OSE SEMINAR 2012

THE QUADRATIC ASSIGNMENT PROBLEM

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OSE OPTIMIZATION AND SYSTEMS ENGINEERING

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Introduction

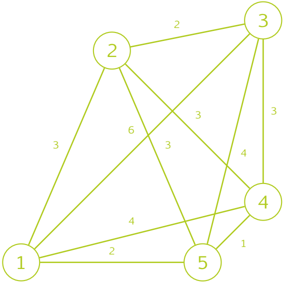
- Introduced by Koopmans and Beckmann in 1957
- Cited by $\approx$ 1500
- Among the hardest combinatorial problems
- Real and test instances easily accessible (QAPLIB - A Quadratic assignment problem Library)
- Instances with N=30 are still unsolved

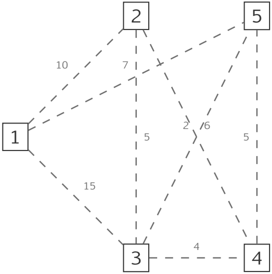
Location theory

- Supply Chains
- Logistics
- Production

Location theory

- Objective: Assign N plants between N given locations in order to minimize total flows.

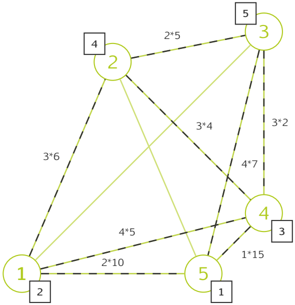




$$\mathbf{A} = \begin{bmatrix} 0 & 3 & 6 & 4 & 2 \\ 3 & 0 & 2 & 3 & 3 \\ 6 & 2 & 0 & 3 & 4 \\ 4 & 3 & 3 & 0 & 1 \\ 2 & 3 & 4 & 1 & 0 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 0 & 10 & 15 & 0 & 7 \\ 10 & 0 & 5 & 6 & 0 \\ 15 & 5 & 0 & 4 & 2 \\ 0 & 6 & 4 & 0 & 5 \\ 7 & 0 & 2 & 5 & 0 \end{bmatrix}$$

Location theory

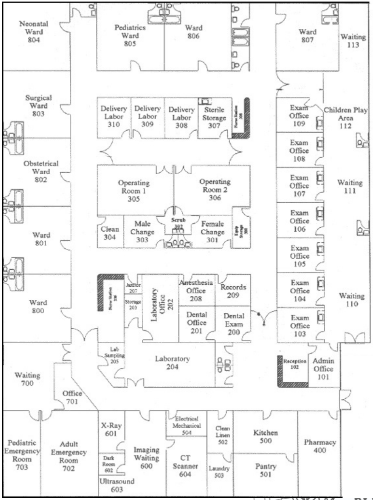


- Optimal solution = 258
- Optimal Permutation=[2 4 5 3 1]

$$A = \begin{bmatrix} 0 & 3 & 6 & 4 & 2 \\ 3 & 0 & 2 & 3 & 3 \\ 6 & 2 & 0 & 3 & 4 \\ 4 & 3 & 3 & 0 & 1 \\ 2 & 3 & 4 & 1 & 0 \end{bmatrix} \qquad B_{24531} = \begin{bmatrix} 0 & 6 & 0 & 5 & 10 \\ 6 & 0 & 5 & 4 & 0 \\ 0 & 5 & 0 & 2 & 7 \\ 5 & 4 & 2 & 0 & 15 \\ 10 & 0 & 7 & 15 & 0 \end{bmatrix}$$

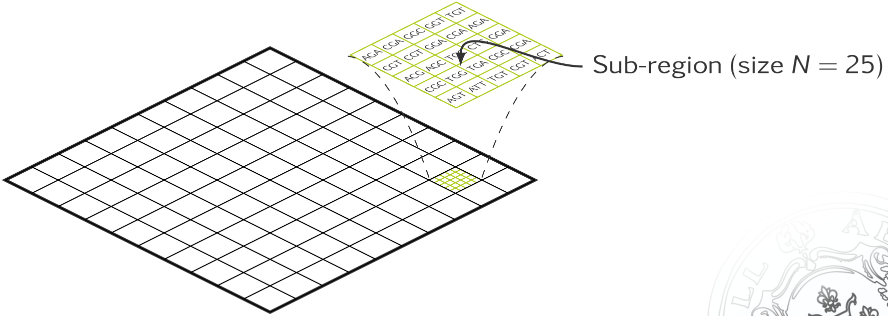
Facility Layout-Real World Examples

- Hospital Layout - German university hospital, Klinikum Regensburg 1972. (Optimality proved in the year 2000)
- Ship Design
- Airport gate Assignment
- Nature Park Layout



DNA MicroArray Layout

- Microarrays can have up to 1.3 million probes
- Small subregions can be solved as QAPs
- Objective: To reduce the risk of unintended illumination of probes



Other applications for QAPs

- Backboard wiring
- Control panel and keyboard layout
- VLSI design
- Computer manufacturing
- Archeology
- Bandwith minimization of a graph
- Economics
- Image processing

Three Objective Functions

- Koopmanns-Beckmann

$$\min A \cdot XBX^T \tag{1}$$

- SDP

$$\min \operatorname{tr}(AXBX^T) \tag{2}$$

- DLR

$$\min XA \cdot BX \tag{3}$$

Koopmans Beckmann form

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{l=1}^{N} a_{ij} b_{kl} \cdot x_{ik} x_{jl}$$

$$\sum_{i=1}^{N} x_{ij} = 1, \qquad j = 1, ..., N;$$

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- This formulation has $N^2(N-1)^2$ bilinear terms.

$$\mathrm{tr}\,(\mathbf{AXBX}^T) = \mathrm{tr}((\mathbf{A} \otimes \mathbf{B})\mathbf{yy}^T)$$

$$\mathbf{Q} = \mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & \cdots & a_{1N}\mathbf{B} \\ \vdots & \ldots & \vdots \\ a_{N1}\mathbf{B} & \cdots & a_{NN}\mathbf{B} \end{bmatrix} \text{ and } \mathbf{y} = \mathrm{vec}\,(\mathbf{X}) = \begin{bmatrix} x_{11} \\ \vdots \\ x_{NN} \end{bmatrix}$$

Semi Definite Programming Relaxation

$$\min_{\mathbf{Y},\mathbf{y}} \mathrm{tr}(\mathbf{Q}'\mathbf{Y})$$

s.t.

$$\mathrm{diag}(\mathbf{Y}) = \mathbf{y}$$

$$\begin{bmatrix} 1 & \mathbf{y}^T \\ \mathbf{y} & \mathbf{Y} \end{bmatrix} \succeq 0$$

- $\otimes$ is the Kronecker product
- The number of continuous variables in $\mathbf{Y}$ is N^4 and the number of binary variables in $\mathbf{y}$ is N^2

$\min XA \cdot BX$

$$\min \sum_{i=1}^{N} \sum_{j=1}^{N} a'_{ij} b'_{ij}$$

$$a'_{ij} = \sum_{k=1}^{n} a_{kj} x_{ik} \qquad \forall i,j$$

$$b'_{ij} = \sum_{k=1}^{n} b_{ik} x_{kj} \qquad \forall i,j$$

$$\mathbf{A} = \begin{bmatrix} 0 & 3 & \mathbf{5} & 9 & 6 \\ 3 & 0 & \mathbf{2} & 6 & 9 \\ 5 & 2 & \mathbf{0} & 8 & 10 \\ 9 & 6 & \mathbf{8} & 0 & 2 \\ 6 & 9 & \mathbf{10} & 2 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 & 4 & 3 & 7 & 7 \\ \mathbf{4} & \mathbf{0} & \mathbf{4} & \mathbf{10} & \mathbf{4} \\ 3 & 4 & 0 & 2 & 3 \\ 7 & 10 & 2 & 0 & 4 \\ 7 & 4 & 3 & 4 & 0 \end{bmatrix}$$

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$$a'_{23} = 5x_{21} + 2x_{22} + 0x_{23} + 8x_{24} + 10x_{25}$$
$$b'_{23} = 4x_{13} + 0x_{23} + 4x_{33} + 10x_{43} + 4x_{53}$$

Discrete Linear Reformulation (DLR)

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{m=1}^{M_i} B_i^m z_{ij}^m$$

$$\left.\begin{array}{ll} z_{ij}^m \leq \overline{A}_j \sum_{k \in \mathbf{K}_i^m} x_{kj} & m = 1, \ldots, M_i \\ \sum_{m=1}^{M_i} z_{ij}^m = a'_{ij} & \end{array}\right\} \forall i, j$$

Example for one bilinear term $a'_{23} b'_{23}$

$$a'_{23} = 5x_{21} + 2x_{22} + 0x_{23} + 8x_{24} + 10x_{25}$$

$$b'_{23} = 4x_{13} + 0x_{23} + 4x_{33} + 10x_{43} + 4x_{53}$$

$$x_{13} + x_{23} + x_{33} + x_{43} + x_{53} = 1$$

$$x_{21} + x_{22} + x_{23} + x_{24} + x_{25} = 1$$

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$$x_{13} + x_{23} + x_{33} + x_{43} + x_{53} = 1$$

$$x_{21} + x_{22} + x_{23} + x_{24} + x_{25} = 1$$

$$4z_{23}^1 + 10z_{23}^2$$

$$z_{23}^1 \le 10(x_{13} + x_{33} + x_{53})$$

$$z_{23}^1 + z_{23}^2 = a'_{23}$$

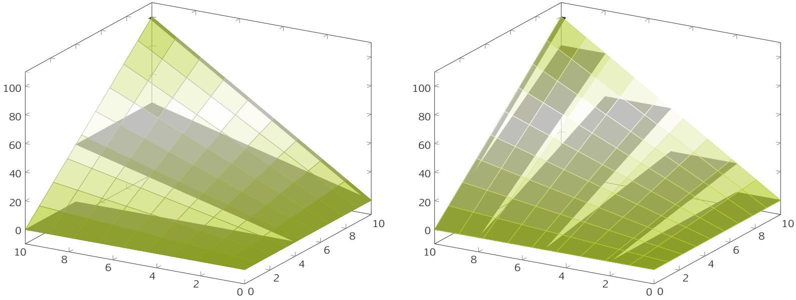

Figure 1: Bilinear term $a'_{23}b'_{23}$ discretized in b'_{23} (to the left) and in a'_{23} (to the right)

- The size of the MILP problem is dependent on the number of unique elements per row.
- Tightness of the MILP problem is dependent on the differences between the elements in each row.

- The size of the DLR is dependent on the number of unique elements per row.
- Tightness of the DLR problem is dependent on the differences between the elements in each row.
- $\mathbf{A}$ can be modified to any matrix $\tilde{\mathbf{A}}$, where $\tilde{a}_{ij} + \tilde{a}_{ji} = a_{ij} + a_{ji}$.

- The size of the DLR is dependent on the number of unique elements per row.
- Tightness of the DLR problem is dependent on the differences between the elements in each row.
- $\mathbf{A}$ can be modified to any matrix $\tilde{\mathbf{A}}$, where $\tilde{a}_{ij} + \tilde{a}_{ji} = a_{ij} + a_{ji}$.

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 2 & 2 & 3 & 4 & 4 & 5 \\ 1 & 0 & 1 & 1 & 2 & 3 & 3 & 4 \\ 2 & 1 & 0 & 2 & 1 & 2 & 2 & 3 \\ 2 & 1 & 2 & 0 & 1 & 2 & 2 & 3 \\ 3 & 2 & 1 & 1 & 0 & 1 & 1 & 2 \\ 4 & 3 & 2 & 2 & 1 & 0 & 2 & 3 \\ 4 & 3 & 2 & 2 & 1 & 2 & 0 & 1 \\ 5 & 4 & 3 & 3 & 2 & 3 & 1 & 0 \end{bmatrix} \qquad \tilde{\mathbf{A}} = \begin{bmatrix} 0 & 2 & 2 & 2 & 6 & 6 & 6 & 6 \\ 0 & 0 & 0 & 0 & 4 & 4 & 4 & 4 \\ 2 & 2 & 0 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 0 & 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & 2 & 2 & 0 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 0 & 2 \\ 4 & 4 & 4 & 4 & 4 & 4 & 0 & 0 \end{bmatrix}$$

Instance	Size	BKS	old LB	DLR	Time(minutes)
esc32a	32	130	103	130	1964
esc32b	32	168	132	168	3500
esc32c	32	642	616	642	254
esc32d	32	200	191	200	10
esc64a	64	116	98	116	48

Table 1: Solution times when solving the instances esc32a, esc32b, esc32c, esc32d and esc64a from the QAPLIB to global optimality

- Previously unsolved instances presented in 1990.
- Nug30 ($N = 30$) solved in 2001 (in 7 days) using 1000 computers in parallel.

A few references

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Thank you for listening!

Questions?