, $\mathcal{H}^0(\mathcal{Z})$, $\mathcal{H}^0(\mathcal{Z}^{[\perp]})$, $\mathcal{U}(\mathcal{Z})$, $\mathcal{U}(\mathcal{Z}^{[\perp]})$, and $\mathcal{X}(\mathcal{Z})$ be the spaces defined above. Then the following conditions are equivalent.*

(1) $\mathcal{Z} = \mathcal{Z}_0\ [\dotplus]\ \mathcal{Z}_+$ *where $\mathcal{Z}_+$ is a uniformly positive subspace of $\mathcal{Z}$.*
(2) $\mathcal{Z}^{[\perp]} = \mathcal{Z}_-\ [\dotplus]\ \mathcal{Z}_0$ *where $\mathcal{Z}_-$ is a uniformly negative subspace of $\mathcal{Z}$.*
(3) $\mathcal{Z}+\mathcal{Z}^{[\perp]} = \mathcal{Z}_-\ [\dotplus]\ \mathcal{Z}_0\ [\dotplus]\ \mathcal{Z}_+$, *where $\mathcal{Z}_-$ and $\mathcal{Z}_+$ are uniformly negative and positive subspaces, respectively, of $\mathcal{X}$.*
(4) $\mathcal{Z}+\mathcal{Z}^{[\perp]}$ *is closed in $\mathcal{X}$.*
(5) $\mathcal{U}(\mathcal{Z})$ *is a Hilbert space.*
(6) $\mathcal{U}(\mathcal{Z}^{[\perp]})$ *is a Hilbert space.*
(7) $\mathcal{H}^0(\mathcal{Z}) = \mathcal{H}(\mathcal{Z})$.
(8) $\mathcal{H}^0(\mathcal{Z}^{[\perp]}) = \mathcal{H}(\mathcal{Z}^{[\perp]})$.
(9) $(\mathcal{Z}+\mathcal{Z}^{[\perp]})/\mathcal{Z}_0$ *is a Kreĭn space with the inner product defined in* (2.20).
(10) $\mathcal{X}(\mathcal{Z})/\mathcal{Z}_0$ *is a Kreĭn space with the inner product defined in* (2.20).
(11) $\mathcal{X}(\mathcal{Z}) = \mathcal{Z}+\mathcal{Z}^{[\perp]}$.

Proof.

Proof of the equivalence (2) ⇔ (5). Let $\mathcal{Z}_-$ be an arbitrary direct complement to $\mathcal{Z}_0$ in $\mathcal{Z}^{\perp}$. Then by, e.g. [7, Lemmas 5.1 and 5.2, p. 11], $\mathcal{Z}_-$ with the inner product inherited from $-\mathcal{X}$ is isometrically isomorphic to $\mathcal{U}(\mathcal{Z})$. Thus, $\mathcal{U}(\mathcal{Z})$ is a complete if and only if $\mathcal{Z}_-$ is complete, and this is true if and only if $\mathcal{Z}_-$ is uniformly negative.

Proof of the equivalence (5) ⇔ (7). The operator S defined in (2.15) is an isometric map of $\mathcal{U}(\mathcal{Z})$ onto $\mathcal{H}^0(\mathcal{Z})$, so $\mathcal{U}(\mathcal{Z})$ is complete if and only $\mathcal{H}^0(\mathcal{Z})$ is complete, and this is true if and only if $\mathcal{H}^0(\mathcal{Z}) = \mathcal{H}(\mathcal{Z})$ (since $\mathcal{H}(\mathcal{Z})$ is a completion of $\mathcal{H}^0(\mathcal{Z})$.

Proof of the equivalence (7) ⇔ (9). The operator Q defined in (2.21) is an isometric map of $(\mathcal{Z} + \mathcal{Z}^{[\perp]})/\mathcal{Z}_0$ onto $-\mathcal{H}^0(\mathcal{Z})\,[\dot{+}]\,\mathcal{H}^0(\mathcal{Z}^{[\perp]})$. Thus, $(\mathcal{Z} + \mathcal{Z}^{[\perp]})/\mathcal{Z}_0$ is complete if and only if $-\mathcal{H}^0(\mathcal{Z})\,[\dot{+}]\,\mathcal{H}^0(\mathcal{Z}^{[\perp]})$ is complete, which is true if and only if $\mathcal{H}^0(\mathcal{Z}) = \mathcal{H}(\mathcal{Z})$.

Proof of the equivalence (7) ⇔ (11). This follows from the definitions of $\mathcal{H}^0(\mathcal{Z})$ and $\mathcal{X}(\mathcal{Z})$.

Proof of the implication (9) ⇒ (10). The operator Q defined in (2.21) is an isometric map of $\mathcal{X}(\mathcal{Z})/\mathcal{Z}_0$ into $-\mathcal{H}(\mathcal{Z})\,[\dot{+}]\,\mathcal{H}(\mathcal{Z}^{[\perp]})$ which contains the image of $(\mathcal{Z} + \mathcal{Z}^{[\perp]})/\mathcal{Z}_0$. If $(\mathcal{Z} + \mathcal{Z}^{[\perp]})/\mathcal{Z}_0$ is complete, then this image coincides with $-\mathcal{H}(\mathcal{Z})\,[\dot{+}]\,\mathcal{H}(\mathcal{Z}^{[\perp]})$ (being dense in $-\mathcal{H}(\mathcal{Z})\,[\dot{+}]\,\mathcal{H}(\mathcal{Z}^{[\perp]})$), and hence also the image of $\mathcal{X}(\mathcal{Z})/\mathcal{Z}_0$ must coincide with $-\mathcal{H}(\mathcal{Z})\,[\dot{+}]\,\mathcal{H}(\mathcal{Z}^{[\perp]})$. This means that $\mathcal{X}(\mathcal{Z})/\mathcal{Z}_0$ is complete.

Proof of the implication (10) ⇒ (7). If (10) holds, then the image of $\mathcal{X}(\mathcal{Z})/\mathcal{Z}_0$ under the isometric operator Q in (2.21) coincides with $-\mathcal{H}(\mathcal{Z})\,[\dot{+}]\,\mathcal{H}(\mathcal{Z}^{[\perp]})$ (being dense in $-\mathcal{H}(\mathcal{Z})\,[\dot{+}]\,\mathcal{H}(\mathcal{Z}^{[\perp]})$). In particular, the range must contain every vector of the form $\left[\begin{smallmatrix} h \\ 0 \end{smallmatrix}\right]$, where h is an arbitrary vector in $\mathcal{H}(\mathcal{Z})$ (and the $\mathcal{H}(\mathcal{Z}^{[\perp]})$-component is zero). However, it is easy to see that the intersection of the image of $\mathcal{X}(\mathcal{Z})/\mathcal{Z}_0$ under Q with the subspace where the $\mathcal{H}(\mathcal{Z}^{[\perp]})$-component is zero is equal to $\mathcal{H}^0(\mathcal{Z})$. This implies that $\mathcal{H}(\mathcal{Z}) \subset \mathcal{H}^0(\mathcal{Z})$, and so $\mathcal{H}^0(\mathcal{Z}) = \mathcal{H}(\mathcal{Z})$.

Proof of the equivalence (1) ⇔ (2)*,* (5) ⇔ (6)*, and* (7) ⇔ (8). These equivalences follow from the equivalence of (1), (5), (7), and (9) and the fact that (9) is invariant under the interchange of $\mathcal{Z}$ and $\mathcal{Z}^{[\perp]}$.

Proof of the implication (1) *&* (2) ⇒ (3). This implication is trivial.

Proof of the implication (3) ⇒ (4). If (3) holds, then $\mathcal{X} = \mathcal{Z}_-\,[\dot{+}]\,\mathcal{Z}_-^{[\perp]}$ and also $\mathcal{X} = \mathcal{Z}_+\,[\dot{+}]\,\mathcal{Z}_+^{[\perp]}$ (see, e.g., [7, Theorem 3.4, p. 104]). This means that there exist bounded orthogonal projections $P_\mp$ of $\mathcal{X}$ onto $\mathcal{Z}_\mp$. Out of these P_- vanishes on $\mathcal{Z}_0\,[\dot{+}]\,\mathcal{Z}_+$ and P_+ vanishes on $\mathcal{Z}_-\,[\dot{+}]\,\mathcal{Z}_0$. For each $x \in \mathcal{X}$, define $P_0 x = x - P_- x - P_+ x$. Then also P_0 is bounded, and for every $x \in \mathcal{Z} + \mathcal{Z}^{[\perp]}$ we have $x = P_- x + P_0 x + P_+ x$, where $P_- x \in \mathcal{Z}_-$, $P_0 x \in \mathcal{Z}_0$, and $P_+ x \in \mathcal{Z}_+$. Let $x_n \in \mathcal{Z} + \mathcal{Z}^\perp$, and let $x_n \to x \in \mathcal{X}$. Then $x_n = P_- x_n + P_0 x_n + P_+ x_n \to P_- x + P_0 x + P_+ x = x$, and hence $x \in \mathcal{Z}_-\,[\dot{+}]\,\mathcal{Z}_0\,[\dot{+}]\,\mathcal{Z}_+ = \mathcal{Z} + \mathcal{Z}^{[\perp]}$. Thus $\mathcal{Z} + \mathcal{Z}^{[\perp]}$ is closed.

Proof of the implication (4) ⇒ (11). This follows from part (2) of Theorem 2.12. □

3. Connection with the complementary space $\mathcal{H}(A)$

In this section we shall discuss the connection between the space $\mathcal{H}(\mathcal{Z})$ for some maximal nonnegative subspace $\mathcal{Z}$ of a Kreĭn space $\mathcal{X}$, and the de Branges complementary space $\mathcal{H}(A)$ induced by the contraction A that appears in the graph representation of $\mathcal{Z}$ with respect to some fundamental decomposition of $\mathcal{X}$. We shall also give alternative proofs of some of the results in Section 2 that depend on the standard graph representation of a maximal nonnegative subspace of a Kreĭn space.

3.1. The graph representation of a maximal nonnegative subspace

Lemma 3.1. *Let A be a linear contraction from a Hilbert spaces $\mathcal{U}$ to a Hilbert space $\mathcal{Y}$, and let*

$$\mathcal{Z} = \left\{ z = \begin{bmatrix} Au \\ u \end{bmatrix} \;\middle|\; u \in \mathcal{U} \right\} \tag{3.1}$$

be the graph of A. Then $\mathcal{Z}$ is a maximal nonnegative subspace of the Kreĭn space $-\mathcal{Y}\,[\dot{+}]\,\mathcal{U}$. Conversely, let $\mathcal{Z}$ be a maximal nonnegative subspace of a Kreĭn space $\mathcal{X}$, and let $\mathcal{X} = -\mathcal{Y}\,[\dot{+}]\,\mathcal{U}$ be a fundamental decomposition of $\mathcal{X}$. Then $\mathcal{Z}$ has the graph representation (3.1) *for a unique linear contraction $A : \mathcal{U} \to \mathcal{Y}$.*

Proof. See, e.g., [7, Theorem 1.7, p. 54 and Theorem 4.2, pp. 105, 106]. □

Lemma 3.2. *The orthogonal companion $\mathcal{Z}^{[\perp]}$ of the maximal nonnegative subspace $\mathcal{Z}$ in Lemma* 3.1 *has the graph representation*

$$\mathcal{Z}^{[\perp]} = \left\{ z = \begin{bmatrix} y \\ A^* y \end{bmatrix} \;\middle|\; y \in \mathcal{Y} \right\}. \tag{3.2}$$

Proof. This is well known, and it is a simple corollary of Lemma 3.1. □

Alternative proof of Lemma 2.1. Let $\mathcal{U} = \mathcal{Y}^{[\perp]}$. Then $\mathcal{X} = \mathcal{Y}\,[\dot{+}]\,\mathcal{U}$ is a fundamental decomposition of $\mathcal{X}$, and hence $\mathcal{Z}$ has the graph representation (3.1) for some $A \in \mathcal{B}(\mathcal{U}; -\mathcal{Y})$. This implies that every $x \in \mathcal{X}$ has the unique decomposition

$$x = \begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} Au \\ u \end{bmatrix} + \begin{bmatrix} y - Au \\ 0 \end{bmatrix}, \tag{3.3}$$

where $\left[\begin{smallmatrix} Au \\ u \end{smallmatrix}\right] \in \mathcal{Z}$ and $\left[\begin{smallmatrix} y-Au \\ 0 \end{smallmatrix}\right] \in \mathcal{Y}$. □

The following lemma is a slight extension of Lemma 2.2.

Lemma 3.3. *Let $\mathcal{Z}$ be a maximal nonnegative subspace of a Kreĭn space $\mathcal{X}$, and let $\mathcal{X} = -\mathcal{Y}\,[\dot{+}]\,\mathcal{U}$ be a fundamental decomposition of $\mathcal{X}$. Then the operator T in Lemma* 2.2 *is a bounded linear operator $\mathcal{X}/\mathcal{Z} \to \mathcal{Y}$ with a bounded inverse, and*

$$T(x + \mathcal{Z}) = y - Au, \qquad z = \begin{bmatrix} y \\ u \end{bmatrix} \in \begin{bmatrix} \mathcal{Y} \\ \mathcal{U} \end{bmatrix}. \tag{3.4}$$

Proof. Most of this follows from Lemma 2.2. The explicit formula (3.4) for $T(x + \mathcal{Z})$ follows from (3.3). □

3.2. The de Branges complement of a range space

Let $A \in \mathcal{B}(\mathcal{U};\mathcal{Y})$. By definition, the *range space* $\mathcal{M}(A)$ of A is the range of A endowed with the *range norm* which makes $A_{|[\mathcal{N}(A)]^\perp}$ a unitary operator $[\mathcal{N}(A)]^\perp \to \mathcal{M}(A)$. In other words, to each $y \in \mathcal{R}(A)$ we can find a unique vector $u \in [\mathcal{N}(A)]^\perp$ such that $y = Au$, and define $\|y\|_{\mathcal{M}(A)} = \|u\|_{\mathcal{U}}$. The basic properties of the space $\mathcal{M}(A)$ can be found in many books, including [12, pp. 2, 3].

We next assume that A is a contraction, and proceed to define the *de Branges complement* $\mathcal{H}(A)$ of $\mathcal{M}(A)$. One starts by defining $\mathcal{H}(A)$ to be the following subset of $\mathcal{Y}$:

$$\mathcal{H}(A) = \{y \in \mathcal{Y} \mid \|y\|_{\mathcal{H}(A)} < \infty\}, \tag{3.5}$$

where

$$\|y\|^2_{\mathcal{H}(A)} = \sup_{u\in\mathcal{U}} \bigl(\|y - Au\|^2 - \|u\|^2_{\mathcal{U}}\bigr). \tag{3.6}$$

It is known from the work of de Branges and Rovnyak [9,10] that $\mathcal{H}(A)$ is a linear subspace of $\mathcal{Y}$, that $\|\cdot\|_{\mathcal{H}(A)}$ defined in (3.6) is a norm in $\mathcal{H}(A)$ induced by a Hilbert space inner product, and that $\mathcal{H}(A)$ with this norm is continuously (but not necessarily densely) contained in $\mathcal{Y}$.

The following well-known facts explain in which sense $\mathcal{H}(A)$ can be interpreted as a complement of $\mathcal{M}(A)$ (see, e.g., [12, Chapter 1] for the proofs):

(1) $\mathcal{Y} = \mathcal{M}(A) + \mathcal{H}(A)$, i.e., every $y \in \mathcal{Y}$ can be written as a sum $y = y_1 + y_2$, where $y_1 \in \mathcal{M}(A)$ and $y_2 \in \mathcal{H}(A)$. The sum is direct (i.e., $\mathcal{M}(A)$ and $\mathcal{H}(A)$ are closed and $\mathcal{M}(A) \cap \mathcal{H}(A) = 0$) if and only if it is orthogonal, i.e., $\mathcal{H}(A) = \mathcal{M}(A)^\perp$, and this is true if and only if A is a partial isometry (i.e., A is an isometry on $[\mathcal{N}(A)]^\perp$).
(2) If $y = y_1 + y_2$ with $y_1 \in \mathcal{M}(A)$ and $y_2 \in \mathcal{H}(A)$, then $\|y\|^2_{\mathcal{Y}} \leqslant \|y_1\|^2_{\mathcal{M}(A)} + \|y_2\|^2_{\mathcal{H}(A)}$. Moreover, for each $y \in \mathcal{Y}$ there exist unique vectors $y_1 \in \mathcal{M}(A)$ and $y_2 \in \mathcal{H}(A)$ such that $y = y_1 + y_2$ and $\|y\|^2_{\mathcal{Y}} = \|y_1\|^2_{\mathcal{M}(A)} + \|y_2\|^2_{\mathcal{H}(A)}$, namely $y_1 = AA^*y$ and $y_2 = (1_{\mathcal{Y}} - AA^*)y$.

The above definition of $\mathcal{H}(A)$ follows the original approach taken by de Branges and Rovnyak in [9,10]. It was later realized that $\mathcal{H}(A)$ also can be characterised in a different way, namely

(3) $\mathcal{H}(A) = \mathcal{M}((1_{\mathcal{Y}} - AA^*)^{1/2})$.

A proof of this fact can be found in, e.g., [12, Note (NI-6), pp. 7, 8].

3.3. The connection between $\mathcal{H}(\mathcal{Z})$ and $\mathcal{H}(A)$

We proceed to investigate the connection between $\mathcal{H}(\mathcal{Z})$ and $\mathcal{H}(A)$.

Lemma 3.4. *The bounded linear operator $T : \mathcal{X}/\mathcal{Z} \to \mathcal{Y}$ defined in Lemma* 2.2 *maps the subset $\mathcal{H}(\mathcal{Z}) \subset \mathcal{X}/\mathcal{Z}$ defined in* (1.1) *one-to-one onto the subspace $\mathcal{H}(A) \subset \mathcal{Y}$ defined in* (3.5)*, and $\|x + \mathcal{Z}\|_{\mathcal{H}(\mathcal{Z})} = \|T(x + \mathcal{Z})\|_{\mathcal{H}(A)}$ for all $x + \mathcal{Z} \in \mathcal{H}(\mathcal{Z})$.*

Proof. Because of (1.1) and (3.5), it suffices to show that $\|x+\mathcal{Z}\|_{\mathcal{H}(\mathcal{Z})} = \|T(x+\mathcal{Z})\|_{\mathcal{H}(A)}$ for all $x \in \mathcal{X}$ with $x+\mathcal{Z} \in \mathcal{H}(\mathcal{Z})$ (finite or infinite), or equivalently, that $\|T^{-1}y\|^2_{\mathcal{H}(\mathcal{Z})} = \|y\|^2_{\mathcal{H}(A)}$ for all $y \in \mathcal{H}(A)$. However, this follows from the fact that the right-hand side is given by (3.6), whereas the left-hand side is given by

$$\begin{aligned}\|T^{-1}y\|^2_{\mathcal{H}(\mathcal{Z})} &= \sup_{z\in\mathcal{Z}}\bigl(-[y-z, y-z]_{\mathcal{X}}\bigr)\\ &= \sup_{u\in\mathcal{U}}\left(-\left[\begin{bmatrix} y\\ 0\end{bmatrix} - \begin{bmatrix} Au\\ u\end{bmatrix}, \begin{bmatrix} y\\ 0\end{bmatrix} - \begin{bmatrix} Au\\ u\end{bmatrix}\right]_{\mathcal{X}}\right)\\ &= \sup_{u\in\mathcal{U}}\bigl(\|y-Au\|^2_{\mathcal{Y}} - \|u\|^2_{\mathcal{U}}\bigr) = \|y\|^2_{\mathcal{H}(A)}. \qquad \Box\end{aligned}$$

Alternative proof of Theorem 2.3. Theorem 2.3 follows from Lemma 3.4 and the fact that $\mathcal{H}(A)$ is a Hilbert space which is continuously (but not necessarily densely) contained in $\mathcal{Y}$. □

Corollary 3.5. *The restriction of the operator T in Lemma* 3.4 *to $\mathcal{H}(\mathcal{Z})$ is a unitary map from $\mathcal{H}(\mathcal{Z})$ to $\mathcal{H}(A)$.*

Proof. This, too, follows from Lemma 3.4. □

Lemma 3.6. *The bounded linear operator $T : \mathcal{X}/\mathcal{Z} \to \mathcal{Y}$ defined in Lemma* 2.2 *maps the subspace $\mathcal{H}^0(\mathcal{Z})$ of $\mathcal{X}/\mathcal{Z}$ defined in* (1.5) *one-to-one onto the range of the operator* $(1_{\mathcal{Y}} - AA^*)$.

Proof. Take any $z^\dagger \in \mathcal{Z}^{[\perp]}$. Then $z^\dagger = \left[\begin{smallmatrix} y\\ A^*y\end{smallmatrix}\right]$ for some $y \in \mathcal{Y}$. Consequently, $T(z^\dagger + \mathcal{Z}) = y - A(A^*y) = (1_{\mathcal{Y}} - AA^*)y$. Thus, $\mathcal{R}(T_{|\mathcal{H}^0(\mathcal{Z})}) \subset \mathcal{R}(1_{\mathcal{Y}} - AA^*)$.

Conversely, suppose that $x \in \mathcal{X}$, and that $T(x+\mathcal{Z}) = (1_{\mathcal{Y}} - AA^*)y$ for some $y \in \mathcal{Y}$. Then $x + \mathcal{Z} = (1_{\mathcal{Y}} - AA^*)y + \mathcal{Z} = \{\left[\begin{smallmatrix} (1_{\mathcal{Y}}-AA^*)y+Au\\ u\end{smallmatrix}\right] \mid u \in \mathcal{U}\}$. In particular, by replacing u by A^*y+u we find that $x + \mathcal{Z} = \left[\begin{smallmatrix} y\\ A^*y\end{smallmatrix}\right] + \mathcal{Z}$. Thus, $x+\mathcal{Z} = z^\dagger + \mathcal{Z}$, where $z^\dagger = \left[\begin{smallmatrix} y\\ A^*y\end{smallmatrix}\right] \in \mathcal{Z}^{[\perp]}$. Thus, $\mathcal{R}(1_{\mathcal{Y}} - AA^*) \subset \mathcal{R}(T_{|\mathcal{H}^0(\mathcal{Z})})$. □

Alternative proof of part (4) of Lemma 2.4. Write $z^\dagger = \left[\begin{smallmatrix} y\\ A^*y\end{smallmatrix}\right]$, and denote $T(x+\mathcal{Z})$ by y'. Then $y' = (1_{\mathcal{Y}} - AA^*)^{1/2}y_1$ for some $y_1 \in [\mathcal{N}((1_{\mathcal{Y}} - AA^*)^{1/2})]^\perp$, and

$$\begin{aligned}\bigl(z^\dagger + \mathcal{Z}, x + \mathcal{Z}\bigr)_{\mathcal{H}(\mathcal{Z})} &= \bigl(T\bigl(z^\dagger + \mathcal{Z}\bigr), Tx + \mathcal{Z}\bigr)_{\mathcal{H}(A)} = \bigl((1_{\mathcal{Y}} - AA^*)y, y'\bigr)_{\mathcal{H}(A)}\\ &= \bigl((1_{\mathcal{Y}} - AA^*)y, (1_{\mathcal{Y}} - AA^*)^{1/2}y_1\bigr)_{\mathcal{M}((1_{\mathcal{Y}}-AA^*)^{1/2})}\\ &= \bigl((1_{\mathcal{Y}} - AA^*)^{1/2}y, y_1\bigr)_{\mathcal{Y}} = \bigl(y, (1_{\mathcal{Y}} - AA^*)^{1/2}y_1\bigr)_{\mathcal{Y}}\\ &= (y, y')_{\mathcal{Y}} = -\bigl[z^\dagger, y'\bigr]_{\mathcal{X}} = -\bigl[z^\dagger, x\bigr]_{\mathcal{X}}. \qquad \Box\end{aligned}$$

Alternative proof of Theorem 2.5. That $\mathcal{H}^0(\mathcal{Z})$ is a dense subset of $\mathcal{H}(\mathcal{Z})$ follows from Corollary 3.5 and the fact that $T(\mathcal{H}^0(\mathcal{Z})) = \mathcal{R}(1_{\mathcal{Y}} - AA^*)$ is a dense subspace of $T(\mathcal{H}(\mathcal{Z})) = \mathcal{H}(A) = \mathcal{M}((1_{\mathcal{Y}} - AA^*)^{1/2})$. That the inner product in $\mathcal{H}^0(\mathcal{Z})$ inherited from $\mathcal{H}(\mathcal{Z})$ is given by (1.6) follows from part (iv) of Lemma 2.4. □

Alternative proof of part (3) of Lemma 2.4. If $h \in \mathcal{H}^0(\mathcal{Z})$, then $h = z^\dagger + \mathcal{Z}$ for some $z^\dagger \in \mathcal{Z}^{[\perp]}$, and by Theorem 2.5,

$$\sup_{x \in h}\bigl(-[x, x]_{\mathcal{X}}\bigr) = -\bigl[z^\dagger, z^\dagger\bigr]_{\mathcal{X}}.$$

Thus, the supremum is achieved for $x = z^\dagger$.

Conversely, suppose that the supremum in (1.1) is achieved for some $x_0 \in h$. Let $y = Th$, where T is the operator in Lemma 3.3. Then $x_0 = y + z_0$ for some $z_0 \in \mathcal{Z}$, and

$$\max\bigl\{-[y + z, y + z]_{\mathcal{X}} \bigm| z \in \mathcal{Z}\bigr\} = -[x + z_0, x + z_0]_{\mathcal{X}}.$$

By using the graph representation (3.1) we can write $z = \left[\begin{smallmatrix} Au \\ u \end{smallmatrix}\right]$ and $z_0 = \left[\begin{smallmatrix} Au_0 \\ u_0 \end{smallmatrix}\right]$ for some $u, u_0 \in \mathcal{U}$ to get

$$\max\bigl\{\|y + Au\|_{\mathcal{Y}}^2 - \|u\|_{\mathcal{U}}^2 \bigm| u \in \mathcal{U}\bigr\} = \|y - Au_0\|_{\mathcal{Y}}^2 - \|u_0\|_{\mathcal{U}}^2. \tag{3.7}$$

We claim that $u_0 \in (\mathcal{N}(A))^\perp$, and prove this as follows. It is always possible to write $u_0 = u_1 + u_2$ where $u_1 \in (\mathcal{N}(A))^\perp$ and $u_2 \in \mathcal{N}(A)$. If $u_2 \neq 0$, then

$$\|y - Au_0\|_{\mathcal{Y}}^2 - \|u_0\|_{\mathcal{U}}^2 = \|y - Au_1\|_{\mathcal{Y}}^2 - \|u_1\|_{\mathcal{U}}^2 - \|u_2\|_{\mathcal{U}}^2 < \|y - Au_1\|_{\mathcal{Y}}^2 - \|u_1\|_{\mathcal{U}}^2,$$

contradicting (3.7). Thus, $u_0 \in (\mathcal{N}(A))^\perp$. Define $y_0 = Au_0$ and $y' = y - y_0$. Then $y' = y + (-y_0)$, $y \in \mathcal{H}(A)$, $-y_0 \in \mathcal{M}(A)$, and

$$\|y'\|_{\mathcal{Y}}^2 = \|y\|_{\mathcal{H}(A)}^2 + \|-y_0\|_{\mathcal{M}(A)}^2.$$

Consequently, by property (2) of the complementary spaces $\mathcal{H}(A)$ and $\mathcal{M}(A)$ listed earlier in this section, $y = (1_{\mathcal{Y}} - AA^*)y'$, and hence $y \in \mathcal{R}(1_{\mathcal{Y}} - AA^*)$. By Lemma 3.6, $h = T^{-1}y \in \mathcal{H}^0(\mathcal{Z})$. $\square$

We end this section with the following addition to Theorem 2.14.

Theorem 3.7. *Let $\mathcal{Z}$ be a maximal nonnegative subspace of a Kreĭn space $\mathcal{X}$, and let $\mathcal{H}(\mathcal{Z})$, $\mathcal{H}(\mathcal{Z}^{[\perp]})$, and $\mathcal{X}(\mathcal{Z})$ be the spaces defined in Section* 2. *Then the following conditions are equivalent to each other, and they are also equivalent to conditions* (1)–(11) *in Theorem* 2.14*,*

(12) *$\mathcal{H}(\mathcal{Z})$ is closed in $\mathcal{X}/\mathcal{Z}$.*
(13) *$\mathcal{H}(\mathcal{Z}^{[\perp]})$ is closed in $-\mathcal{X}/\mathcal{Z}^{[\perp]}$.*
(14) *$\mathcal{X}(\mathcal{Z})$ is closed in $\mathcal{X}$.*

Proof. When we in this proof refer to conditions (1)–(11) we mean the corresponding conditions in Theorem 2.14.

Proof of the equivalence (7) $\Leftrightarrow$ (12). Choose some fundamental decomposition $\mathcal{W} = -\mathcal{Y}\,[\dot{+}]\,\mathcal{U}$ of $\mathcal{W}$, let T be the operator defined in Lemma 2.2, and let A be the contraction in the graph representation (3.1). Then T is an isomorphism $\mathcal{X}/\mathcal{Z} \to \mathcal{Y}$ which maps $\mathcal{H}(\mathcal{Z})$ onto

$\mathcal{H}(A) = \mathcal{R}((1 - AA^*)^{1/2})$ and $\mathcal{H}^0(\mathcal{Z})$ onto $\mathcal{R}(1 - AA^*)$. Thus, the condition $\mathcal{H}^0(\mathcal{Z}) = \mathcal{H}(\mathcal{Z})$ is equivalent to the condition $\mathcal{R}(1 - AA^*) = \mathcal{R}((1 - AA^*)^{1/2})$, whereas the condition that $\mathcal{H}(A)$ is closed in $\mathcal{X}/\mathcal{Z}$ is equivalent to the condition that $\mathcal{R}((1 - AA^*)^{1/2})$ is closed in $\mathcal{Y}$. However, both of these conditions are equivalent to the condition that

$$\mathcal{R}\big((1 - AA^*)^{1/2}\big) = \big[\mathcal{N}(1 - AA^*)\big]^{\perp} = \mathcal{R}\big((1 - AA^*)\big).$$

Thus (7) and (12) are equivalent.

Proof of the implication (4) $\Rightarrow$ (14). This follows from part (2) of Theorem 2.12.

Proof of the implication (14) $\Rightarrow$ (12). If $\mathcal{X}(\mathcal{Z})$ is closed in $\mathcal{X}$, then it follows from part (2) of Theorem 2.12 that $\mathcal{X}(\mathcal{Z}) = \overline{\mathcal{Z} + \mathcal{Z}^{[\perp]}}$. By (2.17), (2.18), and Lemma 2.10, $\mathcal{H}(\mathcal{Z})$ is closed in $\mathcal{X}/\mathcal{Z}$.

Proof of the equivalence (12) $\Leftrightarrow$ (13). Both of these are equivalent to (14) (since $\mathcal{X}(\mathcal{Z}) = \mathcal{X}(\mathcal{Z}^{[\perp]})$), and hence equivalent to each other. $\square$

Above we have given alternative proofs of some of the coordinate free results in Section 2 by appealing to known results about the de Branges complementary spaces $\mathcal{H}(A)$. It is also possible to proceed in the opposite direction and to re-derive results about the spaces $\mathcal{H}(A)$ from the results in Section 2 using the isomorphism T in Corollary 3.5. We leave this to the reader.

4. Application to passive state/signal systems theory

As was mentioned in the introduction, the results presented in this article were obtained as byproducts of our study of the realization problem in passive state/signal systems theory [2–5]. Here we shall only give a short outline of one of the motivating applications.

A passive linear discrete time invariant s/s (state/signal) system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ has a Kreĭn signal space $\mathcal{W}$ (enabling connections to the external environment), a Hilbert state space $\mathcal{X}$ (representing an internal memory), a generating subspace V of the Kreĭn space $\mathfrak{K} = -\mathcal{X}\,[\dotplus]\,\mathcal{X}\,[\dotplus]\,\mathcal{W}$ (defining the dynamics) with the properties (i)–(iv) listed in [2], and the set of trajectories, which consists of sequences $(x(\cdot), w(\cdot)) \in (\mathcal{X} \times \mathcal{W})^{\mathbb{Z}^+}$ satisfying

$$\begin{bmatrix} x(n+1) \\ x(n) \\ w(n) \end{bmatrix} \in V, \quad n \in \mathbb{Z}^+,$$

where $\mathbb{Z}^+ = 0, 1, 2, \ldots$. This system is *passive* if V is a maximal nonnegative subspace of $\mathfrak{K}$. The nonnegativity of V equivalent to the requirement that the trajectories of Σ satisfy

$$\big\|x(n+1)\big\|_{\mathcal{X}}^2 - \big\|x(n)\big\|_{\mathcal{X}}^2 \leqslant \big[w(n), w(n)\big]_{\mathcal{W}}, \quad n \in \mathbb{Z}^+, \tag{4.1}$$

and the *maximal* nonnegativity of V is equivalent to the requirement that, in addition, the adjoint system $\Sigma_* = (V_*; \mathcal{X}, -\mathcal{W})$ defined in a natural way has the same property. See [3] for details.

Trajectories $(x(\cdot), w(\cdot))$ with $x(0) = 0$ (i.e., trajectories whose internal memory is zero at the starting time zero) are called *externally generated*. The *future behavior* $\mathfrak{W}_{\text{fut}}$ of a passive s/s system Σ consists of all sequences $w(\cdot)$ of signals in $\ell^2_+(\mathcal{W}) = \ell^2(\mathbb{Z}^+; \mathcal{W})$ that are obtained from the externally generated trajectories $(x(\cdot), w(\cdot))$ of Σ by ignoring the state component $x(\cdot)$,

$$\mathfrak{W}_{\mathrm{fut}} = \mathfrak{W} \cap \ell^2_+(\mathcal{W}).$$

It is not difficult to show that $\mathfrak{W}_{\mathrm{fut}}$ is a maximal nonnegative subspace of the Kreĭn space $k^2_+(\mathcal{W})$ of sequences $w(\cdot) \in \ell^2_+(\mathcal{W})$ with indefinite inner product

$$\bigl[w_1(\cdot), w_2(\cdot)\bigr]_{k^2_+(\mathcal{W})} = \sum_{k=0}^{\infty}\bigl[w_1(k), w_2(k)\bigr]_{\mathcal{W}}.$$

Moreover, $\mathfrak{W}_{\mathrm{fut}}$ is S_+-shift invariant, where S_+ is the right shift in $k^2_+(\mathcal{W})$.

The *inverse problem* to the one described above is the following: is it true that every maximal nonnegative S_+-invariant subspace in $k^2_+(\mathcal{W})$ can be realized as a future behavior of some passive s/s system? This inverse problem is more difficult to solve, but it turns out that it has a positive answer (given in Theorem 4.1 below), even if we impose some additional constraints on the system Σ, which will be discussed below.

A s/s system Σ is *forward conservative* if (4.1) holds in the form of an equality for all trajectories of Σ, and it is *backward conservative* if the adjoint system Σ_* is forward conservative. Thus, $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is passive and forward conservative if and only if V is maximal nonnegative and $V \subset V^{[\perp]}$ (this inclusion means that V is neutral), and Σ is passive and backward conservative if and only if V is maximal nonnegative and $V^{[\perp]} \subset V$. Both of these conditions hold if and only if V is a Lagrangian subspace of $\mathfrak{K}$, in which case Σ is called *conservative*.

The subspace of $\mathcal{X}$ that we get by taking the closure in $\mathcal{X}$ of all states $x(n)$ that appear in externally generated trajectories $(x(\cdot), w(\cdot))$ of Σ is called the (approximately) *reachable subspace*, and we denote it by $\mathfrak{R}_\Sigma$. If $\mathfrak{R}_\Sigma = \mathcal{X}$, then Σ is called *controllable*. The subspace of all $x_0 \in \mathcal{X}$ with the property that $(x(\cdot), w(\cdot))$ with $x(0) = x_0$ and $w(n) = 0$ for all $n \in \mathbb{Z}^+$ is a trajectory of Σ is called the *unobservable subspace*, and it is denoted by $\mathfrak{U}_\Sigma$. If $\mathfrak{U}_\Sigma = \{0\}$, then Σ is called (approximately) *observable*. A s/s system is called *simple* if $\mathcal{X} = \overline{\mathfrak{R}_\Sigma + \mathfrak{U}_\Sigma^\perp}$, or equivalently, if $\mathfrak{U}_\Sigma \cap \mathfrak{R}_\Sigma^\perp = \{0\}$.

The following solution to the inverse problem is given in [3] (see [3, Theorem 3.8] and its proof).

Theorem 4.1. *Let $\mathcal{W}$ be a Kreĭn space, and let $\mathcal{Z}$ be an arbitrary maximal nonnegative S_+-invariant subspace of the Kreĭn space $k^2_+(\mathcal{W})$. Then there exists a passive s/s system Σ with future behavior $\mathcal{Z}$ satisfying one of the following sets of additional conditions*:

(1) *Σ is observable and backward conservative.*
(2) *Σ is controllable and forward conservative.*
(3) *Σ is simple conservative s/s system.*

Each of the above three s/s systems are defined by $\mathcal{Z}$ up to unitary similarity.

The notion of unitary similarity of s/s systems used above is defined in a natural way; see [3].

The idea behind the proof of Theorem 4.1 given in [3] is the following. First one chooses a fundamental decomposition $\mathcal{W} = -\mathcal{Y}\ [\dot{+}]\ \mathcal{U}$ of $\mathcal{W}$, which induces the fundamental decomposition $k^2_+(\mathcal{W}) = -\ell^2_+(\mathcal{Y})\ [\dot{+}]\ \ell^2_+(\mathcal{U})$ of $k^2_+(\mathcal{W})$. A maximal nonnegative right-shift invariant subspace $\mathcal{Z}$ of $k^2_+(\mathcal{W})$ has the graph representation (3.1) with respect to this fundamental decomposition of $k^2_+(\mathcal{W})$, where the operator A is a contractive linear block Toeplitz operator from $\ell^2_+(\mathcal{U})$ to $\ell^2_+(\mathcal{Y})$. The symbol of this operator is a $\mathcal{B}(\mathcal{U};\mathcal{Y})$-valued Schur function (i.e., an analytic and contractive-valued function) $\mathfrak{D}(z)$ in the unit disk. There exist three different de Branges–Rovnyak i/s/o (input/state/output) models with the same scattering matrix (characteristic function) equal to the given Schur function $\mathfrak{D}(z)$. All of these three models are passive discrete-time invariant i/s/o scattering systems, with one of the following sets of additional properties: (1) the first one is observable and backward conservative, (2) the second one is controllable and forward conservative, and (3) the third one is simple and conservative. In operator theory one calls systems with the above properties "operator colligations" (nodes) that are (1) "co-isometric and closely outer connected," or (2) "isometric and closely inner connected," or (3) "unitary and closely connected"; see, e.g., [1, Chapter 2]. The state space of the observable and backward conservative de Branges–Rovnyak model is the de Branges–Rovnyak space $\mathcal{H}(A)$, where A is the contractive shift-invariant operator of multiplication by $\mathfrak{D}(z)$, acting from the Hardy space $H^2_+(\mathcal{U})$ to the Hardy space $H^2_+(\mathcal{Y})$, and the main operator in this model is the incoming shift operator $y(z) \mapsto [y(z) - y(0]/z$. The three passive s/s systems constructed in the proofs of parts (1)–(3) of Theorem 4.1 are the unique passive s/s systems whose i/s/o representations corresponding to the fundamental decomposition $\mathcal{W} = -\mathcal{Y}\,[\dotplus]\,\mathcal{U}$ are the time domain versions of the three de Branges–Rovnyak models (1)–(3) described above. By the time domain versions of these models we mean the models that one gets by mapping the Hardy spaces $H^2_+(\mathcal{U})$ and $H^2_+(\mathcal{Y})$ isometrically onto the sequence spaces $\ell^2_+(\mathcal{U})$ and $\ell^2_+(\mathcal{Y})$ by means of the inverse Fourier transform. In the time domain the inverse i/s/o problem becomes the problem of realizing a contractive right-shift invariant map from $\ell^2_+(\mathcal{U})$ to $\ell^2_+(\mathcal{Y})$ as the i/o (input/output) map of a scattering passive systems.

The main disadvantage with the proofs outlined above is that they do not in each case produce just one single s/s realization but infinitely many, all of which are unitarily equivalent to each other. In all cases the realizations that we obtain depend on the fundamental decomposition $\mathcal{W} = -\mathcal{Y}\,[\dotplus]\,\mathcal{U}$ that we start with. This is obvious from the fact that, for example, in case (1) the state space is a subspace of $\ell^2_+(\mathcal{Y})$, so different choices of $\mathcal{Y}$ result in different (albeit unitarily similar) s/s realizations.

The results presented in Section 2 were obtained in our search for a *one and only canonical* (*or coordinate free*) *s/s realization* in each of cases (1)–(3). By "canonical" we mean that this realization should be uniquely determined by the given data, i.e., by the original maximal nonnegative shift-invariant subspace $\mathcal{Z}$ of $k^2_+(\mathcal{W})$ that we want to realize. In particular, it *must not depend on some arbitrary choice of a fundamental decomposition of the signal space* $\mathcal{W}$. Indeed, our search was successful, and in case (1) it led to the following result.

Theorem 4.2. *Let $\mathcal{W}$ be a Kreĭn space, and let $\mathcal{Z}$ be a maximal nonnegative S_+-invariant subspace of $k^2_+(\mathcal{W})$. Let $\mathcal{X}_{\mathrm{obc}} = \mathcal{H}(\mathcal{Z})$, and let*

$$V_{\mathrm{obc}} = \left\{ \begin{bmatrix} S_+^* w + \mathcal{Z} \\ w + \mathcal{Z} \\ w(0) \end{bmatrix} \in \begin{bmatrix} \mathcal{H}(\mathcal{Z}) \\ \mathcal{H}(\mathcal{Z}) \\ \mathcal{W} \end{bmatrix} \;\middle|\; w \in \mathcal{X}(\mathcal{Z}) \right\},$$

where $\mathcal{X}(\mathcal{Z})$ is the space defined in (2.18) *with $\mathcal{X} = k^2_+(\mathcal{W})$. Then $\Sigma_{\mathrm{obc}} = (V_{\mathrm{obc}}; \mathcal{X}_{\mathrm{obc}}, \mathcal{W})$ is a passive observable backward conservative s/s system with future behavior $\mathfrak{W}_{\mathrm{fut}} = \mathcal{Z}$.*

As a part of the proof of this theorem one shows thatV_{obc} is well defined, i.e., that $S_+^* w + \mathcal{Z} \in \mathcal{H}(\mathcal{Z})$ whenever $w \in \mathcal{X}(\mathcal{Z})$.

Analogous canonical shift realization models can also be obtained for cases (2) and (3) based on the results presented in Section 2. The proof of Theorem 4.2 and the corresponding passive s/s realizations of the types (2) and (3) will be given elsewhere.

Remark 4.3. As we have seen above, our construction in Theorem 2.3 of the Hilbert space $\mathcal{H}(\mathcal{Z})$ (contained in the quotient of a Kreĭn space $\mathcal{X}$ over the maximally nonnegative subspace $\mathcal{Z}$ of $\mathcal{X}$) is related to the corresponding construction in [10] of the Hilbert space $\mathcal{H}(A)$, where A is a contraction between two Hilbert spaces. That construction was extended by Louise de Branges in [8] to the case where A is a contraction between two Kreĭn spaces, in which case the resulting space $\mathcal{H}(A)$ is a Kreĭn space. The primary motivation for our interest in $\mathcal{H}(\mathcal{Z})$ was explained above: we need the space $\mathcal{H}(\mathcal{Z})$ in our construction of a canonical model of an observable and backward conservative passive state/signal system with a *Hilbert* state space. However, it seems plausible that there also exists a coordinate free version of the construction in [8] that would lead to canonical models of state/signal systems with a *Kreĭn* state space. We leave this as an open question.

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References

[1] Daniel Alpay, Aad Dijksma, James Rovnyak, Henrik de Snoo, Schur Functions, Operator Colligations, and Reproducing Kernel Hilbert Spaces, Operator Theory Adv. Appl., vol. 96, Birkhäuser, Basel, 1997.

[2] Damir Z. Arov, Olof J. Staffans, State/signal linear time-invariant systems theory. Part I: Discrete time systems, in: The State Space Method, Generalizations and Applications, in: Operator Theory Adv. Appl., vol. 161, Birkhäuser, Basel, 2005, pp. 115–177.

[3] Damir Z. Arov, Olof J. Staffans, State/signal linear time-invariant systems theory. Passive discrete time systems, Internat. J. Robust Nonlinear Control 17 (2007) 497–548.

[4] Damir Z. Arov, Olof J. Staffans, State/signal linear time-invariant systems theory. Part III: Transmission and impedance representations of discrete time systems, in: Operator Theory, Structured Matrices, and Dilations, Tiberiu Constantinescu Memorial Volume, Bucharest, Romania, in: Theta Foundation, Amer. Math. Soc., Providence, RI, 2007, pp. 101–140.

[5] Damir Z. Arov, Olof J. Staffans, State/signal linear time-invariant systems theory. Part IV: Affine representations of discrete time systems, Complex Anal. Oper. Theory 1 (2007) 457–521.

[6] Tomas Ya. Azizov, Iosif S. Iokhvidov, Linear Operators in Spaces with an Indefinite Metric, Wiley, New York, 1989.

[7] János Bognár, Indefinite Inner Product Spaces, Ergeb. Math. Grenzgeb., vol. 78, Springer-Verlag, Berlin, 1974.

[8] Louis de Branges, Complementation in Kreĭn spaces, Trans. Amer. Math. Soc. 305 (1) (1988) 277–291.

[9] Louis de Branges, James Rovnyak, Canonical models in quantum scattering theory, in: Perturbation Theory and Its Applications in Quantum Mechanics, Proc. Adv. Sem. Math. Res. Center, U.S. Army, Theoret. Chem. Inst., Univ. of Wisconsin, Madison, WI, 1965, Wiley, New York, 1966, pp. 295–392.

[10] Louis de Branges, James Rovnyak, Square Summable Power Series, Holt, Rinehart & Winston, New York, 1966.

[11] Rudolf E. Kalman, Peter L. Falb, Michael A. Arbib, Topics in Mathematical System Theory, McGraw–Hill, New York, 1969.

[12] Donald Sarason, Sub-Hardy Hilbert Spaces in the Unit Disk, Univ. Arkansas Lecture Notes in Math. Sci., vol. 10, Wiley–Interscience, New York, 1994.