Centre for Process Systems Engineering

Imperial College London

Multi-Parametric Programming & Explicit MPC
a progress report

Stratos Pistikopoulos

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Acknowledgements

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  - EU - MOBILE, PRISM, PROMATCH, DIAMANTE, HY2SEPS
  - CPSE Industrial Consortium, KAUST
  - Air Products

CONNECT
PRISM
EPSRC
erc MOBILE
MARIE CURIE ACTIONS
PROMATCH
AIR PRODUCTS

- People
  - J. Acevedo, V. Dua, V. Sakizlis, P. Dua, N. Bozinis, N. Faisca
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  - Imperial & ParOS R&D Teams

ParOS
Advanced control technology on a chip

Outline

- Key concepts & historical overview
- Recent developments in multi-parametric programming and mp-MPC
- MPC-on-a-chip applications
- Concluding remarks & future outlook

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- Key concepts & historical overview
- Recent developments in multi-parametric programming and mp-MPC
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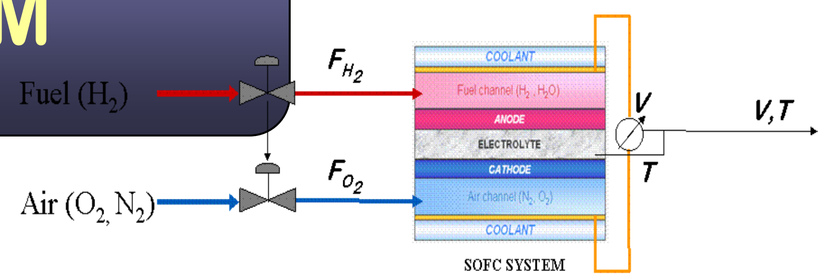
What is On-line Optimization?

MODEL/OPTIMIZER

Control Actions

Data - Measurements

SYSTEM



What is Multi-parametric Programming?

- Given:
  - a performance criterion to minimize/maximize
  - a vector of constraints
  - a vector of parameters

$$z(x) = \min_{u} \; f(u,x)$$

$$\text{s.t.} \; g(u,x) \leq 0$$

$$x \in \mathbb{R}^n$$

$$u \in \mathbb{R}^s$$

What is Multi-parametric Programming?

$$z(x) = \min_{u} f(u,x)$$
$$\text{s.t. } g(u,x) \le 0$$
$$x \in \mathbb{R}^n$$
$$u \in \mathbb{R}^s$$

- Given:
  - a performance criterion to minim
  - a vector of constraints
  - a vector of parameters
- Obtain:
  - the performance criterion and the optimization variables as a function of the parameters
  - the regions in the space of parameters where these functions remain valid

Multi-parametric programming

$$z(x) = \min_{u} \; f(u,x)$$

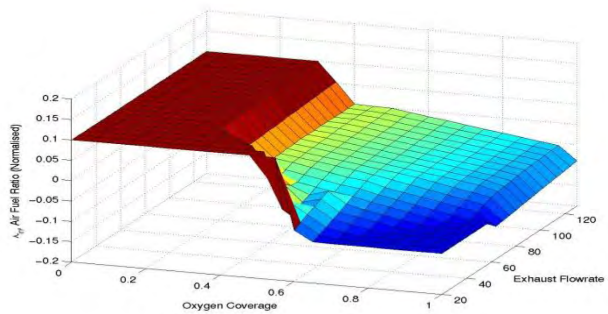
$$\text{s.t.} \;\; g(u,x) \leq 0$$

$$x \in \mathbb{R}^{n}$$

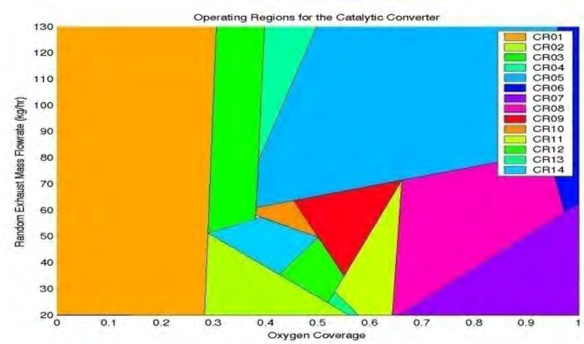
$$u \in \mathbb{R}^{s}$$

$$u(x)$$

(1) Optimal look-up function



(2) Critical Regions



Obtain optimal solution $u(x)$ **as a function of the parameters** x

Multi-parametric programming

Problem Formulation

$$\min_{\mathbf{u}_1,\mathbf{u}_2} \left(-3\mathbf{u}_1 - 8\mathbf{u}_2\right)$$

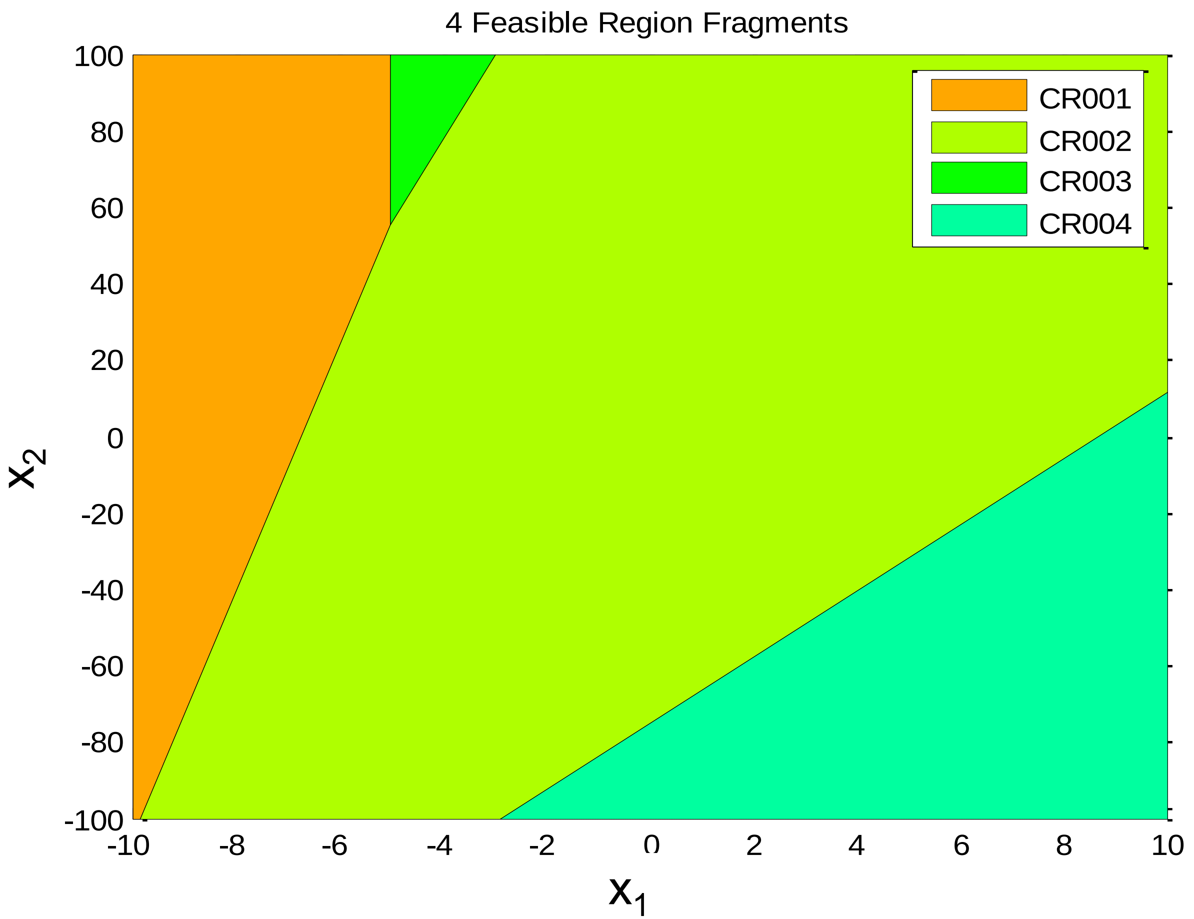
st.

$$\begin{bmatrix} 1 & 1 \\ 5 & -4 \\ -8 & 22 \\ -4 & -1 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} + \begin{bmatrix} -13 \\ -20 \\ -121 \\ 8 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-10 \leq \mathbf{x}_1 \leq 10 \qquad -100 \leq \mathbf{x}_2 \leq 100$$

Multi-parametric programming

Critical Regions



Multi-parametric programming

Multi-parametric Solution

$$
\mathbf{U} = \begin{cases}
\begin{bmatrix} -0.33 & 0 \\ 1.33 & 0 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} + \begin{bmatrix} -1.67 \\ 14.67 \end{bmatrix} & \textbf{if} \begin{bmatrix} 1 & -0.031 \\ 1 & 0 \\ -1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} \leq \begin{bmatrix} -6.71 \\ -5 \\ 10 \\ 100 \\ 100 \end{bmatrix} \\
\begin{bmatrix} 0.73 & -0.03 \\ 0.26 & 0.03 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} + \begin{bmatrix} 5.5 \\ 7.5 \end{bmatrix} & \textbf{if} \begin{bmatrix} 1 & -0.115 \\ -1 & 0.031 \\ -1 & 0.045 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} \leq \begin{bmatrix} 8.65 \\ 6.71 \\ 7.5 \\ 10 \\ 100 \\ 100 \end{bmatrix} \\
\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 13 \end{bmatrix} & \textbf{if} \begin{bmatrix} 1 & -0.045 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} \leq \begin{bmatrix} -7.5 \\ 5 \\ 100 \end{bmatrix} \\
\begin{bmatrix} 0 & 0.05 \\ 0 & 0.06 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} + \begin{bmatrix} 11.8 \\ 9.8 \end{bmatrix} & \textbf{if} \begin{bmatrix} -1 & 0.11 \\ 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} \leq \begin{bmatrix} -8.65 \\ 10 \\ 100 \end{bmatrix}
\end{cases}
$$

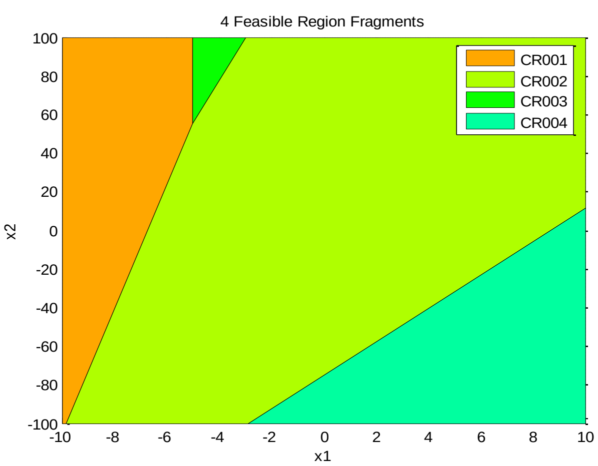
Multi-parametric programming

$$\min_u \left(-3u_1 - 8u_2\right)$$

$st.$

$$\begin{bmatrix} 1 & 1 \\ 5 & -4 \\ -8 & 22 \\ -4 & -1 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -13 \\ -20 \\ -121 \\ 8 \end{bmatrix} \le \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

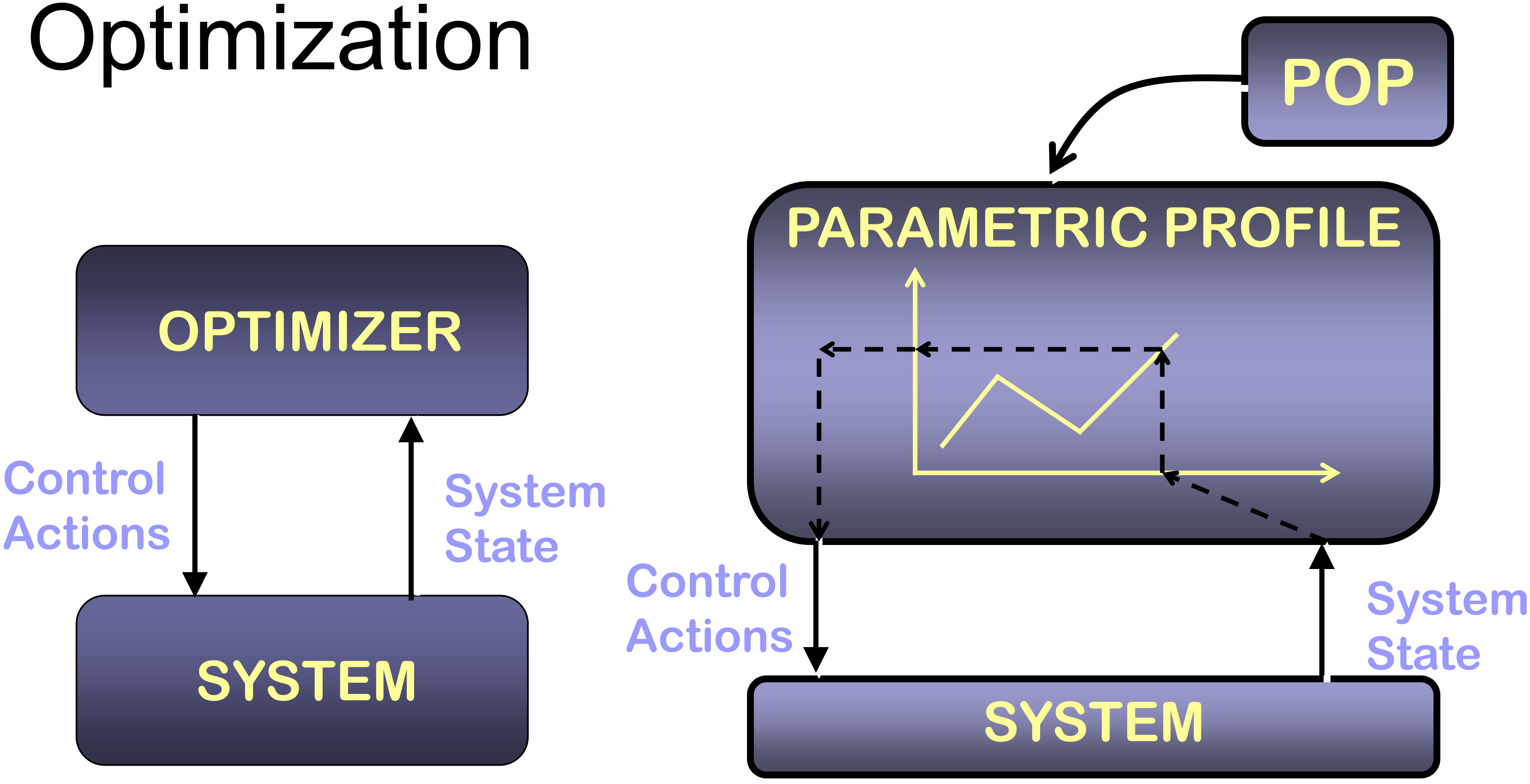
$$-10 \le x_1 \le 10, -100 \le x_2 \le 100$$



$$U = \begin{cases} \begin{bmatrix} -0.333 & 0 \\ 1.333 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -1.6667 \\ 14.6667 \end{bmatrix} & \textbf{if} \begin{bmatrix} 1 & -0.03125 \\ 1 & 0 \\ -1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \begin{bmatrix} -6.71875 \\ -5 \\ 10 \\ 100 \\ 100 \end{bmatrix} \\ \begin{bmatrix} 0.7333 & -0.0333 \\ 0.26667 & 0.03333 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 5.5 \\ 7.5 \end{bmatrix} & \textbf{if} \begin{bmatrix} 1 & -0.115385 \\ -1 & 0.03125 \\ -1 & 0.0454545 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \begin{bmatrix} 8.65385 \\ 6.71875 \\ 7.5 \\ 10 \\ 100 \\ 100 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 13 \end{bmatrix} & \textbf{if} \begin{bmatrix} 1 & -0.0454545 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \begin{bmatrix} -7.5 \\ 5 \\ 100 \end{bmatrix} \\ \begin{bmatrix} 0 & 0.05128 \\ 0 & 0.0641 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 11.8462 \\ 9.80769 \end{bmatrix} & \textbf{if} \begin{bmatrix} -1 & 0.115385 \\ 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \begin{bmatrix} -8.65385 \\ 10 \\ 100 \end{bmatrix} \end{cases}$$

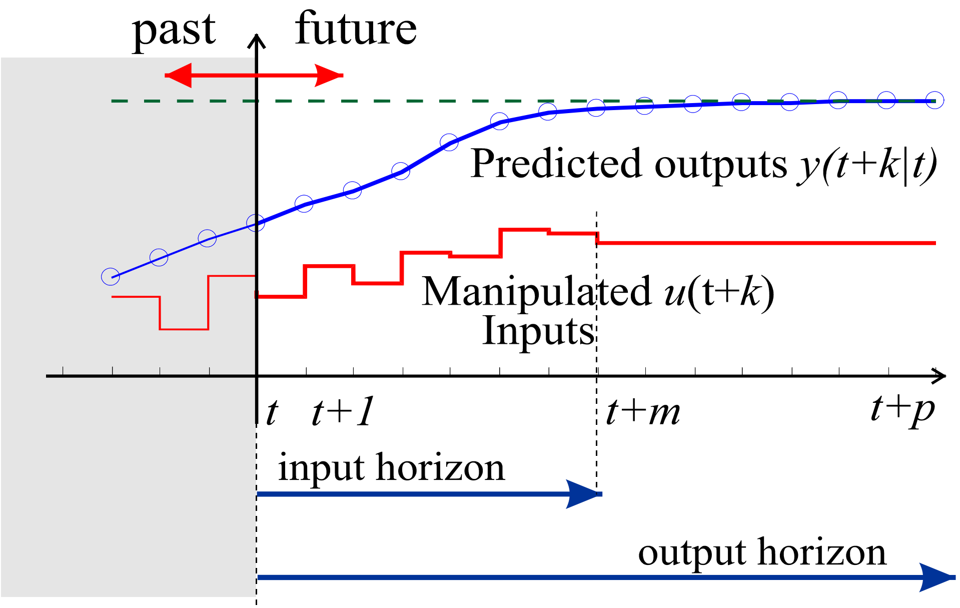
Only 4 optimization problems solved!

On-line Optimization via off-line Optimization



Function Evaluation!

Multi-parametric/Explicit Model Predictive Control



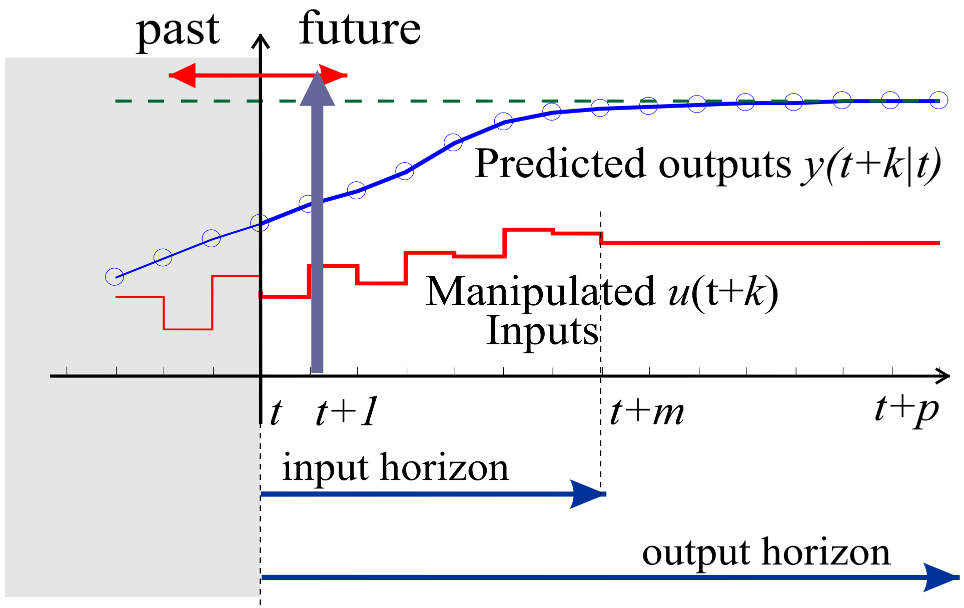
- Compute the optimal sequence of manipulated inputs which minimizes

tracking error = output – reference

subject to constraints on inputs and outputs

- On-line re-planning: Receding Horizon Control

Multi-parametric/Explicit Model Predictive Control



- Compute the optimal sequence of manipulated inputs which minimizes

Solve a QP at each time interval

- On-line re-planning: Receding Horizon Control

Multi-parametric Programming Approach

- State variables → Parameters
- Control variables → Optimization variables

- MPC → Multi-Parametric Programming problem
- Control variables → F(State variables)

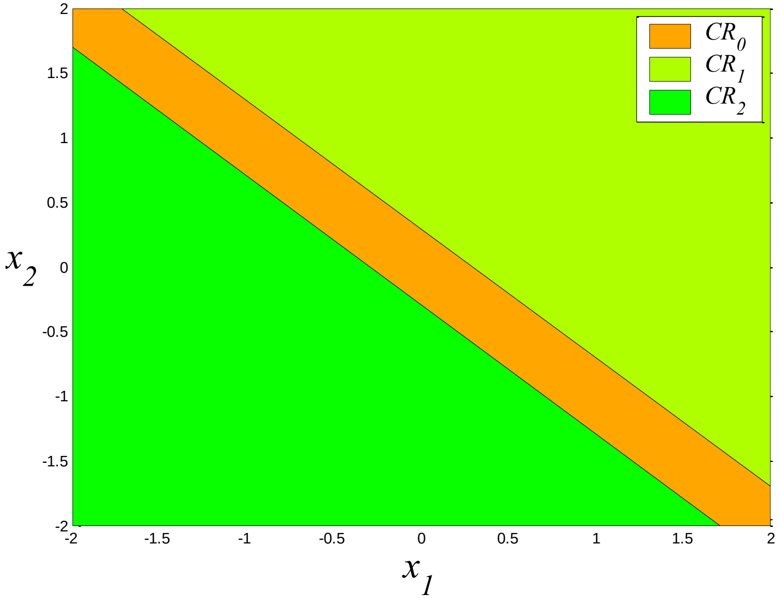
Multi-parametric Quadratic Program

Explicit Control Law

$$J(x(t)) = \min_{u_{t|t}, u_{t+1|t}} \sum_{j=0}^{1} \left\{ \mathbf{x}_{t+j|t}^T \mathbf{x}_{t+j|t} + 0.01\, \mathbf{u}_{t+j|t}^2 \right\} + \mathbf{x}_{t+2|t}^T P \mathbf{x}_{t+2|t}$$

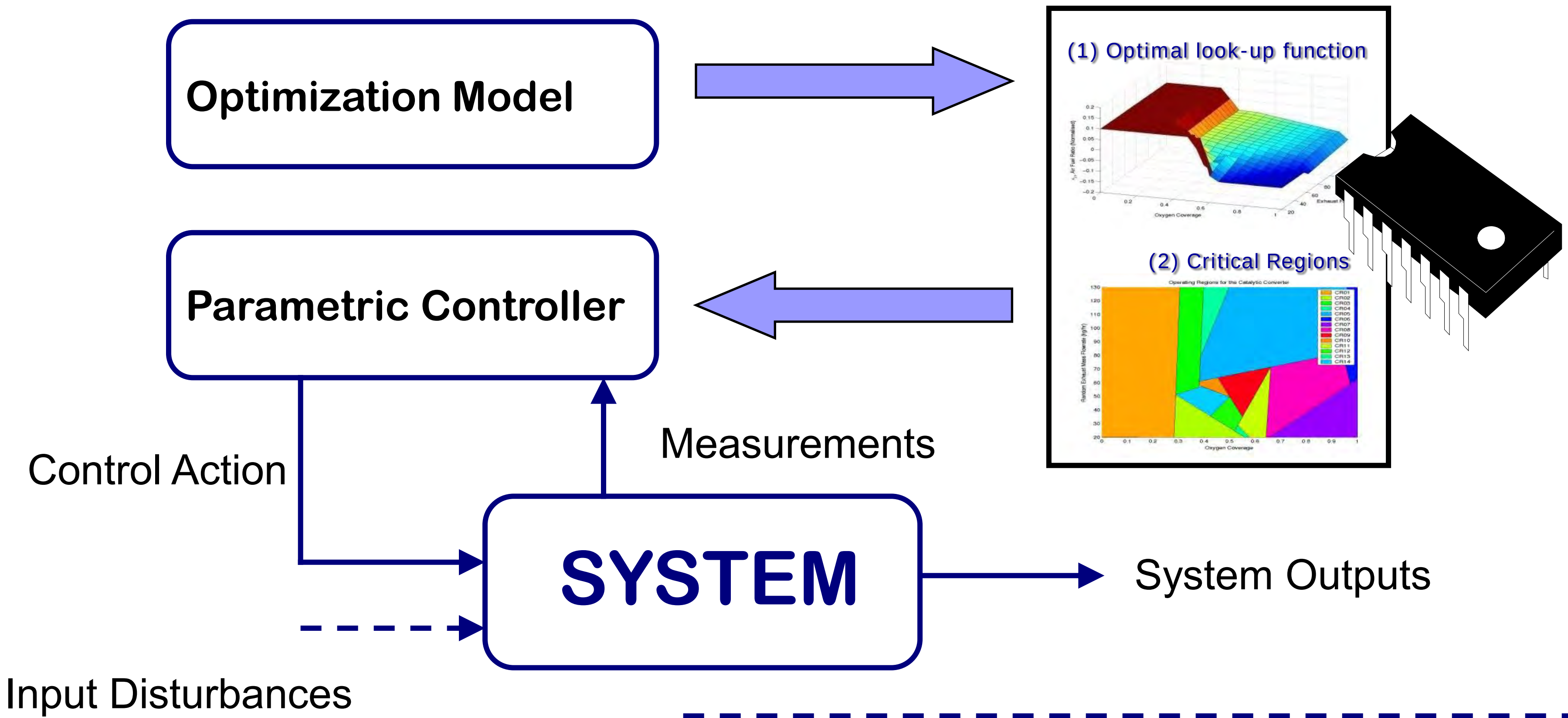
$$\text{s.t} \quad \mathbf{x}_{t+j+1|t} = \begin{bmatrix} 0.7326 & -0.0861 \\ 0.1722 & 0.9909 \end{bmatrix} \mathbf{x}_{t+j|t} + \begin{bmatrix} 0.0609 \\ 0.0064 \end{bmatrix} \mathbf{u}_{t+j|t}$$

$$-2 \le \mathbf{u}_{t+j|t} \le 2 \quad \text{j} = 1,2 \quad \mathbf{x}_{t|t} = \mathbf{x}(t)$$



$$\mathbf{u}(t) = \begin{cases} [-6.8355 \;\; -6.8585]\,\mathbf{x}(t) & \textbf{if} \begin{bmatrix} 0.7059 & 0.7083 \\ -0.7059 & -0.7083 \end{bmatrix} \mathbf{x}(t) \le \begin{bmatrix} 0.2065 \\ 0.2065 \end{bmatrix} \\ -2 & \textbf{if} \;\; [-0.7059 \;\; -0.7083]\,\mathbf{x}(t) \le -0.2065 \\ 2 & \textbf{if} \;\; [0.7059 \;\; 0.7083]\,\mathbf{x}(t) \le -0.2065 \end{cases}$$

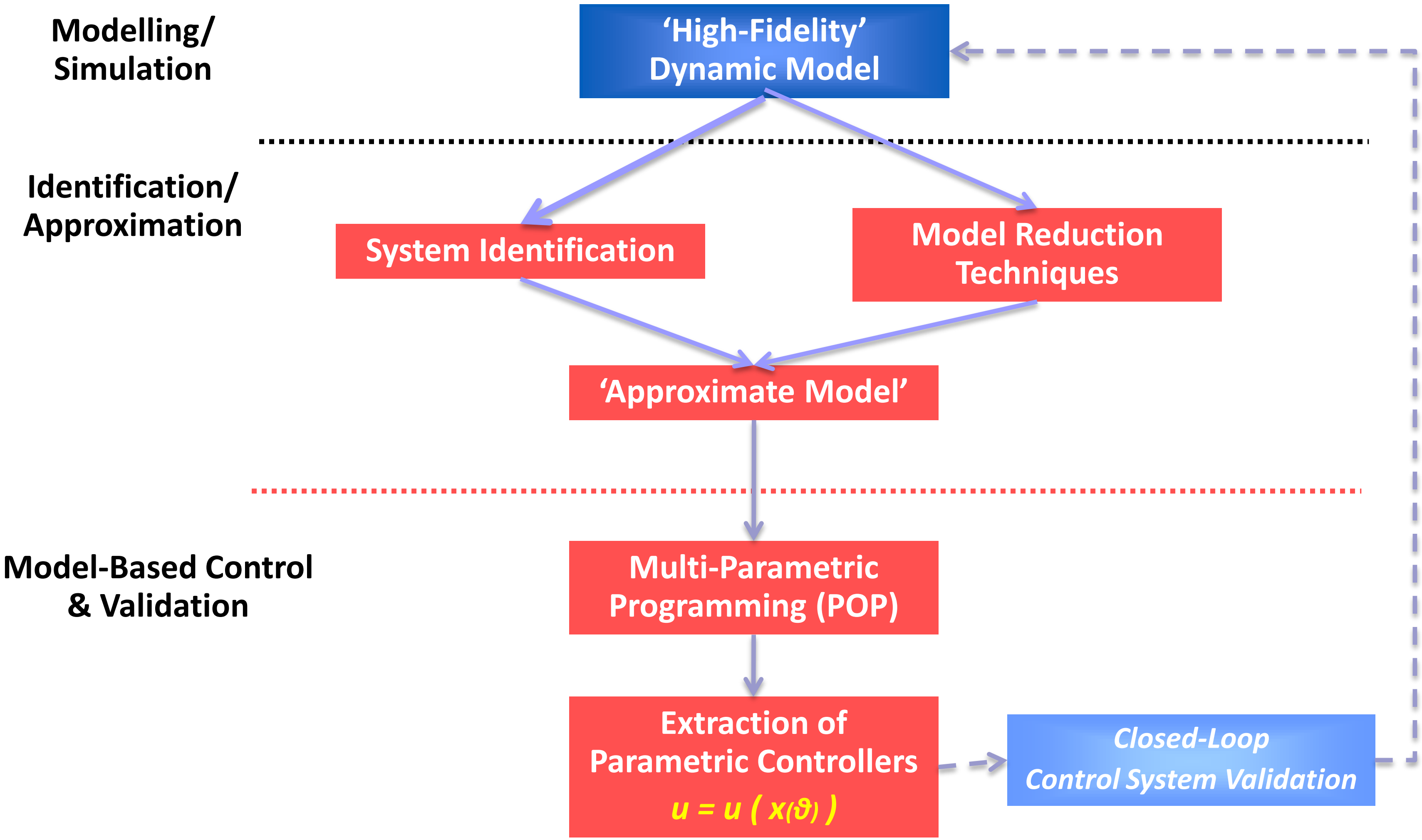
Multi-parametric Controllers



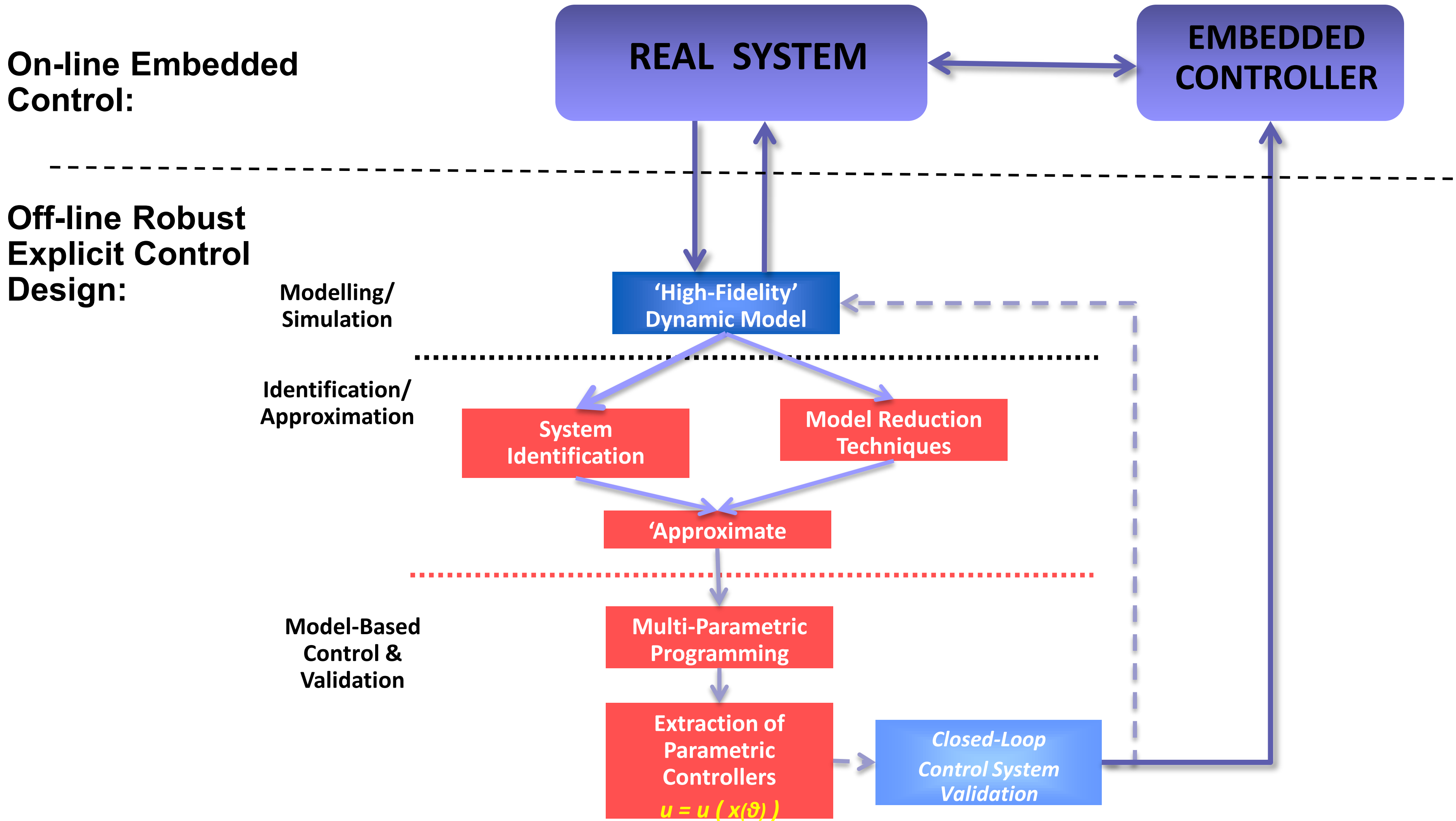
MPC-on-a-chip!

- **Explicit Control Law**
- **Eliminate expensive, on-line computations**
- **Valuable insights !**

A framework for multi-parametric programming & MPC *(Pistikopoulos 2008, 2009)*



A framework for multi-parametric programming and MPC *(Pistikopoulos 2010)*



Key milestones-Historical Overview

AIChE J.,Perspective (2009)

- Number of publications

	Multi-Parametric Programming	Multi-Parametric MPC & applications
Pre-1999	**>100**	**0**
Post-1999	**~70**	**250+**

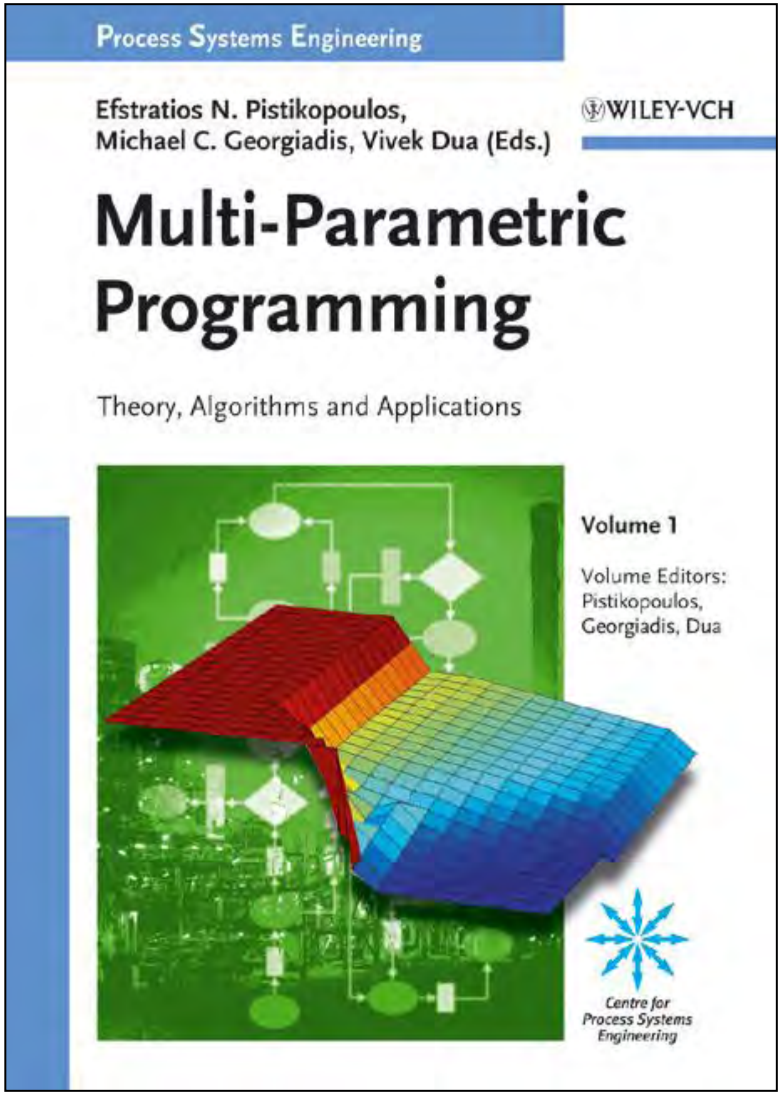
- 2002 Automatica paper ~ 650+ citations
- Multi-parametric programming – until 1992 mostly analysis & linear models
- Multi-parametric/explicit MPC – post-2002 much wider attention

Multi-parametric Programming Theory

mp-LP	Gass & Saaty [**1954**], Gal & Nedoma [**1972],** Propoi [1975], Adler and Monterio [1992], Gal [1995], Acevedo and Pistikopoulos[1997], Dua et al [2002], Pistikopoulos et al [2007]
mp-QP	Townsley [**1972**], Propoi [1978], Best [1995], Dua et al [2002], Pistikopoulos et al [2002,2007]
mp-NLP	Fiacco [**1976**],Kojima [1979], Bank et al [1983], Fiacco [1983], Fiacco & Kyoarisis [1986], Acevedo & Pistikopoulos [1996], Dua and Pistikopoulos [1998], Pistikopoulos et al [2007]
mp-DO	Sakizlis et al.[2002], Bansal [2003], Sakizlis et al [2005], Pistikopoulos et al [2007]
mp-GO	Fiacco [1990], Dua et al [1999,2004], Pistikopoulos et al [2007]
mp-MILP	Marsten & Morin [**1975**], Geoffrion & Nauss [1977], Joseph [1995], Acevedo & Pistikopoulos [1997,1999], Dua & Pistikopoulos[2000]
mp-MINLP	McBride & Yorkmark [**1980**], Chern [1991], Dua & Pistikopoulos [1999], Hene et al [2002], Dua et al [2002]

Multi-parametric/Explicit Model Predictive Control Theory

mp-MPC	Pistikopoulos [**1997, 2000],** Bemporad, Morari, Dua & Pistikopoulos [2000], Sakizlis & Pistikopoulos [2001], Tondel et al [2001], Pistikopoulos et al [2002], Bemporad et al [2002], Johansen and Grancharova [2003], Sakizlis et al [2003], Pistikopoulos et al [2007]
mp-Continuous MPC	Sakizlis et al [**2002**], Kojima & Morari[2004], Sakizlis et al [2005], Pistikopoulos et al [2007]
Hybrid mp-MPC	Bemporad et al [**2000**], Sakizlis & Pistikopoulos [2001], Pistikopoulos et al [2007]
Robust mp-MPC	Kakalis & Pistikopoulos [**2001**], Bemporad et al [2001], Sakizlis et al [2002], Sakizlis & Pistikopoulos [2002], Sakizlis et al [2004], Olaru et al [2005], Faisca et al [2008]
mp-DP	Nunoz de la Pena et al [2004],Pistikopoulos et al [2007],Faisca et al [2008]
mp-NMPC	Johansen [2002], Bemporad [2003], Sakizlis et al [2007], Dobre et al [2007], Narciso & Pistikopoulos [2009]


Process Systems Engineering
Efstratios N. Pistikopoulos,
Michael C. Georgiadis, Vivek Dua (Eds.)
WILEY-VCH
Multi-Parametric Programming
Theory, Algorithms and Applications
Volume 1
Volume Editors:
Pistikopoulos,
Georgiadis, Dua
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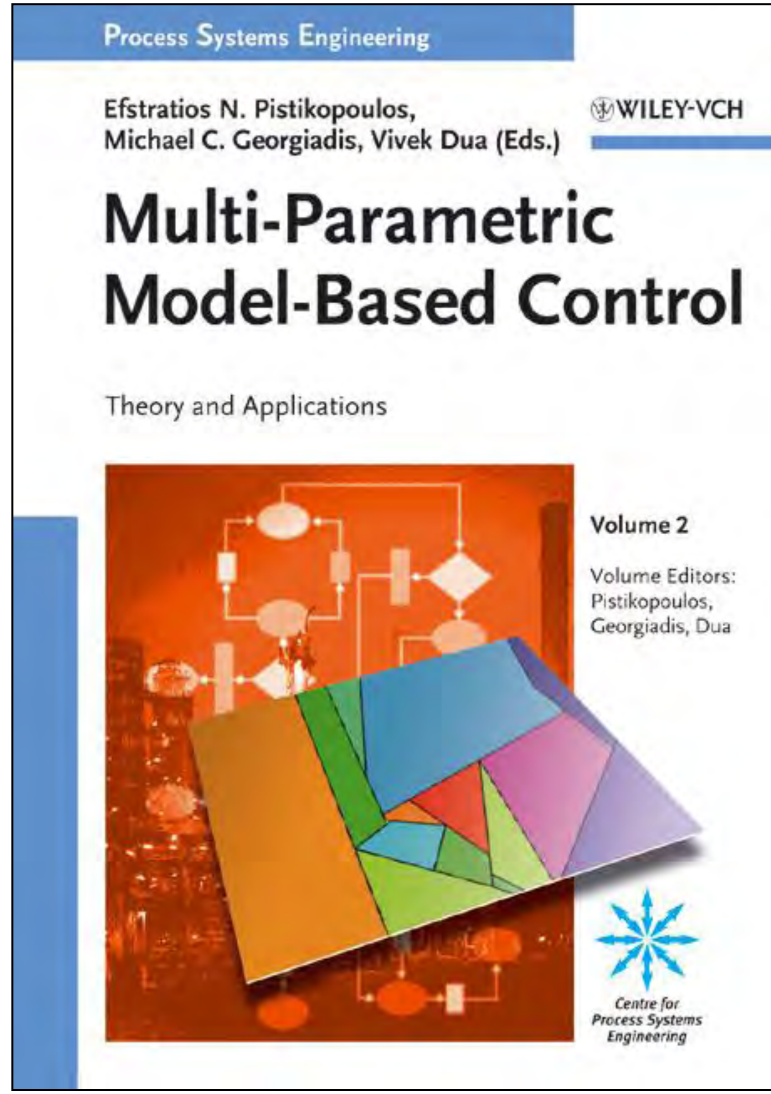
Process Systems Engineering
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WILEY-VCH
Multi-Parametric Model-Based Control
Theory and Applications
Volume 2
Volume Editors:
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Georgiadis, Dua
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Patented Technology

- Improved Process Control

  European Patent No EP1399784, 2004

- Process Control Using Co-ordinate Space

  United States Patent No US7433743, 2008

ParOS
Advanced control technology on a chip

Outline

- Key concepts & historical overview
- Recent developments in multi-parametric programming and mp-MPC
  - Model reduction/approximation
  - mp-NLP & explicit nonlinear mp-MPC
  - mp-MILP
  - State estimation and mp-MPC
  - Focus on **Robust Explicit mp-MPC**

Model Reduction/Approximation

$$\dot{x} = f(x,u)$$
$$x = g(\dot{x},x,u)$$
$$y = g(x,u)$$

Replace discrete dynamical System with a set of affine algebraic models

N-step ahead prediction- enables use of Linear MPC routines

$$\underset{\Delta u}{\text{Min}} \left\{ y_{t+N}^{*T} P y_{t+N}^{*T} + \sum_{k=0}^{N-1} y_{t+k}^{*T} Q y_{t+k}^{*} + \delta u_{t+k}^{T} R \delta u_{t+k} \right\}$$

st:

$$y_{t+k}^{*} = y_{t+k} - y_{sp},\ k = 1 \dots N$$

$$y_{t+k} = A_k x_t + B_k U + C_k,\ k = 1 \dots N$$

$$u_{t+k} = u_{t+k-1} + \delta u_{t+k},\ k = 1 \dots N$$

$$y_k \in \mathcal{Y},\ k = 0 \dots N$$

$$u_k \in \mathcal{U},\ k = 0 \dots N$$

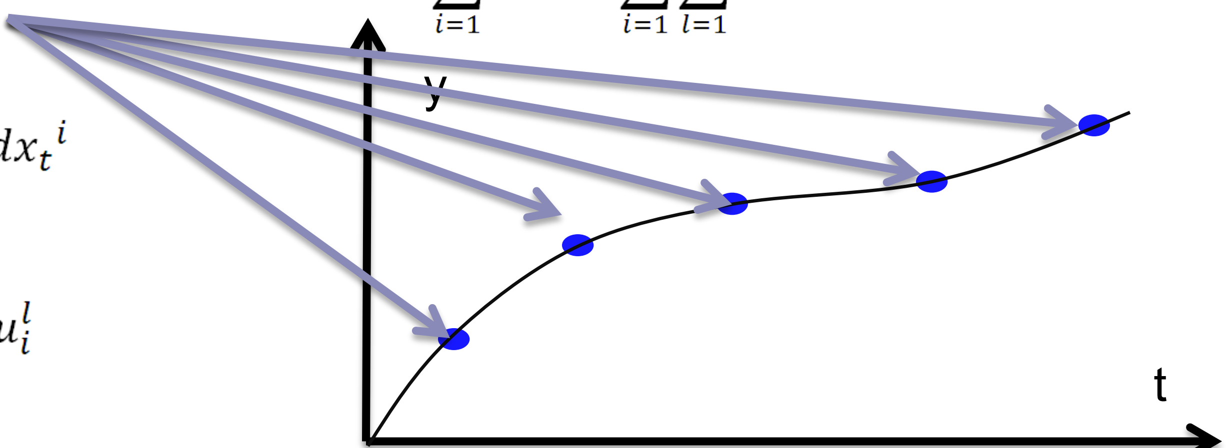
Approximation Method

- N-step-ahead approximation based on initial conditions (measurements) and sequence of controls (constant control vector parameterization). Set of affine algebraic models
- For all j point over the time horizon - approximations are constructed as follows

$$y_j = f_j\left(x_t{}^1, x_t{}^2, \ldots, x_t{}^n, u_1, u_2, \ldots, u_j\right) \approx \alpha_0 + \sum_{i=1}^{n} \alpha_i x_t{}^i + \sum_{i=1}^{j} \sum_{l=1}^{m} \alpha_i^l u_i^l$$

$$\alpha_i = \int_0^1 f_i(x_t{}^i)\varphi_1(x_t{}^i)dx_t{}^i$$

$$\alpha_j^l = \int_0^1 f_j^l(u_i^l)\varphi_1(u_i^l)du_i^l$$



mp-NLP Algorithms for Explicit NMPC

Strategy: **Direct Approach**

Multi-parametric Nonlinear Dynamic Optimization Problem

Discretize state and controls via Orthogonal Collocation Techniques

Approximate

Multi-parametric Nonlinear Programming Problem (mp-NLP)

Quadratic Approximation Based
Solve sequence of mp-QP's

Nonlinear Sensitivity based
Solve sequence of NLP's

Partition state space recursively

mp-NLP Algorithms for Explicit NMPC

Key features: Two implementations for the characterization of the Parameter space

Quadratic Approximation based (General mp-NLP)

- Characterizes the parameter space by sub-partitioning CRs where the QA approximation provides "poor" solutions.

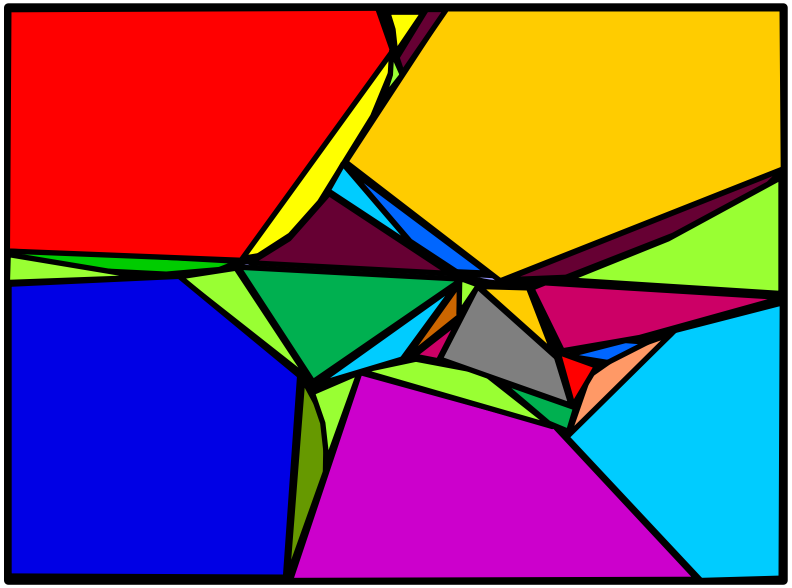
NLP Sensitivity Based (NMPC mp-NLP)

- Characterizes the parameter space using NLP sensitivity information and linearization of the constraints.

$$\begin{pmatrix} v(x) \\ \lambda(x) \\ \mu(x) \end{pmatrix} = \begin{pmatrix} v_0 \\ \lambda_0 \\ \mu_0 \end{pmatrix} - (M_0)^{-1} N_0 + (x - x_0) + \phi(||x||)$$

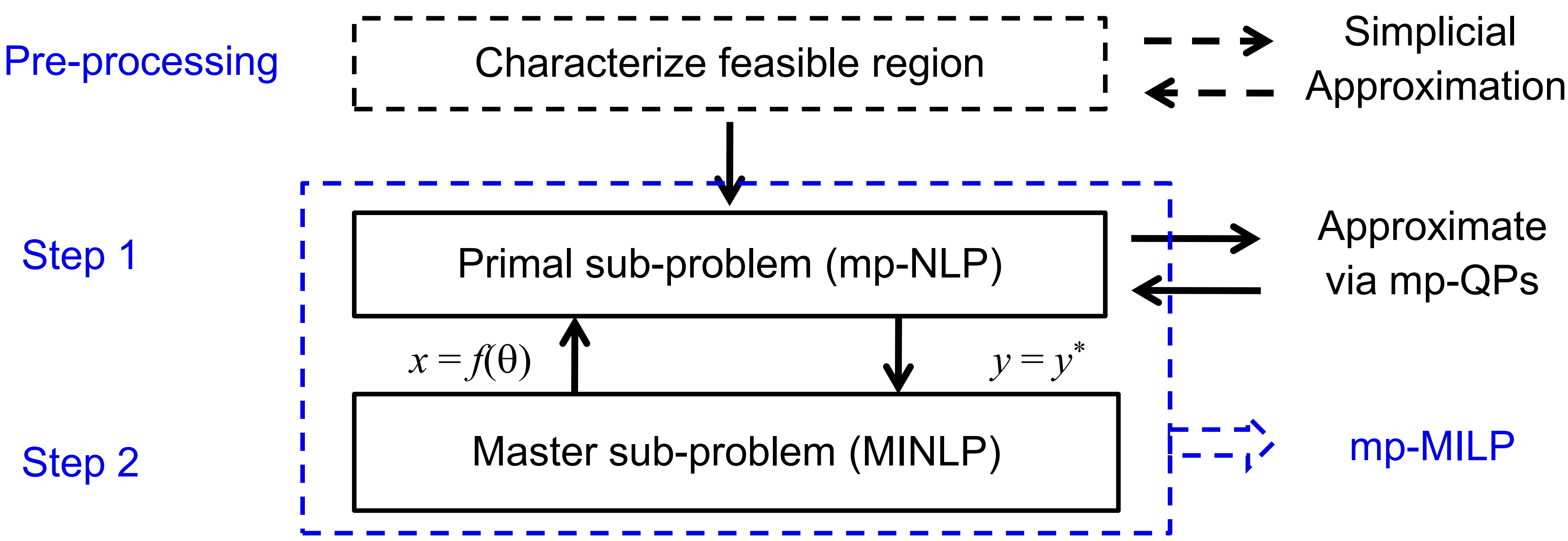
Validity of approximation:

$\phi(x) = O(||x||) \Rightarrow (x)/||x|| \rightarrow 0$ as $x \rightarrow 0$.



Multiparametric Mixed-Integer Nonlinear Programming

Strategy: **Decompose mp-MINLP into two sub-problems**



Iterate until master sub-problem is infeasible

MHE & mp-MPC

$$\min_{\bar{x}_0, \bar{u}_k} \left\| \bar{x}_{N_{MPC}} \right\|^2_{P_{MPC}} + \sum_{k=0}^{N_{MPC}} \left\| \bar{x}_k \right\|^2_{Q_{MPC}} + \sum_{k=0}^{N_{MPC}-1} \left\| \bar{u}_k \right\|^2_{R_{MPC}}$$

$$\text{s.t. } x_{k+1} = Ax_k + Bu_k + Gw_k \text{ (actual system)},$$

$$\bar{x}_{k+1} = A\bar{x}_k + B\bar{u}_k \text{ (nominal system)},$$

$$\hat{x}_{k+1} = A\hat{x}_k + Bu_k + t \text{ (estimated system step 1.3)},$$

$$u_0 = \bar{u}_0^* - K\left(\hat{x}_0^* - \bar{x}_0^*\right), \bar{u} \in \mathbf{U} \ominus K \cdot \bar{\mathbf{S}}, \bar{x}_{N_{MPC}} \in \bar{\mathbf{X}}_{\bar{f}},$$

$$\bar{x}_k \in \bar{\mathbf{X}} = \mathbf{X} \ominus \mathbf{S}, k = 1 \ldots N_{MPC} - 1, \mathbf{S} = \mathbf{E}_x \oplus \bar{\mathbf{S}}, \hat{x}_0 \in \bar{x}_0 \oplus \bar{\mathbf{S}},$$

$$\bar{\mathbf{S}} \text{ is mRPI of } \hat{x}_{k+1} - \bar{x}_{k+1} = \left(A - BK\right)\left(\hat{x}_k - \bar{x}_k\right) + t.$$

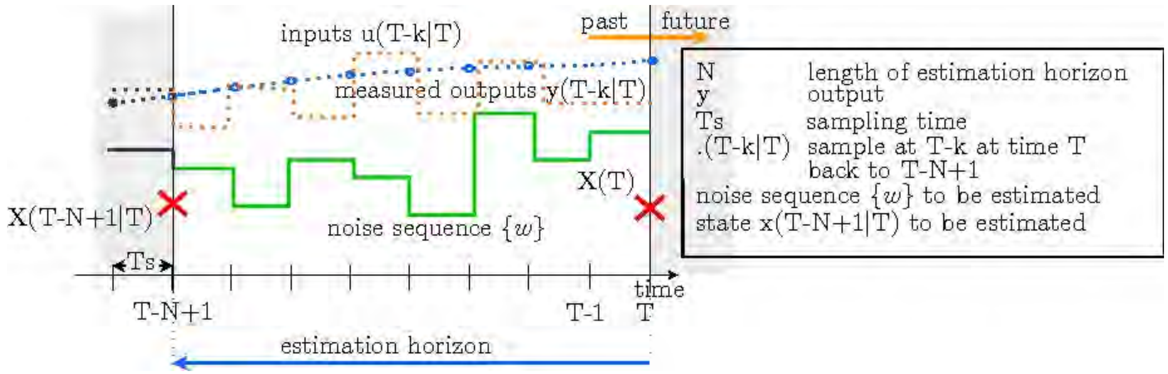
Main idea:

Step 1. Formulate the dynamics that govern the estimation error $e_T = f\left(e_{T-1}, w_{T-1}\right)$

Step 2. Use these dynamics to find the set that bounds the estimation error $e_T \in \bar{S}$

Step 3. Incorporate the bounding set into the controller to ‘robustify’ against the estimation error

Moving Horizon Estimation (MHE)



$$\min_{\hat{x}_{T-N|T},\hat{W}_{T-N|T}} \left\|\hat{x}_{T-N|T} - \underline{x}_{T-N|T}\right\|^2_{P^{-1}} - \left\|Y_{T-N}^{T-1} - \mathcal{O}\hat{x}_{T-N|T} - c\bar{b}U_{T-N}^{T-2}\right\|^2_{\mathcal{W}^{-1}} + \sum_{k=T-N}^{T-1}\left\|\hat{w}_k\right\|^2_{Q^{-1}} + \sum_{k=T-N}^{T}\left\|\hat{v}_k\right\|^2_{R^{-1}}$$

$$\text{s.t.} \quad \hat{x}_{k+1} = A\hat{x}_k + Bu_k + G\hat{w}_k, \quad \hat{y}_k = C\hat{x}_k + \hat{v}_k, \quad \hat{x}_k \in \mathbf{X}, \quad \hat{w}_k \in \mathbf{W}, \quad \hat{v}_k \in \mathbf{W}$$

$$\underline{x}_{T-N|T} = A\hat{x}^*_{T-N-1|T-1} + Bu_{T-N-1|T-1} + G\hat{w}^*_{T-N-1|T-1} \text{ (smoothed update of arrival cost)}$$

- Model-based state estimator
- Obtains current state estimate x_T
- Main advantage: incorporates system constraints
- MHE is dual to MPC: *backwards MPC*

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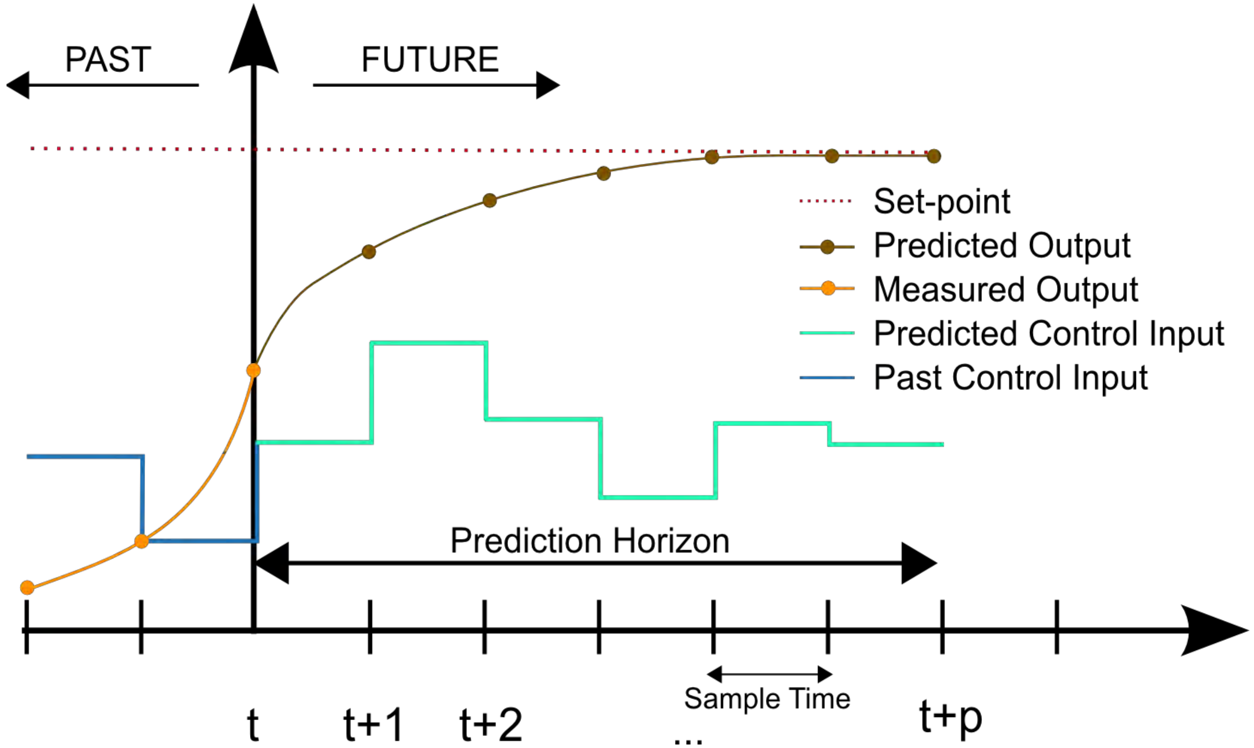
Robust Explicit MPC of continuous & hybrid systems

Model predictive control (MPC)

Overview



Closed-loop control strategy

The optimal control sequence is computed at each time step **t**

Model predictive control (MPC)

Online optimization

Control input: Optimization variable

System state

$$\min_{U} \; J\{U, x\} = x'_N P x_N + \sum_{k=0}^{N-1} [x'_k Q x_k + u'_k R u_k],$$

$$\text{s.t.} \quad x_{k+1} = f(x_k, u_k), \qquad u_k \in \mathcal{U}, \qquad x_k \in \mathcal{X}, \qquad k = 0, 1, \ldots, N-1$$

$$U = [u_0, u_1, \ldots, u_N - 1]^T$$

Solution: Optimal control action, U^* for a given value of $\mathbf{x}$
(optimization performed **online**)

Model predictive control (MPC)

Multi-parametric MPC

Control input: Optimization variable

System state: Parameters

$$\min_{U} \; J\{U,x\} = x'_N P x_N + \sum_{k=0}^{N-1} [x'_k Q x_k + u'_k R u_k],$$

$$\text{s.t.} \quad x_{k+1} = f(x_k, u_k),$$
$$u_k \in \mathcal{U}, \qquad k = 0, 1, \ldots, N-1$$
$$x_k \in \mathcal{X},$$
$$U = [u_0, u_1, \ldots, u_N - 1]^T$$

Solution: Optimal control law, $U^* = f(x)$
(optimization performed **offline**)

Framework for robust explcit MPC

- Constrained dynamic programming of linear/quadratic and mixed integer problems by multi-parametric programming
- MPC – dynamic programming representation
- Robustification step for LHS uncertainty
- Global solution of mp-LP/QP or mp-MILP/MIQP – RHS and objective function uncertainty

Key concepts

Dynamic programming representation

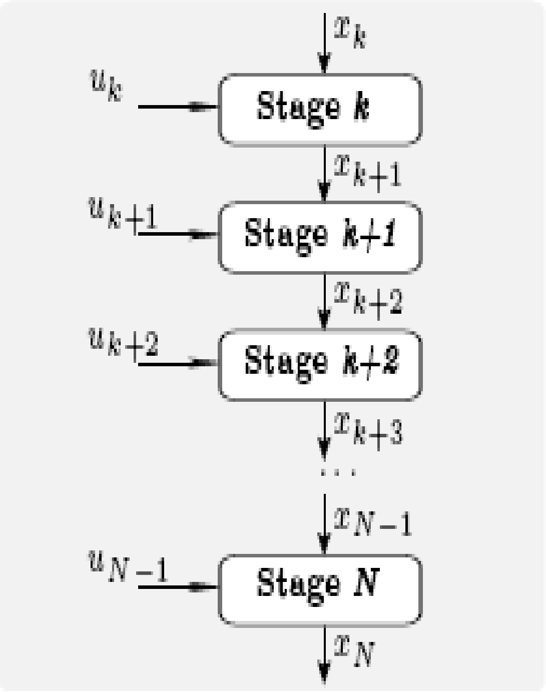
applied to

(Multi-parametric) model predictive control

towards

Robust multi-parametric model predictive control

Constrained Dynamic Programming by multi-parametric programming



$$V_k(x_k) = \min_{u_i \in U_i} \; g_N(x_N) + \sum_{i=k}^{N-1} g_i(x_i, u_i)$$

- The multi-stage optimisation problem is disassembled into a set of lower dimensionality problems

$$V_k(x_k) = \min_{u_i \in U_i} \; g_k(u_k, x_k) + V_{k+1}(x_{k+1})$$

Bellman, 1962; Bertsekas, 2005; Bazaraa & Shetty, 1979

- At each stage k the decision u_k is obtained given current x_k and provided that future decisions are optimized
- Obtain the optimal decision sequence $\{u_k, u_{k+1}, \ldots, u_{k+N-1}\}$

Constrained Dynamic Programming by multi-parametric programming

- Step 1: ($j=1$) Solve N^{th} stage as a multi-parametric programming problem with x_N as parameters

- Step 2: ($j=j+1$) Solve $(N\text{-}j+1)^{th}$ stage as a multi-parametric problem with parameters $x_{N\text{-}j}$ and $u_{N\text{-}j+1}$, … , $u_{N\text{-}1}$

$$u_{N-j} = f_{N-j}(x_{N-j}, u_{N-j+1}, \ldots, u_{N-1})$$

$$u_{N-j+1} = f_{N-j+1}(x_{N-j+1}, u_{N-j+2}, \ldots, u_{N-1})$$

$$\vdots$$

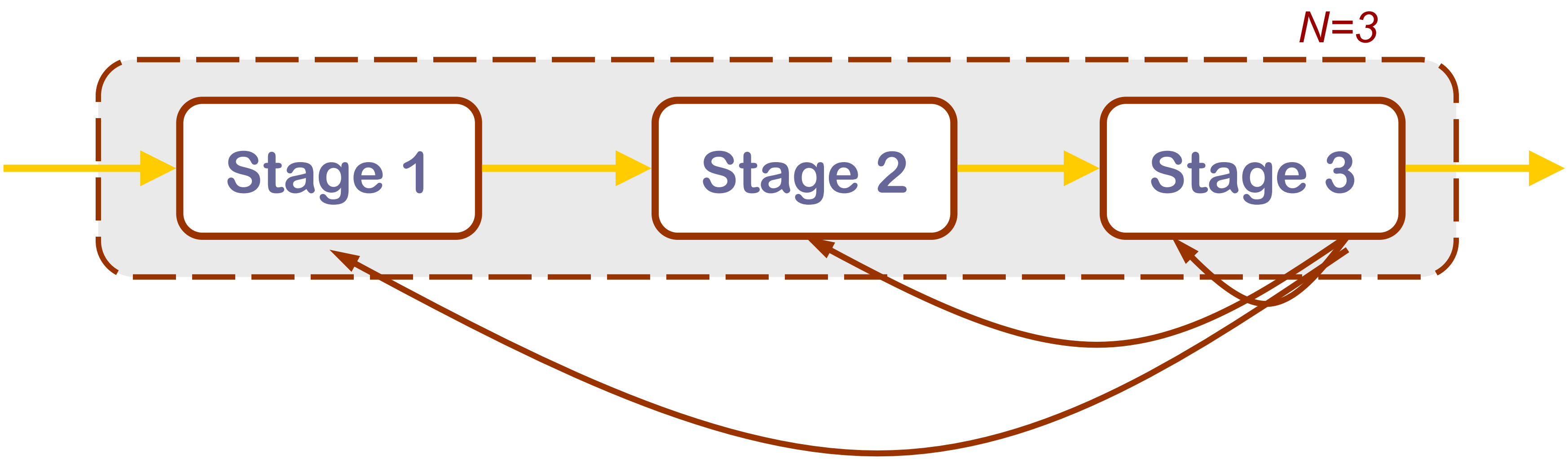
$$u_{N-1} = f_{N-1}(x_{N-1})$$

- Step 3: ($j=j+1$) Replace previous multi-parametric solutions in the current solution to obtain

$$u_{N-j} = f_{N-j}(x_{N-j})$$

- Step 4: Repeat or stop if $j=N$.

mp-MPC via Dynamic Programming – nominal case



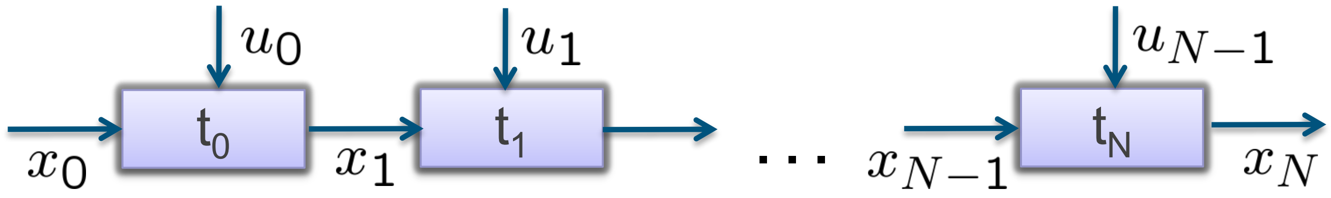
mp-QP solved at each stage with only constraints corresponding to the current level!

- No extra complexity is introduced – no need for global optimization
- Optimization variables are obtained as explicit functions of the states

MPC via dynamic programming

MPC as a stage-wise process

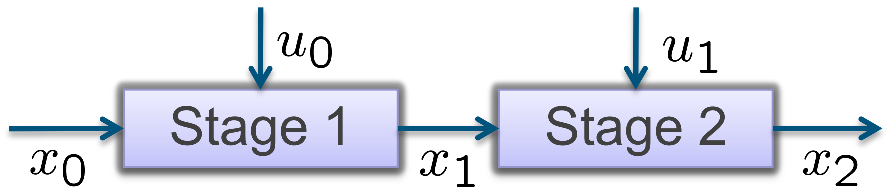
For a horizon N, the MPC problem to be solved may be represented by the block diagram



Standard MPC algorithms do not explore this particular structure

mp-MPC via dynamic programming

Hybrid system, N=2



$$\min_{u_0,u_1} \begin{cases} x_0^T Q x_0 + x_1^T Q x_1 + x_2^T P x_2 \\ + u_0^T R u_0 + u_1^T R u_1 \end{cases}$$

$$\text{s.t.} \quad x_2 = A^i x_1 + B^i u_1 + f^i \ \text{if} \begin{bmatrix} x_k \\ u_k \end{bmatrix} \in \mathcal{P}_i, i = 1, \ldots, s$$

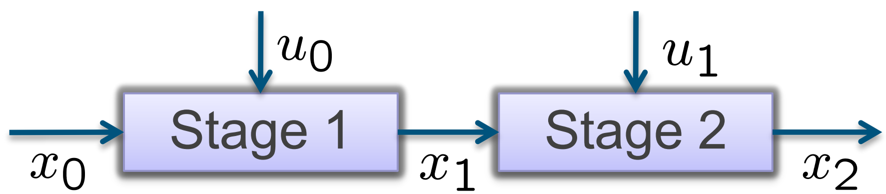
$$x_1 = A^i x_0 + B^i u_0 + f^i$$

$$x_1, x_2 \in \mathcal{X}; u_0, u_1 \in \mathcal{U}$$

Logical condition

mp-MPC via dynamic programming

Hybrid system, N=2



$$\min_{u_0,u_1} \begin{cases} x_0^T Q x_0 + x_1^T Q x_1 + x_2^T P x_2 \\ + u_0^T R u_0 + u_1^T R u_1 \end{cases}$$

$$\text{s.t.} \quad x_2 = B_3 z_1$$

$$x_1 = B_3 z_0$$

$$E_2 d_k + E_3 z_k \leq E_1 u_k + E_4 x_k + E_5$$

$$x_1, x_2 \in \mathcal{X};\ u_0, u_1 \in \mathcal{U}$$

Integer variables (d_k)

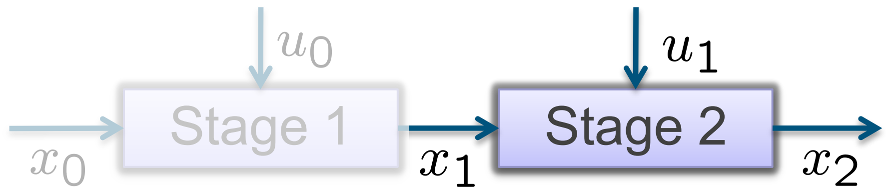
Continuous variables (z_k)

- mp-MILP/MIQP re-formulation
- Solved by using mp-MILP or mp-MIQP algorithms

mp-MPC via dynamic programming

Hybrid system, N=2

Stage 2



$$\min_{u_0, u_1} \begin{cases} x_0^T Q x_0 + x_1^T Q x_1 + x_2^T P x_2 \\ + u_0^T R u_0 + u_1^T R u_1 \end{cases}$$

$$\text{s.t.} \quad x_2 = B_3 z_1$$
$$x_1 = B_3 z_0$$
$$E_2 d_k + E_3 z_k \leq E_1 u_k + E_4 x_k + E_5$$
$$x_1, x_2 \in \mathcal{X}; u_0, u_1 \in \mathcal{U}$$

Optimisation variables: u_1, d, z

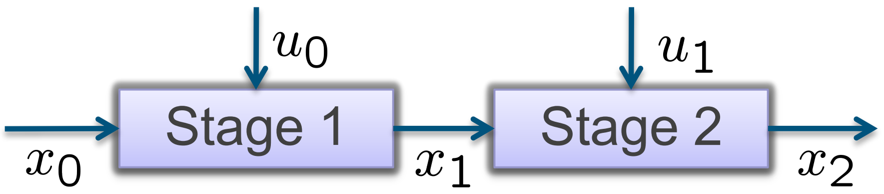
Parameter: x_1

Solution for Stage 2 obtained: $u_1 = f(x_1)$

mp-MPC via dynamic programming

Hybrid system, N=2

Stage 1



$$\min_{u_0,u_1} \begin{cases} x_0^T Q x_0 + x_1^T Q x_1 + x_2^T P x_2 \\ + u_0^T R u_0 + u_1^T R u_1 \end{cases}$$

$$\text{s.t.} \quad \begin{aligned} & x_2 = B_3 z_1 \\ & x_1 = B_3 z_0 \\ & E_2 d_k + E_3 z_k \le E_1 u_k + E_4 x_k + E_5 \\ & x_1, x_2 \in \mathcal{X};\, u_0, u_1 \in \mathcal{U} \end{aligned}$$

Optimisation variable: u_0, d, z

Parameters: x_0, u_1

Solution for Stage 2 obtained: $u_0 = f(x_0, u_1)$

Introduce solution from stage 2 to obtain: $u_0 = f(x_0)$

Robust mp-MPC

- Dynamic System with Model Uncertainty

$$V(x) = \min_U \left\{ \sum_{k=0}^{N-1} (x_k' Q x_k + u_k' R u_k) + x_N' P x_N \right\}$$
$$x_{k+1} = \mathbf{A} x_k + \mathbf{B} u_k + W\theta_k$$
$$Cx_k + Du_k \le d$$
$$Mx_N \le \mu$$
$$u_{min} \le u \le u_{max}$$
$$x_0 = x$$

Parametric Uncertain System

$$A = [a_{ij}] \in \mathbb{R}^{n\times n},\ B = [b_{ij}] \in \mathbb{R}^{n\times m}$$
$$a_{ij} \in \left\{ a_{ij} : |a_{ij} - a_{ij,0}| \le \varepsilon a_{ij,0} \right\}$$
$$b_{ij} \in \left\{ b_{ij} : |b_{ij} - b_{ij,0}| \le \varepsilon b_{ij,0} \right\}$$

Exogenous Disturbance

$$\theta_{min} \le \theta \le \theta_{max}$$

x : system states
u : control inputs
$U = [u_0' \quad \ldots \quad u_{N-1}']'$

- Uncertainty due to modelling, identification errors, measurement errors etc.
- Constraints represent safety, operational constraints
- It is very *critical* that the system does not violate them
- Immunize against uncertainty

Robust mp-MPC via Dynamic & Multi-parametric Programming

- Dynamic Programming framework
- Robustification – robust reformulation step (*Ben-Tal & Nemirovski, 2000)*
- Novel Multi-parametric Programming algorithm to constrained Dynamic Programming
  - Small mp-QP at each stage
  - No need for global optimisation

Robust mp-MPC via Dynamic & Multi-parametric Programming

- (Mixed-integer) Linear Programming with Uncertainty

$$\min_x J(x) = c' \cdot x$$
$$\text{s.t. } E \cdot x = e,$$
$$A \cdot x \le b,$$
$$A = A_0 + \Delta A,$$
$$-\varepsilon \cdot |A_0| \le \Delta A \le \varepsilon \cdot |A_0|$$
$$l \le x \le u$$

- Robust re-formulation:

$$\min_x J(x) = c' \cdot x$$
$$\text{s.t. } E \cdot x = e,$$
$$A_0 \cdot x \le b,$$
$$A_0 \cdot x + \varepsilon \cdot |A_0| \cdot |x| \le b + \delta \cdot \max[1, |b|],$$
$$l \le x \le u$$

Immunization against worst-case uncertainty

Infeasibility tolerance

Explicit Solution of the general mp-MILP Problem

$$
\begin{aligned}
& z(\theta) := \min_{x,y} c(\theta)^T x + d(\theta)^T y \\
s.t. \quad & A(\theta)x + E(\theta)y \leq b(\theta) \\
& x \in \mathbb{R}^n, x_{min} \leq x \leq x_{max}, \;\; y \in \{0,1\}^q \\
& \theta \in \mathbb{R}^p, \theta_{min} \leq \theta \leq \theta_{max}
\end{aligned}
$$

Applications

- **Pro-active Scheduling** under price, demand and processing time uncertainty *(seee poster & paper)*
- **Explicit Model Predictive Control** of **Hybrid Systems**: Control actions as *optimization variables*, states as *parameters*, input and model disturbances *as parameters*

Hybrid Approach - Two-Stage Method for mp-MILP[1]

Stage 1 – Reformulation

Partially robust RIM-mp-MILP* model;

Solutions are immunized against all *immeasurable parameters* and *complicating constraint matrix uncertainty*

↓

Stage 2 – Solution

Suitable multi-parametric programming algorithms *(e.g. Faisca et al. (2009))*

↓

Optimal partially robust solution; Upper bound on optimal objective function value

[1] *Wittmann-Hohlbein, Pistikopoulos (2011)*

*objective function coefficient and right hand side vector uncertainty

Global Optimization of mp-MILP[1]

Constraint matrix uncertainty poses major challenge $\longrightarrow$ mp-MINLP

Multi-Parametric Global Optimization:

- Adaptation of strategies from the deterministic case to multi-parametric framework: *Parametric B&B procedure*
- Globally optimal solution is a piecewise affine function over polyhedral convex critical regions

Challenges in Global Optimization of mp-MILP Problems:

- *Comparison of parametric profiles*, not scalar values
- High computational requirements

$\downarrow$

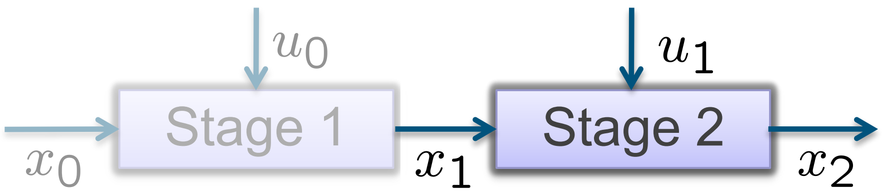
Can we find *"good solutions"* of an mp-MILP problem with less effort?

[1] *Wittmann-Hohlbein, Pistikopoulos; JOGO, submitted , 2011*

MPC via dynamic programming

Illustrative example

Stage 2

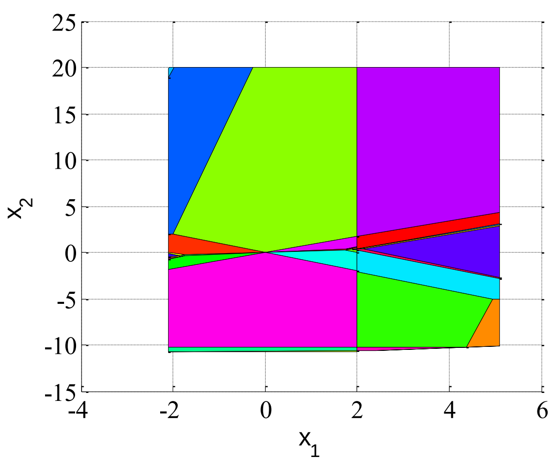


Optimisation variables:

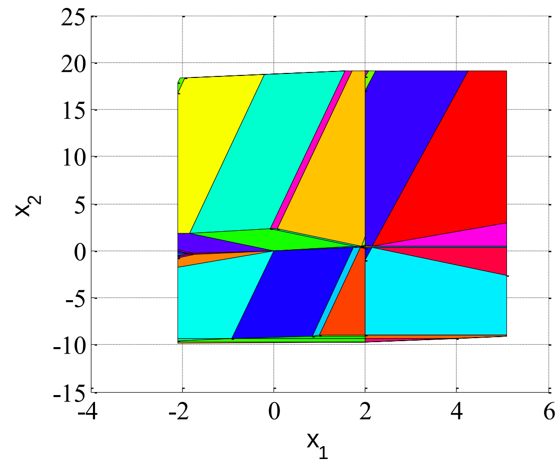
$$u_1, z_1, d_1, \epsilon_2^x, \epsilon_1^u, w_1$$

Parameters:

$$x_1$$



Nominal



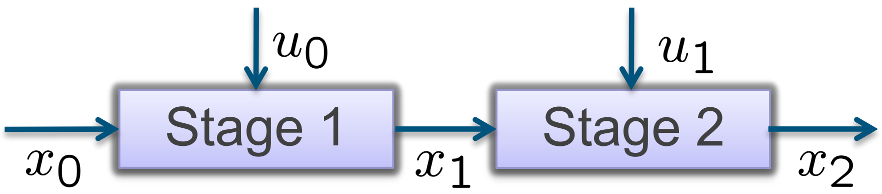
Robust

$$u_1 = f(x_1)$$

MPC via dynamic programming

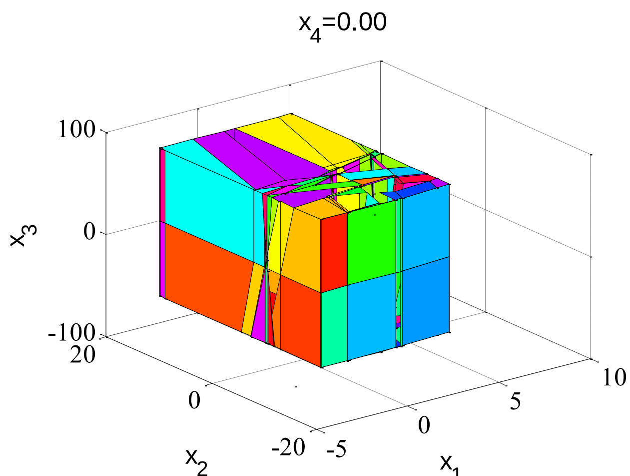
Illustrative example

Stage 1

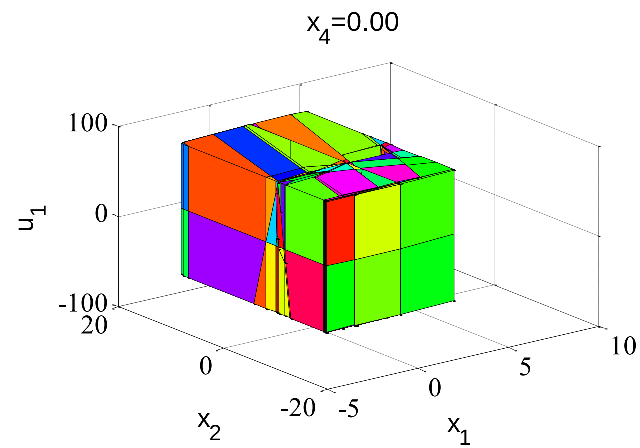


Optimisation variables:
$u_0, z_0, d_0, z_1, d_1, \epsilon_1^x, \epsilon_2^x, \epsilon_0^u, \epsilon_1^u, w_0$

Parameters:
x_0, u_1



Nominal



Robust

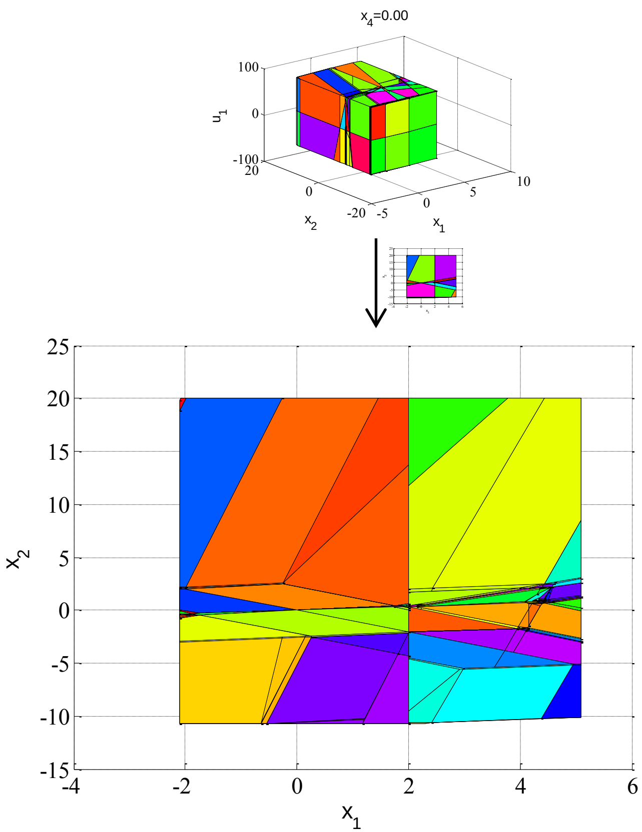
$$u_0 = f(x_0, u_1)$$

MPC via dynamic programming

Illustrative example

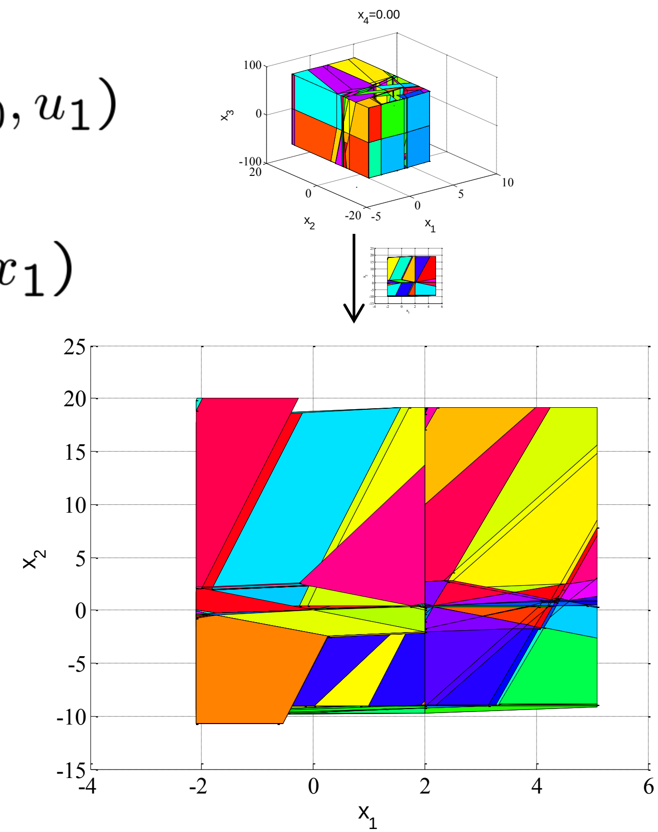
$$u_0 = f(x_0, u_1)$$

$$u_1 = f(x_1)$$



Nominal

$$u_0 = f(x_0)$$

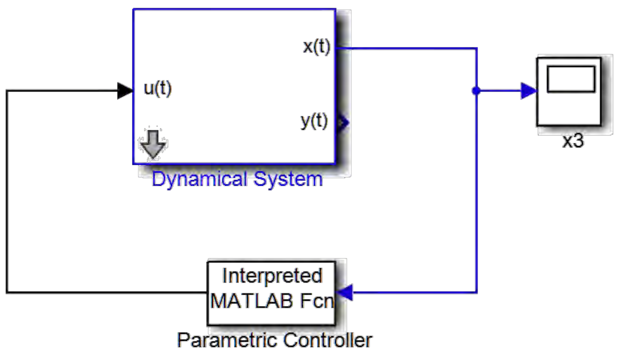


Robust

MPC via dynamic programming

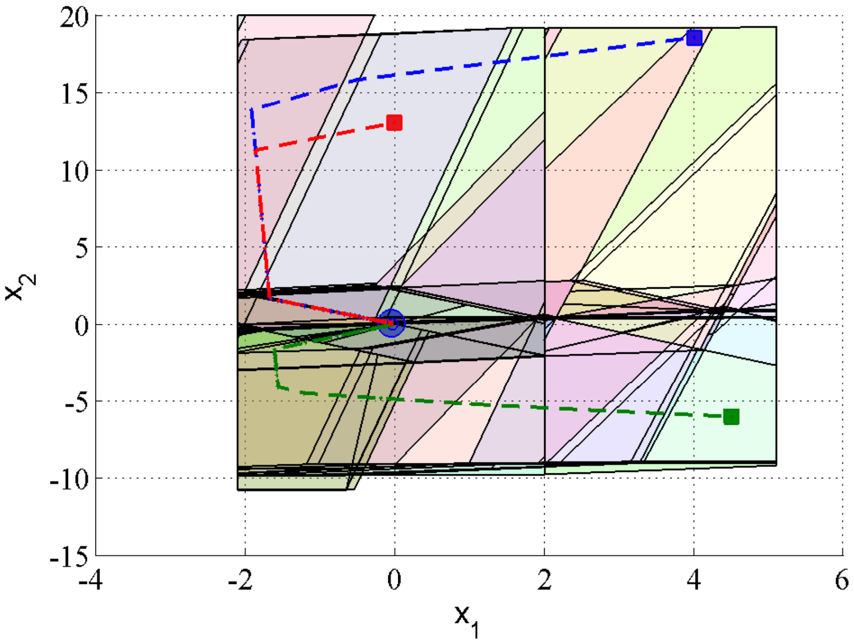
Illustrative example

Simulation with disturbed system



ε = 10%

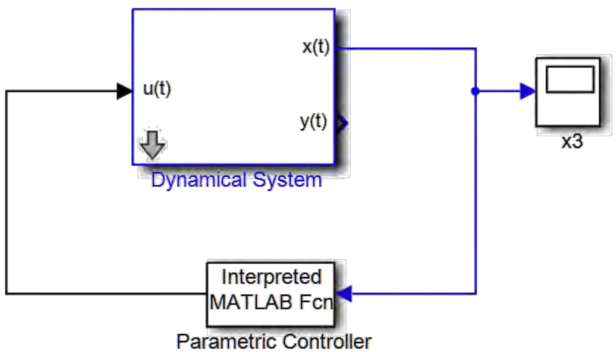
Robust controller



MPC via dynamic programming

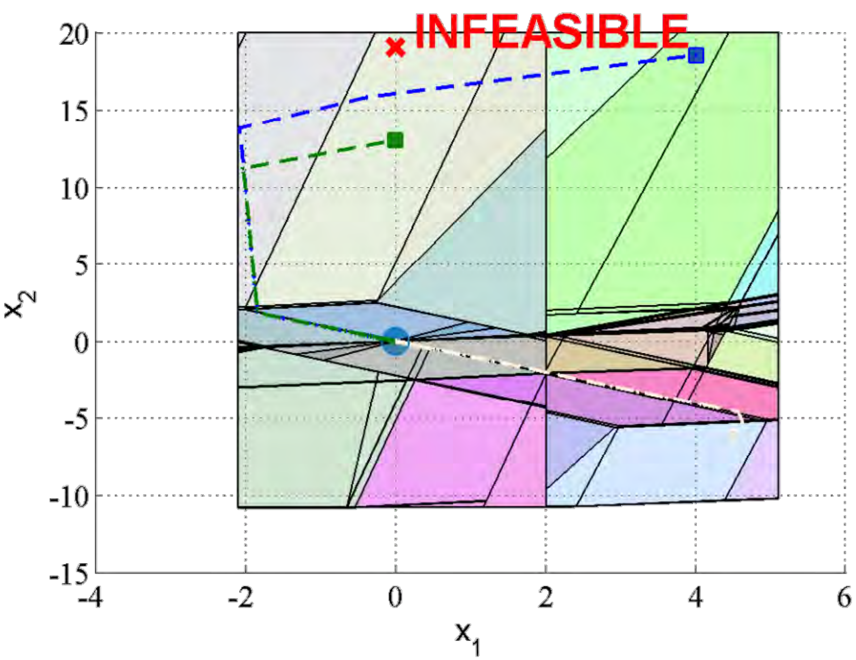
Illustrative example

Simulation with disturbed system



ε = 10%

Nominal controller



Outline

- Key concepts & historical overview
- Recent developments in multi-parametric programming and mp-MPC
- MPC-on-a-chip applications
  - PSA system
  - Fuel Cell system
  - Biomedical systems
  - Other applications

MPC-on-a-chip Applications – Recent Developments

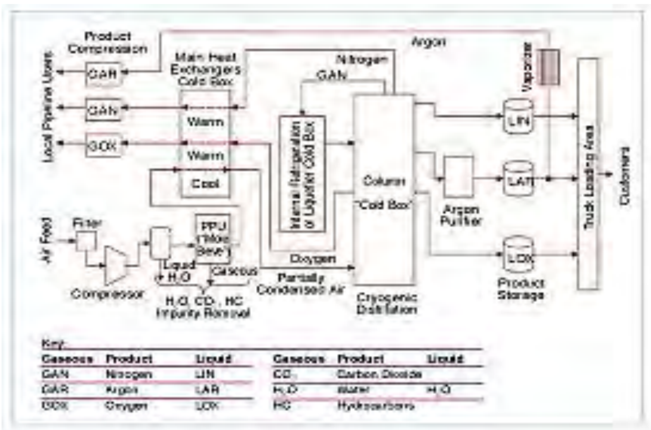
- Process Control
  - Air Separation (Air Products)
  - Hybrid PSA/Membrane Hydrogen Separation (EU/HY2SEPS, KAUST)
- Automotive
  - Active Valve Train Control (Lotus Engineering)
- Energy Systems
  - Hydrogen Storage (EU/DIAMANTE)
  - Fuel Cell

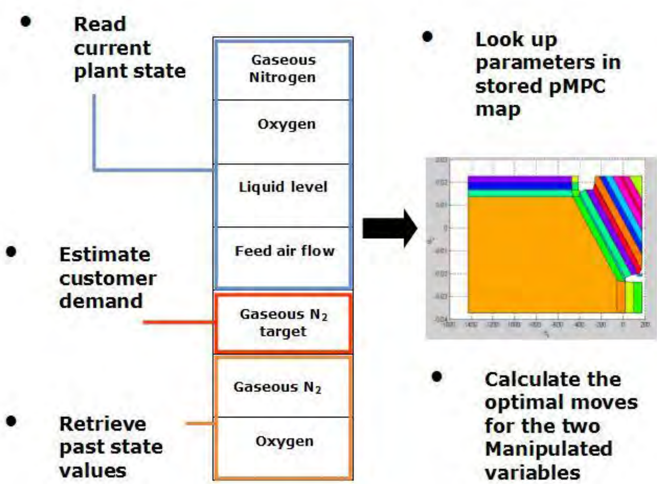
MPC-on-a-chip Applications – Recent Developments

- Biomedical Systems (MOBILE - ERC Advanced Grant Award)
  - Drug/Insulin, Anaesthesia and Chemotherapeutic Agents Delivery Systems
- Imperial Racing Green
  - Fuel cell powered Student Formula Car
- Aeronautics (EPSRC)
  - (Multiple) Unmanned Air Vehicles – with Cranfield University

Small Air Separation Units

(Air Products, Mandler et al,2006)





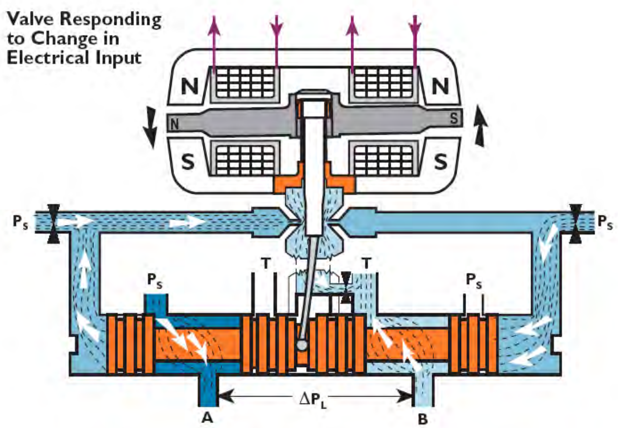
Implementation of lookup operation of pMPC for a small Nitrogen generator.

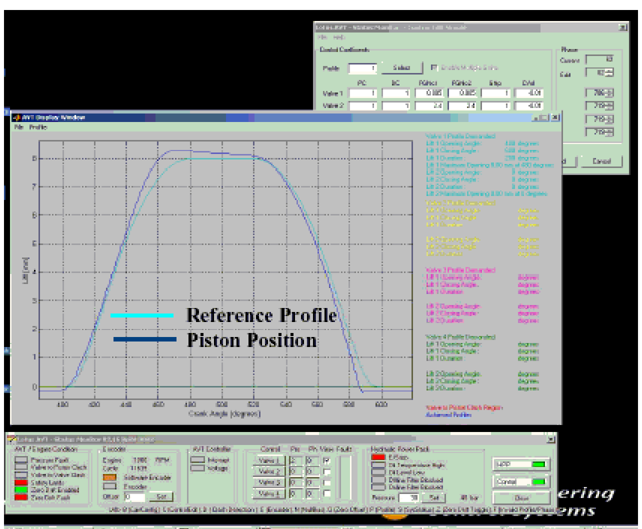
- Enable advanced MPC for small separation units
  - Optimize performance
  - Minimize operating costs
  - Satisfy product and equipment constraints
- Parametric MPC ideally suited
  - Supervises existing regulatory control
  - Off-line solution with minimum on-line load
  - Runs on existing PLC
  - Rapid installation compared to traditional MPC
- Advantages of Parametric MPC
  - 5% increased throughput
  - 5% less energy usage
  - 90% less waste
  - Installation on PLC in 1-day

Active Valve Train Control

(Lotus Engineering, Kosmidis et al, 2006)

ParOS
Advanced control technology on a chip





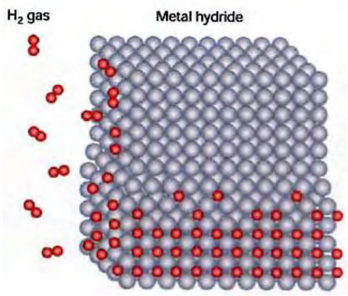
(b) 8mm trapezoidal profile

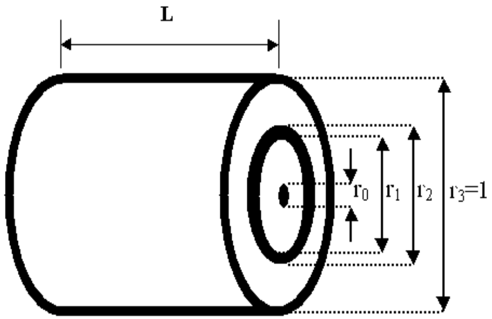
- Active Valve Trains (AVT):
  - Optimum combustion efficiency, Reduced Emissions, Elimination of butterfly valve, Cylinder deactivation, Controlled auto-ignition (CAI), Quieter operation
- Basic idea:
  - Control System sends signal to valve
  - This actuates piston attached to engine valve
  - Enables optimal control of valve timing over entire engine rpm range

Challenges for the AVT control

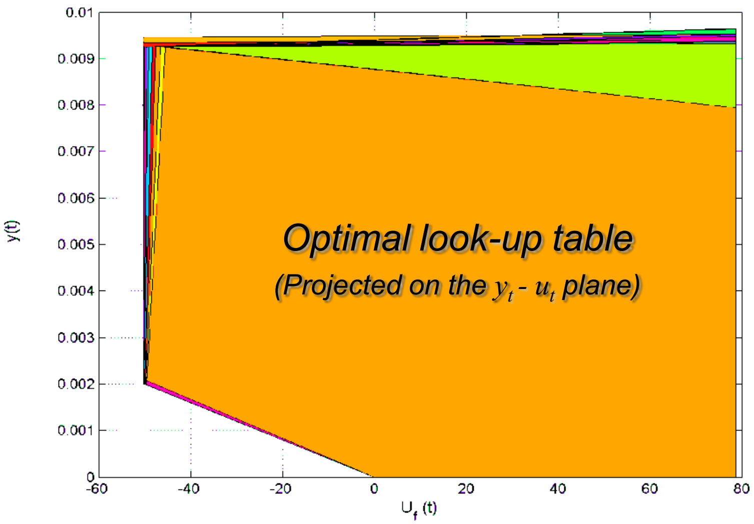
- **Nonlinear system dynamics:** Saturation, flow non-linearity, variation in fluid properties, non-linear opening of the orifices
- **Robustness** to various valve lift profiles
- **Fast dynamics** and sampling times (0.1ms)

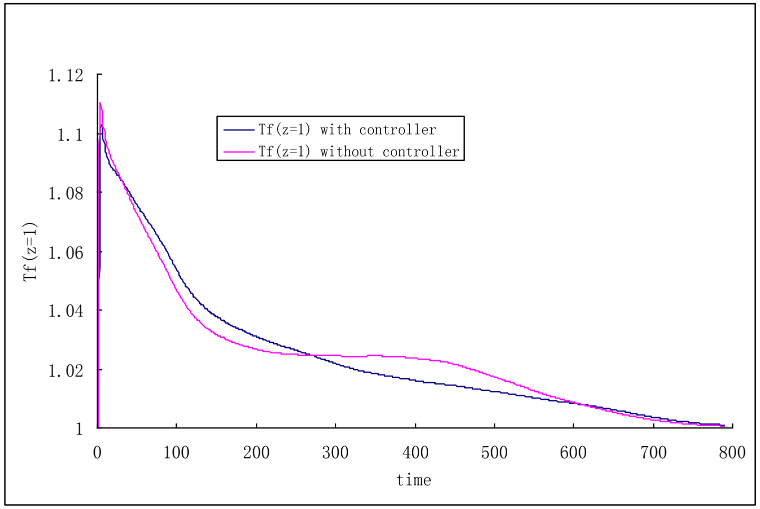
Multi-parametric Control of H_2 Storage in Metal-Hydride Beds *(EU-DIAMANTE, Georgiadis et al, 2008)*



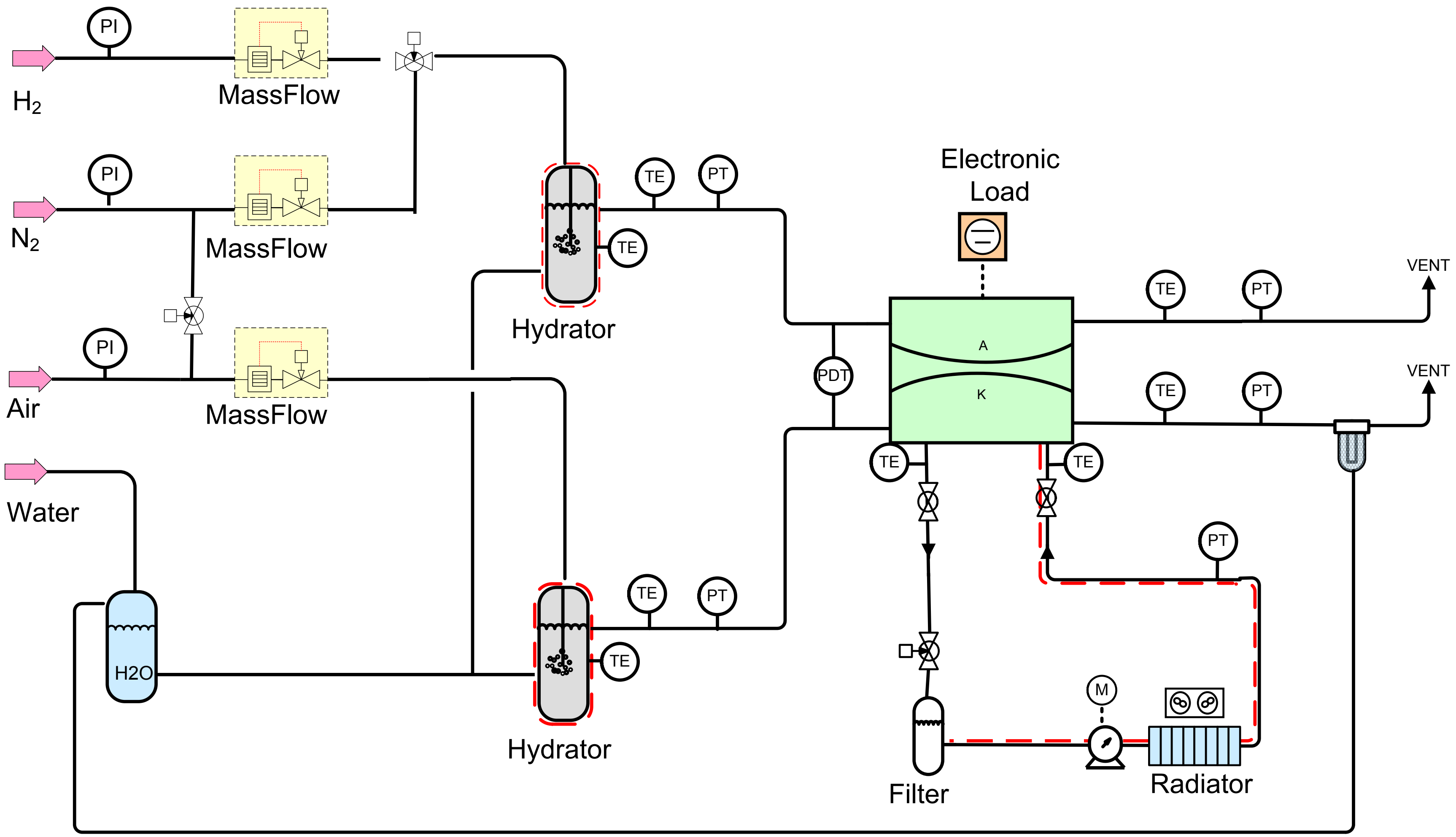


- Tracking the optimal temperature profile
- Ensure economic storage – expressed by the total required storage time
- Satisfy temperature and pressure constraints





PEM Fuel Cell Unit

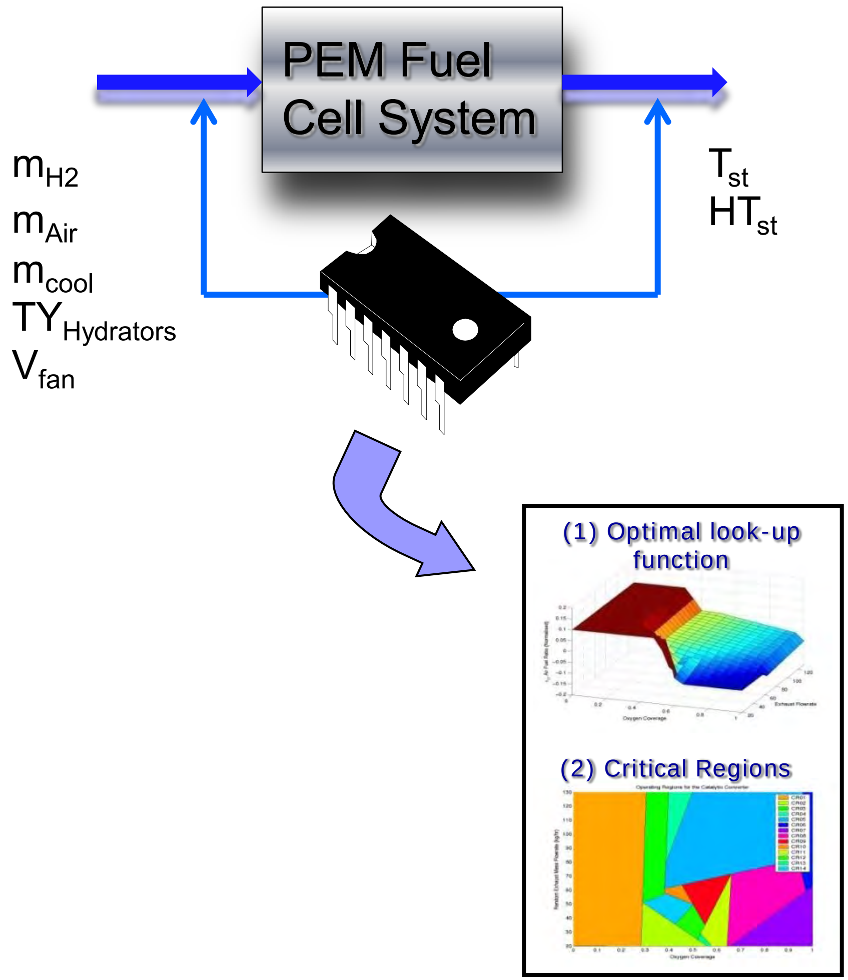


Collaborative work with Process Systems Design & Implementation Lab (PSDI) at **CERTH - Greece**

PEM Fuel Cell Unit

Unit Specifications

- Fuel Cell : 1.2kW
- Anode Flow : 5..10 lt/min
- Cathode Flow : 8..16 lt/min
- Operating Temperature : 65 – 75 °C
- Ambient Pressure



Control Strategy

<u>Start-up Operation</u>

- Heat-up Stage : Control of coolant loop

<u>Nominal Operation</u>

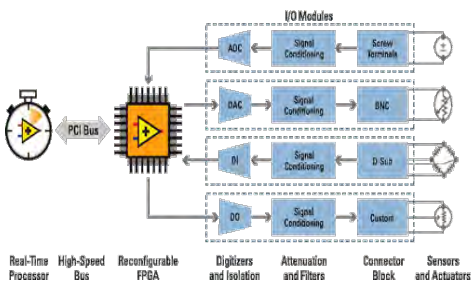
- Control Variables :
  - Mass Flow Rate of Hydrogen & Air
  - Humidity via Hydrators temperature
  - Cooling system via pump regulation
- Known Disturbance : Current

Unit Design : Centre For Research & Technology Hellas (**CERTH**)

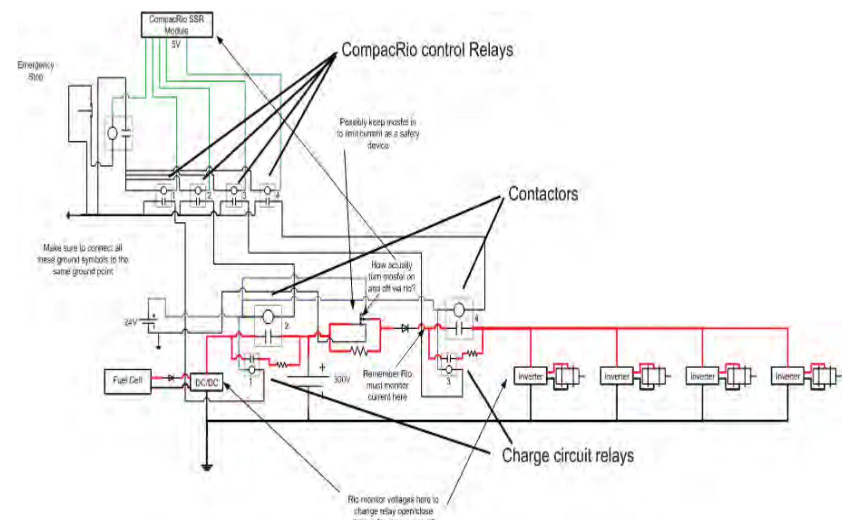
Imperial Racing Green Car


- Student Formula Project
- Control of Start-up/Shut-down of the FC
- Traction Motion Control

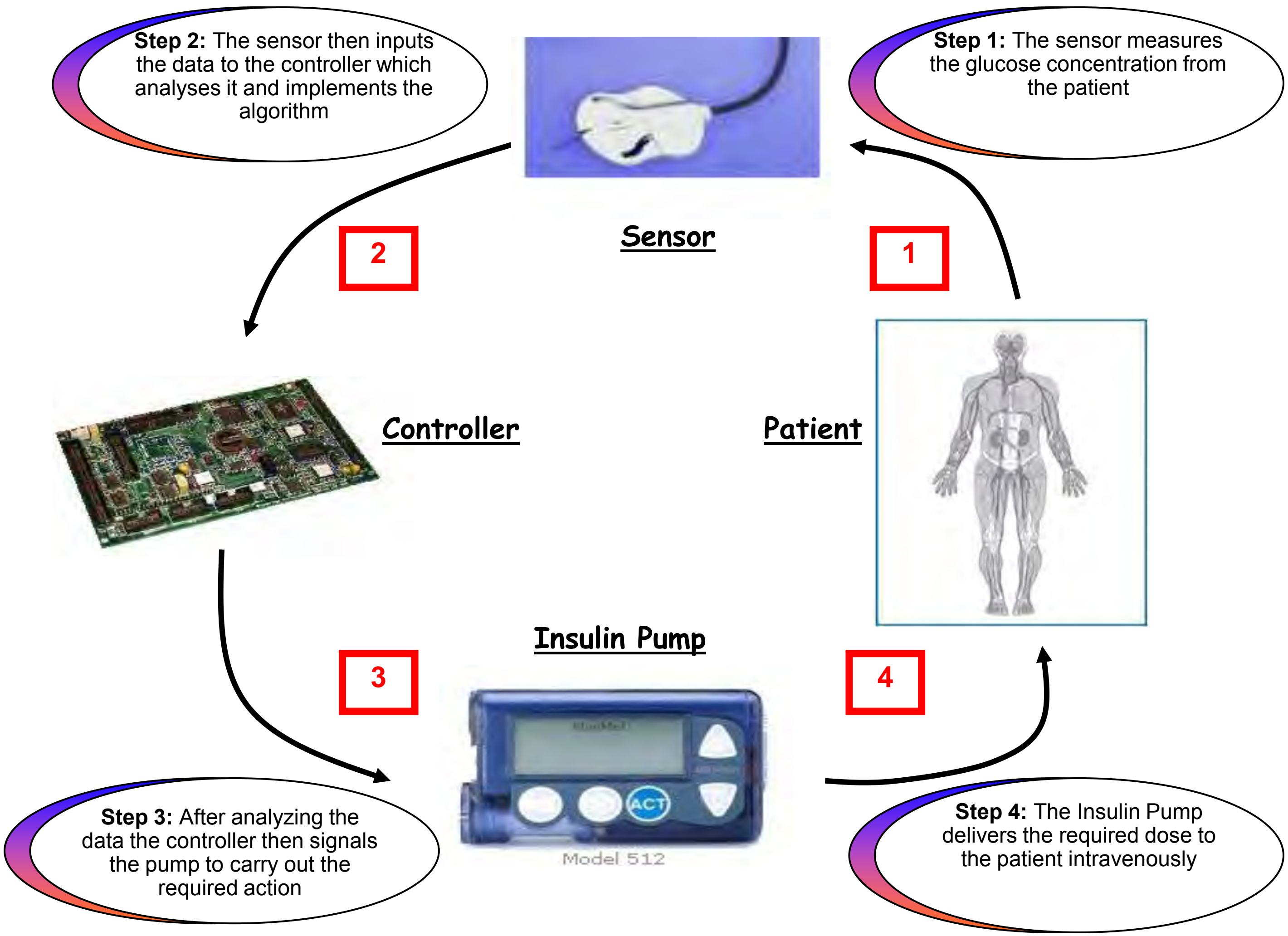
FPGA
(MPC-on-a-Chip)



Control & Acquisition System



Biomedical Systems *(MOBILE ERC Advanced Grant)*

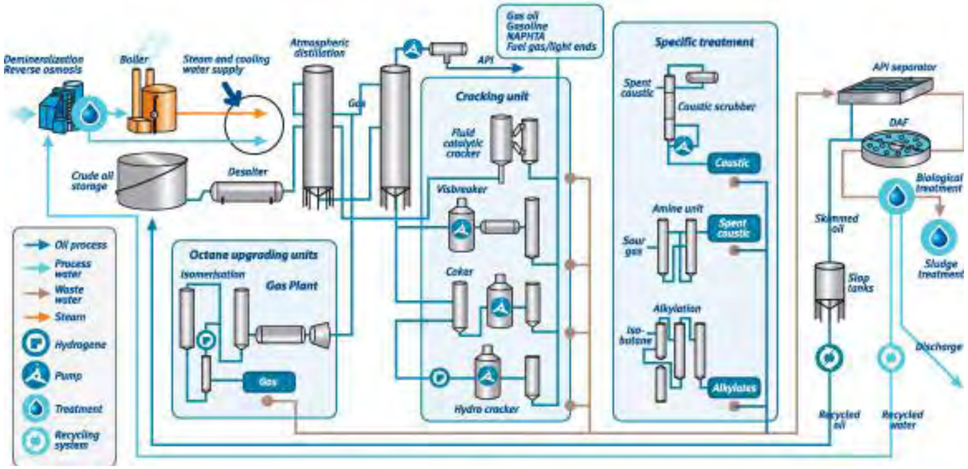


MPC-on-a-chip – Perspectives

- Application types for Multi-parametric Programming & MPC
  - **Type 1** - Large scale and expensive industrial processes with slow/medium dynamics
  - **Type 2** - Medium scale and cost industrial processes with medium/fast dynamics
  - **Type 3** - Small scale and inexpensive processes/equipment with medium/fast dynamics

MPC-on-a-chip – Future Directions

- **Type 1** – Large scale and expensive industrial processes with slow/medium dynamics

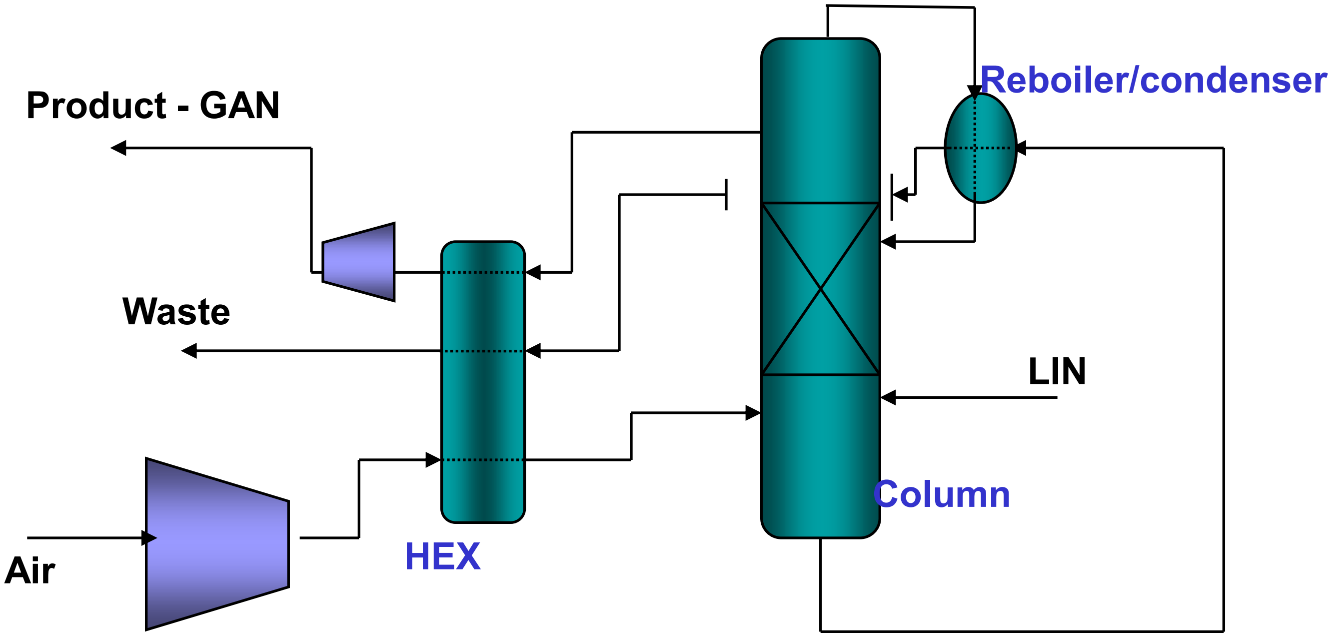


MPC-on-a-chip – Future Directions

- **Type 1** - Large scale and expensive industrial processes with slow/medium dynamics
  - Control hardware/software availability
  - MPC implementation mainly via online optimization
  - Explicit MPC **can play a role** for low level process control
  - Hybrid (on-line + off-line) approach possible

MPC-on-a-chip – Future Directions

- **Type 2** – medium scale and cost industrial processes with medium/fast dynamics

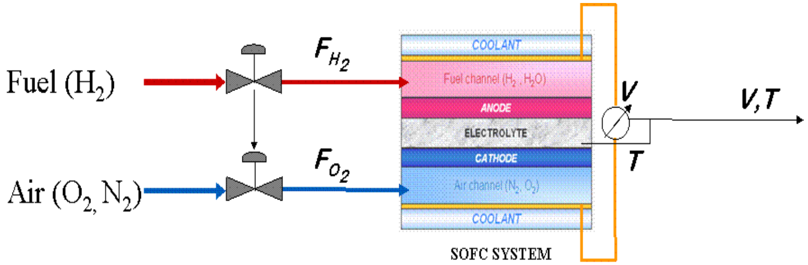


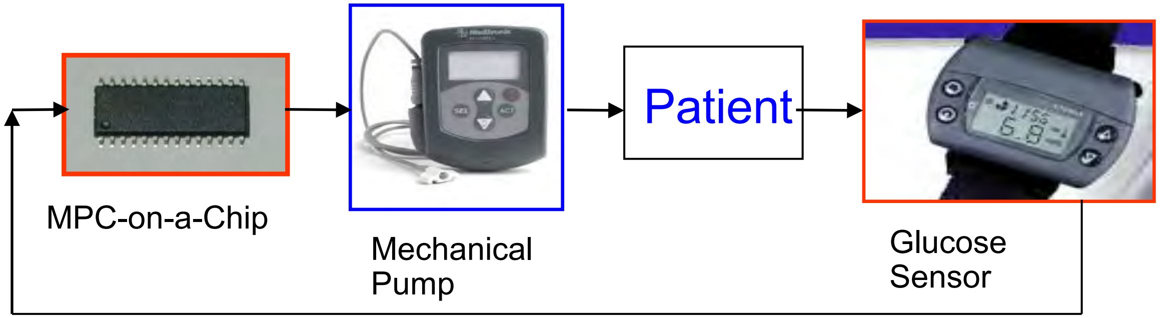
MPC-on-a-chip – Future Directions

- **Type 2** – medium scale and cost industrial processes with medium/fast dynamics
  - Limited Control hardware/software availability
  - Online optimization/MPC usually prohibitive
  - Multi-parametric MPC **ideal** – proved in previous applications (Air Separation, Automotive)

MPC-on-a-chip – Future Directions

- **Type 3** – small scale and inexpensive processes/equipment with medium/fast dynamics





MPC-on-a-chip – Future Directions

- **Type 3** – small scale and inexpensive processes/equipment with medium/fast dynamics
  - Available control hardware/software limited - not suitable for online MPC
  - Multi-parametric MPC technology **essential**
  - MPC-on-a-Chip part of embedded (all-in-one) system
  - Suitable for new technologies (FPGA, wireless)

Centre for Process Systems Engineering

Imperial College London

Multi-Parametric Programming & Explicit MPC

a progress report

Stratos Pistikopoulos

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