Implementation of an αBB-type underestimator in the SGO-algorithm

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Refining without branching

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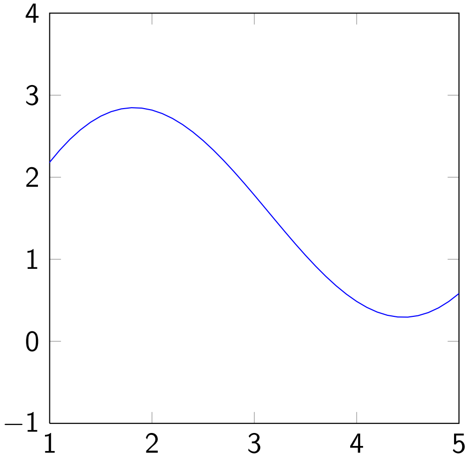
- We developed a convex formulation that handles breakpoints with binary variables instead of direct branching
- "Why?"
  - It can readily be integrated with the SGO algorithm
  - It could turn out to be especially well-suited for some types of mixed-integer problems
  - As a convex reformulation it could be of interest in automated reformulation procedures

Refining without branching



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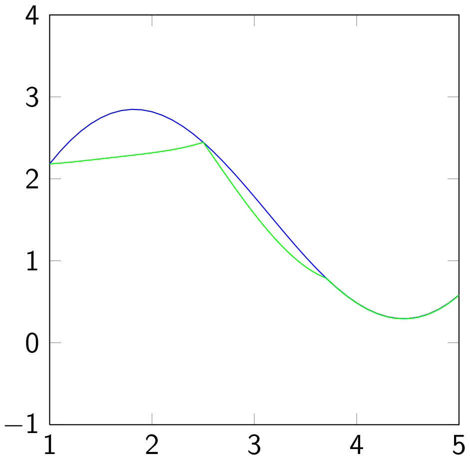
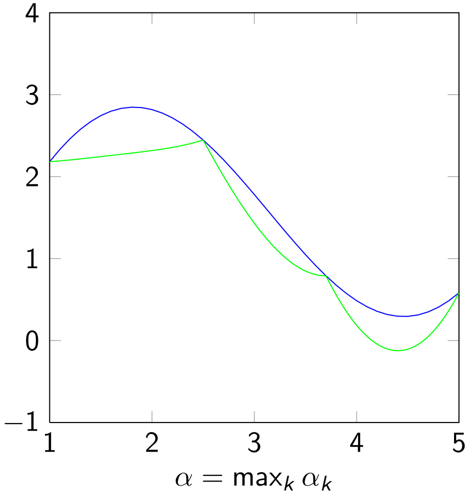


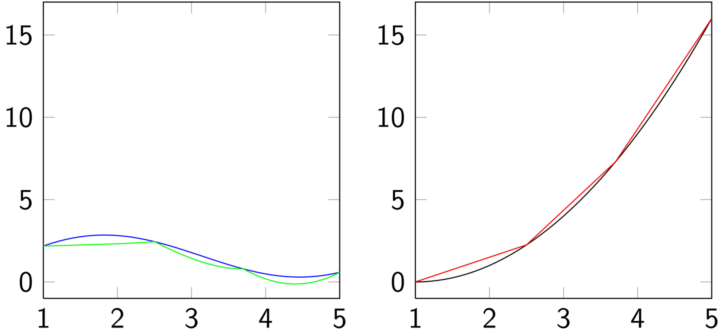
$\alpha_1 = 1, \alpha_2 = 0.6, \alpha_3 = 0$

Refining without branching



Refining without branching

The underestimation error can be described as the difference of a parabola and a piecewise linear function.



Refining without branching

Our formulation: (1D for clarity)

$$f(x) \leq 0 \tag{1}$$

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$$f(x) + \alpha x^2 - W \leq 0 \quad (1)$$

$$W = \alpha x^2 \quad (2)$$

Overestimating W will relax the feasible domain, we replace W with a piecewise linear function

$$\hat{W} = \sum_{k=1}^{K} A_k b_k + (B_k - A_k) s_k \quad (3)$$

where

$$A_k = \alpha \underline{x_k}^2$$
$$B_k = \alpha \overline{x_k}^2$$

($\underline{x_k}$ and $\overline{x_k}$ denote the interval endpoints)

Refining without branching

The b_k are binary variables, s_k are continuous and nonnegative.

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We relate x to b_k and s_k with the constraints

$$\sum_{k=1}^{K} b_k = 1$$

$$s_k \leq b_k, \; \forall k$$

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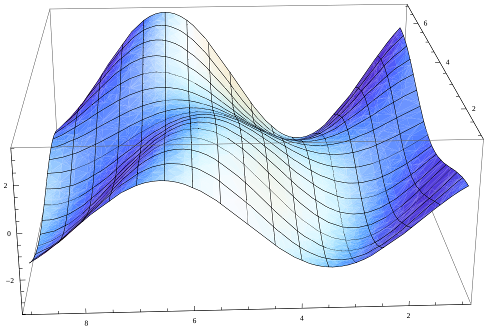
Every constraint is convex and the feasible set is relaxed

In two dimensions

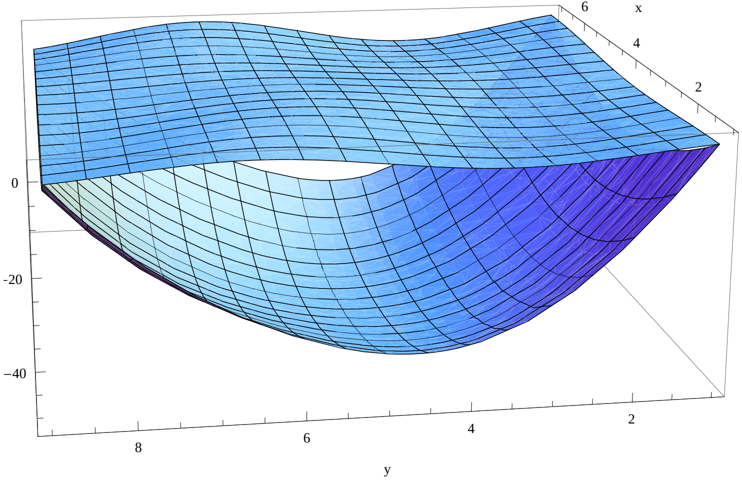
An example:

$$f(x, y) = \sin(x + y) + \sqrt{x} \cos y$$

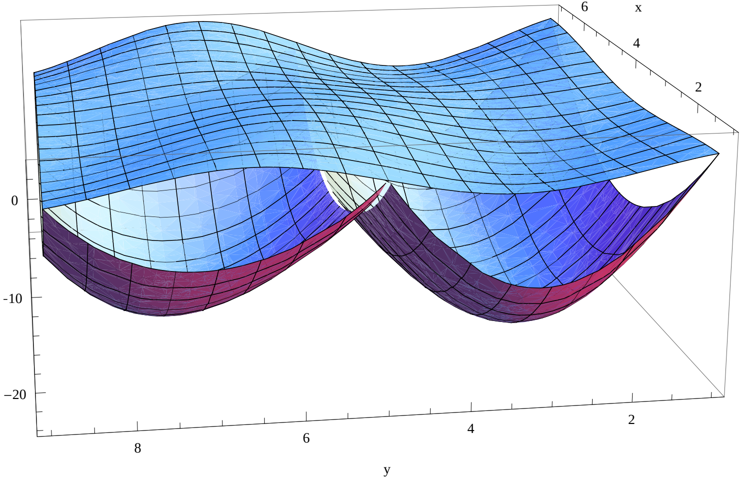
$$1 \leq x \leq 7, \quad 1 \leq y \leq 9$$



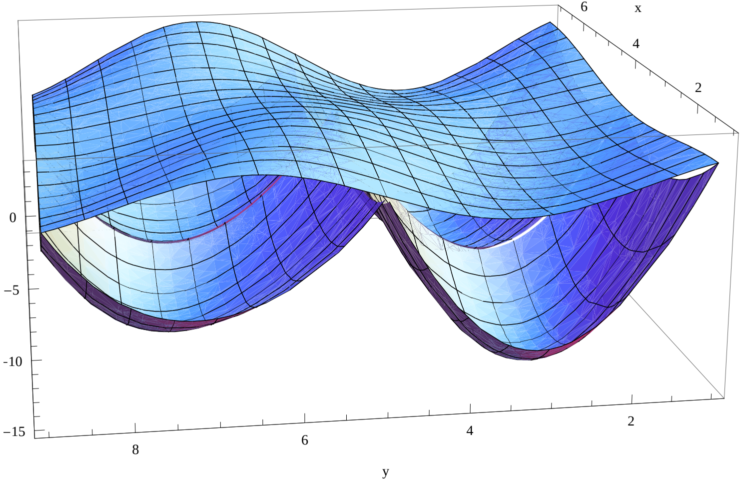
Underestimator - no breakpoints



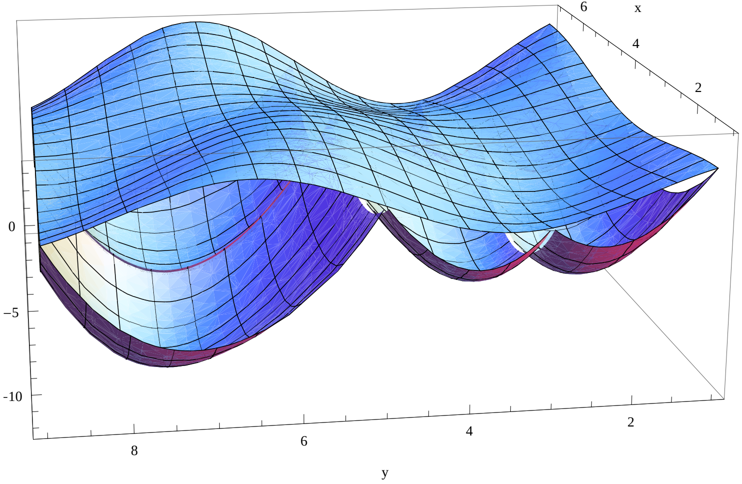
Underestimator - 1 breakpoint



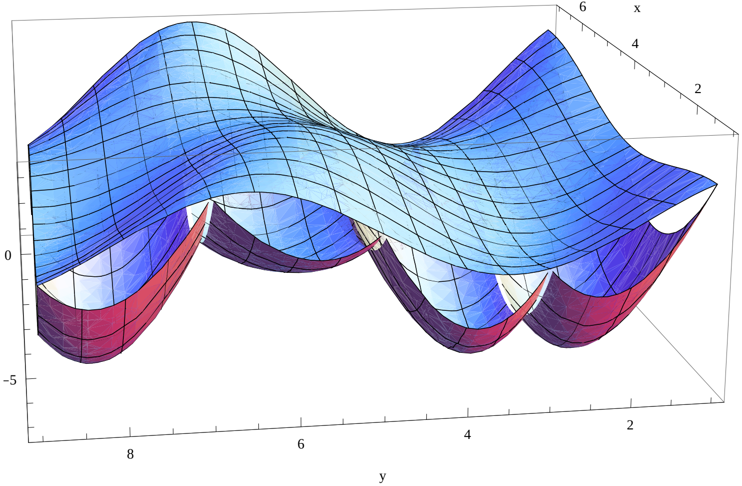
Underestimator - 1+1 breakpoints



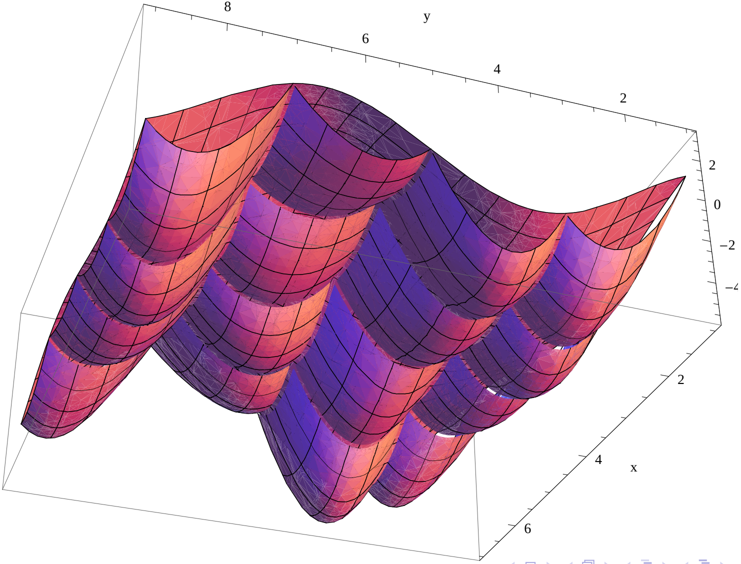
Underestimator - 1+2 breakpoints



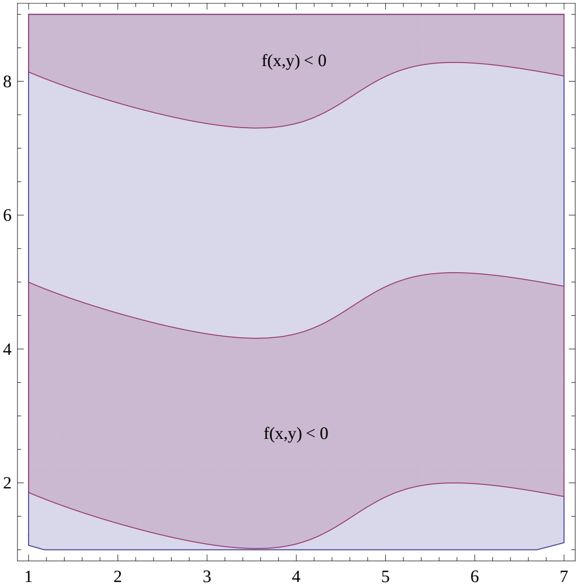
Underestimator - 1+3 breakpoints



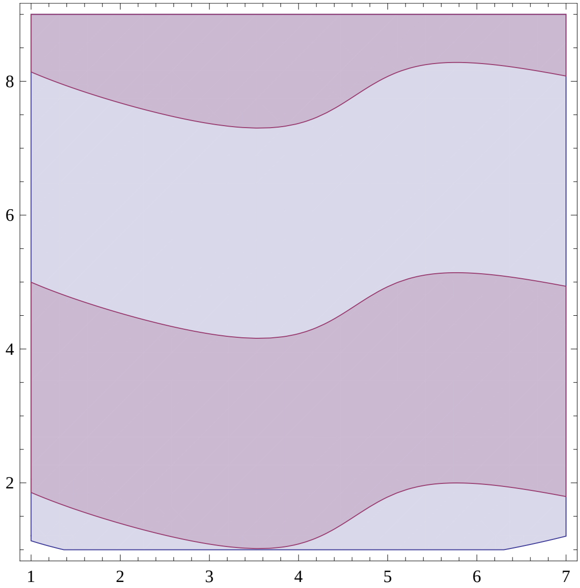
Underestimator - 3+3 breakpoints



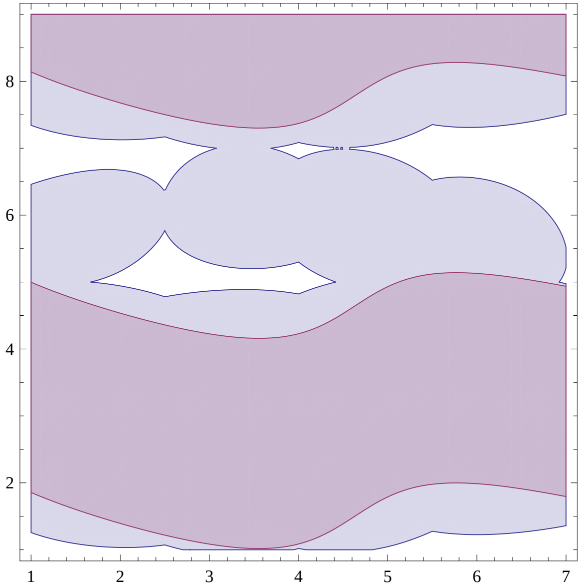
Constraint feasibility



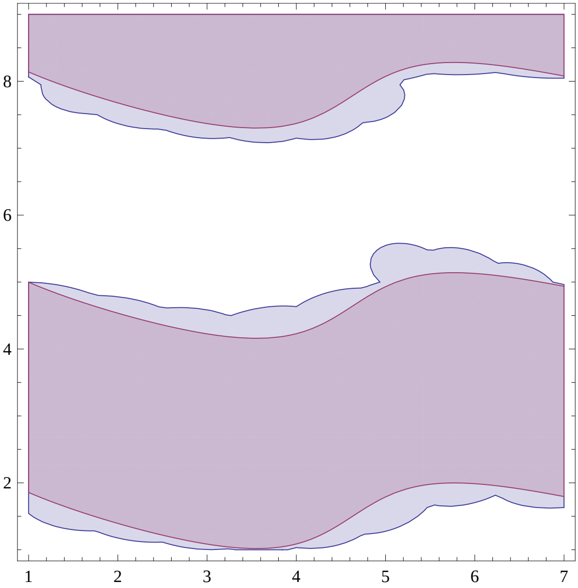
Constraint feasibility - 1+1 breakpoints



Constraint feasibility - 3+3 breakpoints



Constraint feasibility - 7+7 breakpoints



About convergence

The largest underestimation error in a subdomain depends only on the value of $\alpha_i, i = 1, \ldots n$ and the size of the subdomain: (1D)

$$\max_{x \in [\underline{x_k}, \overline{x_k}]} \hat{W} - \alpha x^2 = \max_{x \in [\underline{x_k}, \overline{x_k}]} -\alpha(x - \underline{x_k})(x - \overline{x_k}) = \alpha \left(\frac{\overline{x_k} - \underline{x_k}}{2} \right)^2$$

An ϵ precision is guaranteed if the width of the interval

$$\overline{x_k} - \underline{x_k} \leq \sqrt{\frac{4\epsilon}{\alpha}}.$$

$\Rightarrow$ The algorithm will converge

Challenges & Ideas

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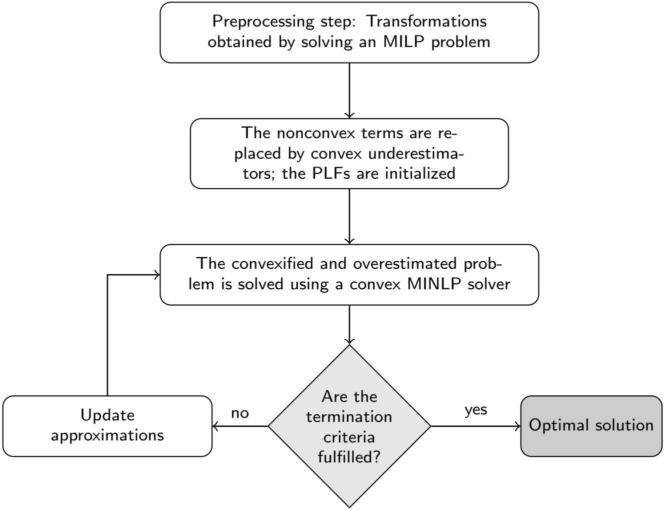
- A type of "minor" breakpoint halving every interval can be introduced without too much cost

Integration with SGO



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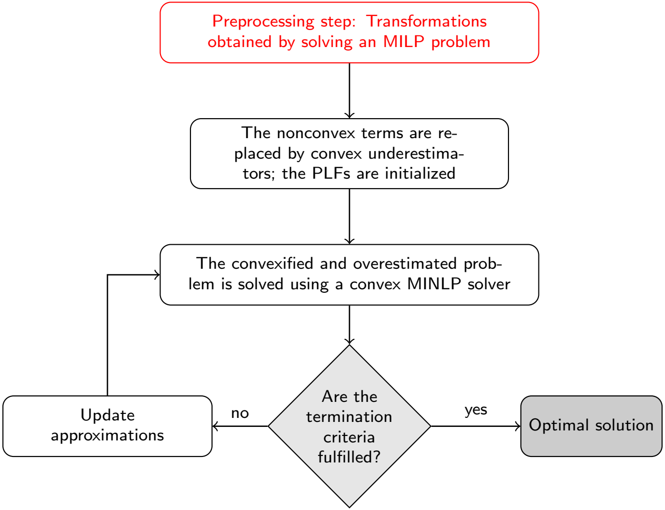

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Convex underestimation strategies for signomial functions.
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Thank you

C^2 constraints

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$$f(x) \leq 0$$

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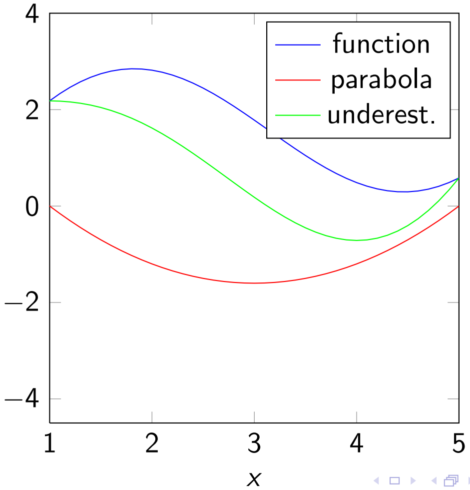
- In addition C^2 objective functions can be handled by rewriting

$$\min \quad f(x) \qquad \text{as} \qquad \begin{array}{ll} \min & \mu \\ \text{s.t.} & f(x) - \mu \leq 0 \end{array}$$

The αBB underestimator

A C^2 function f on the domain $[x_L, x_U] \subset \mathbb{R}$ can always be convexified by adding a parabola $p(x) = \alpha(x - x_L)(x - x_U)$ with a large enough α.



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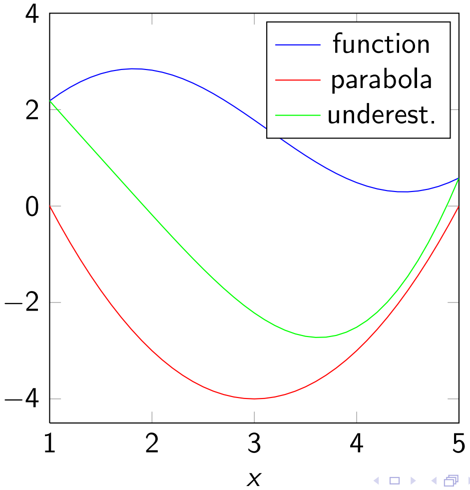

This convex underestimator can be extended to multiple dimensions. Let f be a C^2 function on $\mathbb{R}^n$. For a large enough α the function $g = f + q$ where

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is convex. Tighter underestimators can be found by letting α depend on i

$$q(x_1, \ldots, x_n) = \sum_{i=1}^{n} \alpha_i (x_i - x_i^L)(x_i - x_i^U)$$

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$$H_g = H_f + 2 \cdot \mathrm{diag}(\alpha_i) =$$

$$\begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{pmatrix} + 2 \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_n \end{pmatrix}$$

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- Choose α_i such that the eigenvalues of H_g are nonnegative **on the relevant domain**

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- Smaller valid choices usually exist, but finding the optimal α_i is in general as hard as the optimization problem itself
- Branch-and-bound methods can be used to solve the original problem

Branch-and-bound search

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- Branch to split a domain in two and get tighter underestimators
- Any feasible solution to the original problem gives an upper bound on the optimal objective value
- Any branch with a lower bound greater than the best found upper bound is fathomed (cut)
- A number of techniques are used to speed up the search, e.g. *bound reduction*