OSE SEMINAR 2012

A Metaheuristic Optimization Algorithm for Binary Quadratic Problems

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$$\min_{x \in X} \quad \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} f_{ik} d_{jl} x_{ij} x_{kl} + \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

$$\begin{aligned} X = \{x \mid \sum_{j=1}^{n} x_{ij} &= 1 \quad i \in N \\ \sum_{i=1}^{n} x_{ij} &= 1 \quad j \in N \\ x_{ij} &\in \{0,1\} \quad i,j \in M\} \end{aligned}$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} f_{ik}\, d_{jl}\, x_{ij}\, x_{kl} = \mathrm{trace}(\mathbf{DXFX}^T)$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} f_{ik} d_{jl} x_{ij} x_{kl} = \mathrm{trace}(\mathbf{DXFX}^T)$$

$$\mathbf{F}=\mathbf{qq}^T$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} f_{ik}\, d_{jl}\, x_{ij}\, x_{kl} = \mathrm{trace}(\mathbf{DXFX}^T)$$

$$\mathbf{F} = \mathbf{qq}^T$$

$$\begin{aligned}
&= \mathrm{trace}(\mathbf{DXqq}^T\mathbf{X}^T) \\
&= \mathrm{trace}(\mathbf{q}^T\mathbf{X}^T\mathbf{DXq}) \\
&= \mathrm{trace}(\mathbf{Xq}^T\mathbf{DXq})
\end{aligned}$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} f_{ik} d_{jl} x_{ij} x_{kl} = \mathrm{trace}(\mathbf{DXFX}^T)$$

$$\mathbf{F}\text{=}\mathbf{qq}^T$$

$$
\begin{aligned}
&= \mathrm{trace}(\mathbf{DXqq}^T\mathbf{X}^T) \\
&= \mathrm{trace}(\mathbf{q}^T\mathbf{X}^T\mathbf{DXq}) \\
&= \mathrm{trace}(\mathbf{Xq}^T\mathbf{DXq})
\end{aligned}
$$

$$= \mathrm{trace}(\mathbf{y}^T\mathbf{Dy}) = \mathbf{y}^T\mathbf{Dy}$$

$$\min_{x \in X, y \in \mathbb{R}^n} \quad \mathbf{y}^T \mathbf{D} \mathbf{y}$$

subject to

$$y_i = \sum_{j=1}^{n} x_{ij} q_j \quad \forall\, i$$

$$\sum_{i=1}^{n} y_i = \sum_{j=1}^{n} q_j$$

$$\min_{x \in X,\, y, z \in \mathbb{R}^n} \quad \mathbf{y}^T(\mathbf{D} + \mathrm{Diag}(\mathbf{u}))\mathbf{y} - \mathbf{u}^T\mathbf{z}$$

subject to

$$y_i = \sum_{j=1}^{n} x_{ij} q_j \quad \forall\, i$$

$$z_i = \sum_{j=1}^{n} x_{ij} q_j^2 \quad \forall\, i$$

$$\sum_{i=1}^{n} y_i = \sum_{j=1}^{n} q_j$$

$$\min \mathbf{x}^T(\mathbf{D} + \mathrm{diag}(\mathbf{u}))\mathbf{x} - \mathbf{u}^T\mathbf{x}$$

subject to

$$\sum_{i=1}^{n} x_i = k$$

iteration constraint

$$x_{iter} = 1$$

$$\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \qquad \mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad \mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad \mathbf{x}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$
x = \begin{pmatrix} 1\\0\\1\\1\\0\\0\\0\\1\\0\\0\\0\\1\\1\\0 \end{pmatrix}
\quad
x_1 = \begin{pmatrix} 1\\0\\1\\1\\0\\0\\0\\1\\0\\0\\0\\1\\1\\0 \end{pmatrix}
\quad
x_2 = \begin{pmatrix} 1\\0\\1\\1\\0\\0\\0\\1\\0\\0\\0\\1\\1\\0 \end{pmatrix}
\quad
x_3 = \begin{pmatrix} 1\\0\\1\\1\\0\\0\\0\\1\\0\\0\\0\\1\\1\\0 \end{pmatrix}
$$

$$\mathbf{x} = \begin{pmatrix} 1\\0\\1\\1\\0\\0\\0\\1\\0\\0\\0\\1\\1\\0 \end{pmatrix} \quad \mathbf{x}_1 = \begin{pmatrix} 1\\0\\1\\1\\0\\0\\0\\1\\0\\0\\0\\1\\1\\0 \end{pmatrix} \quad \mathbf{x}_2 = \begin{pmatrix} 1\\0\\1\\1\\0\\0\\0\\1\\0\\0\\0\\1\\1\\0 \end{pmatrix} \quad \mathbf{x}_3 = \begin{pmatrix} 1\\0\\1\\1\\0\\0\\0\\1\\0\\0\\0\\1\\1\\0 \end{pmatrix} \quad \mathbf{x}_{r1} = \begin{pmatrix} 1\\0\\1\\1\\0\\0\\0\\1\\0\\0\\0\\1\\1\\0 \end{pmatrix}$$

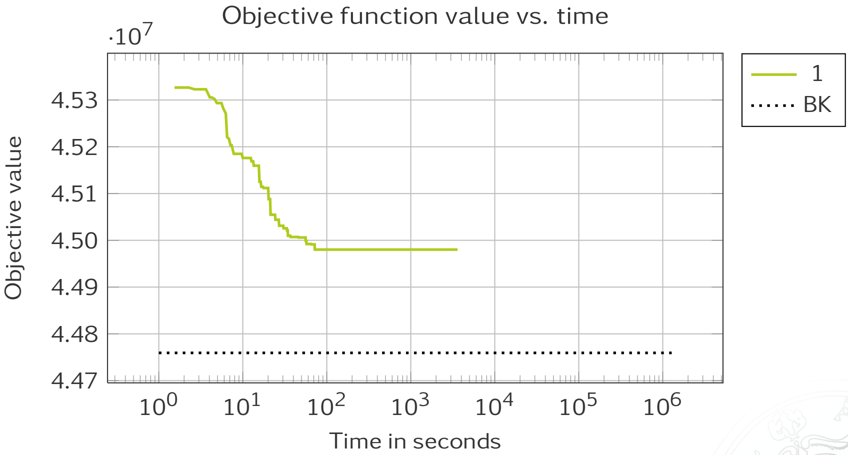
$$\mathbf{x} = \begin{pmatrix} 1\\0\\1\\1\\0\\0\\0\\1\\0\\0\\0\\1\\1\\0 \end{pmatrix} \quad \mathbf{x}_1 = \begin{pmatrix} 1\\0\\1\\1\\0\\0\\0\\1\\0\\0\\0\\1\\1\\0 \end{pmatrix} \quad \mathbf{x}_2 = \begin{pmatrix} 1\\0\\1\\1\\0\\0\\0\\1\\0\\0\\0\\1\\1\\0 \end{pmatrix} \quad \mathbf{x}_3 = \begin{pmatrix} 1\\0\\1\\1\\0\\0\\0\\1\\0\\0\\0\\1\\1\\0 \end{pmatrix} \quad \mathbf{x}_{r1} = \begin{pmatrix} 1\\0\\1\\1\\0\\0\\0\\1\\0\\0\\0\\1\\1\\0 \end{pmatrix} \quad \mathbf{x}_{r2} = \begin{pmatrix} 1\\0\\1\\1\\0\\0\\0\\1\\0\\0\\0\\1\\1\\0 \end{pmatrix}$$

$$T_{rstu} = \max_{v,w \in \{-1,0,1\}} \frac{1}{(r-t+nv)^2 + (s-u+nw)^2}$$

$$f_{ij} = \begin{cases} 1 & \text{if } i \leq m \text{ and } j \leq m \\ 0 & \text{otherwise} \end{cases}$$

$$d_{ij} = d_{n(r-1)+s\; n(t-1)+u} = T_{rstu}$$

where (r,s) are the coordinates for i and (t,u) are the coordinates for j


Objective function value vs. time
·10^7
4.53
4.52
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4.50
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4.48
4.47
Objective value
1
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10^0
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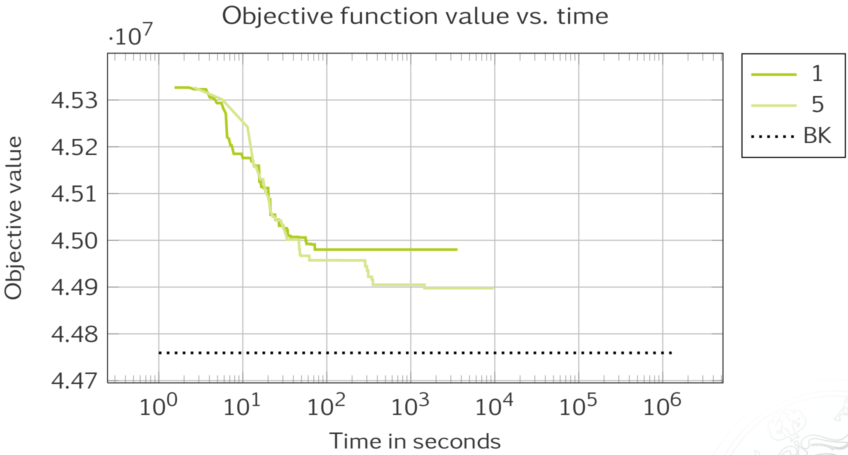
10^4
10^5
10^6
Time in seconds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Time in seconds
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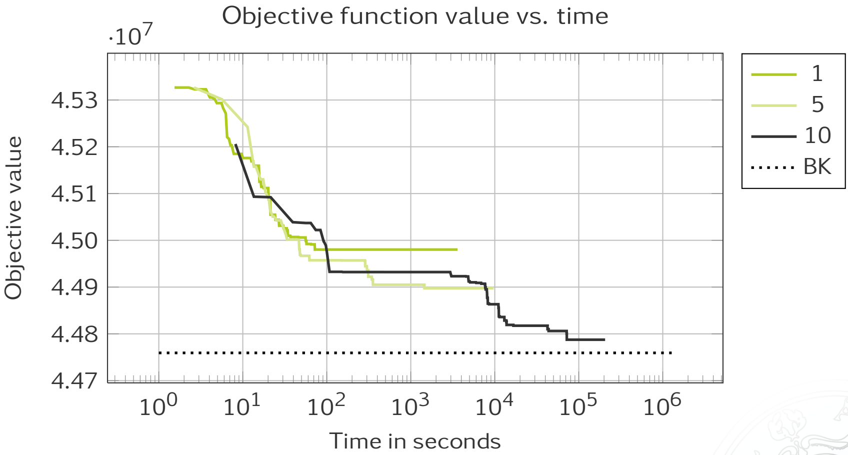
5
BK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Time in seconds
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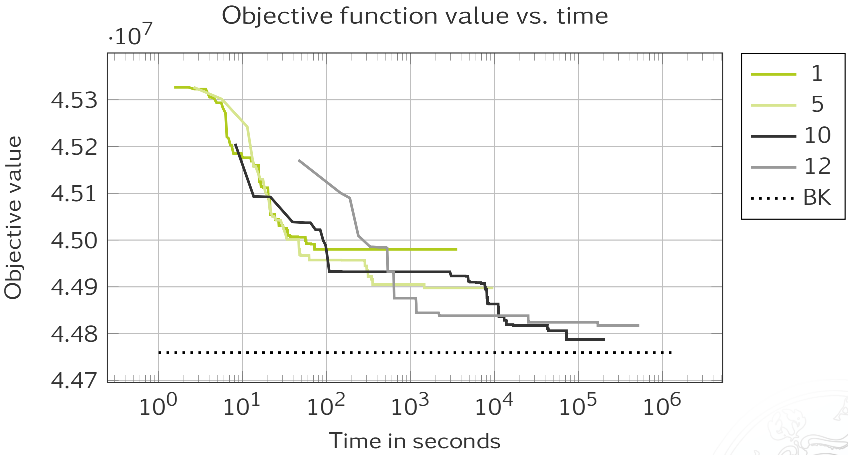
BK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Time in seconds
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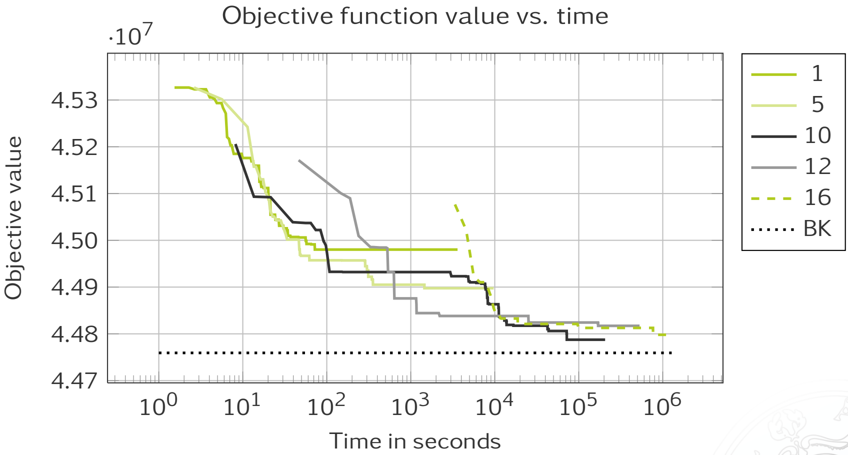
5
10
12
BK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Time in seconds
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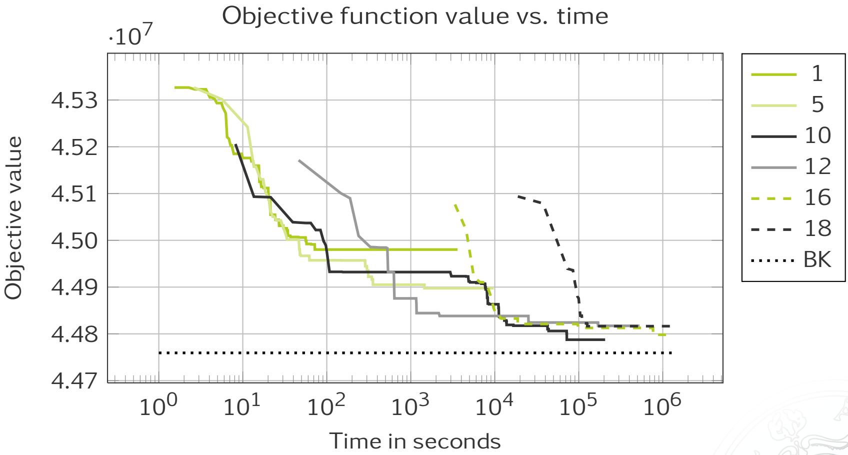
5
10
12
16
BK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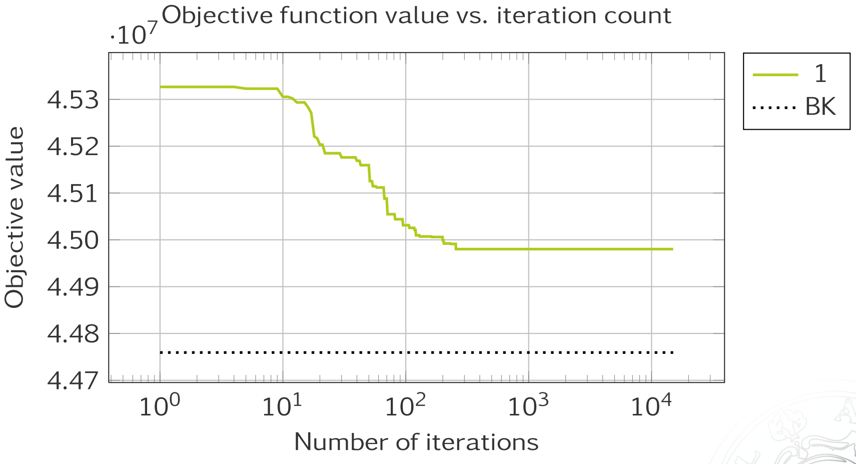

Objective function value vs. iteration count
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4.53
4.52
4.51
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Objective value
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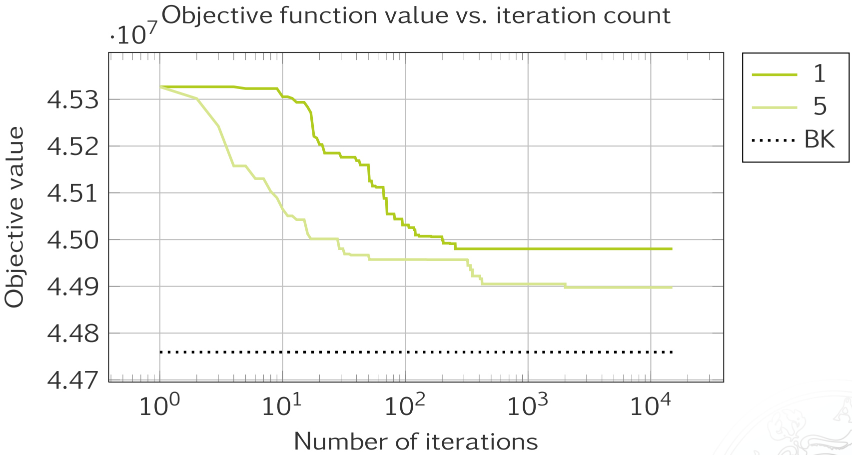
10^1
10^2
10^3
10^4
Number of iterations
1
BK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Number of iterations
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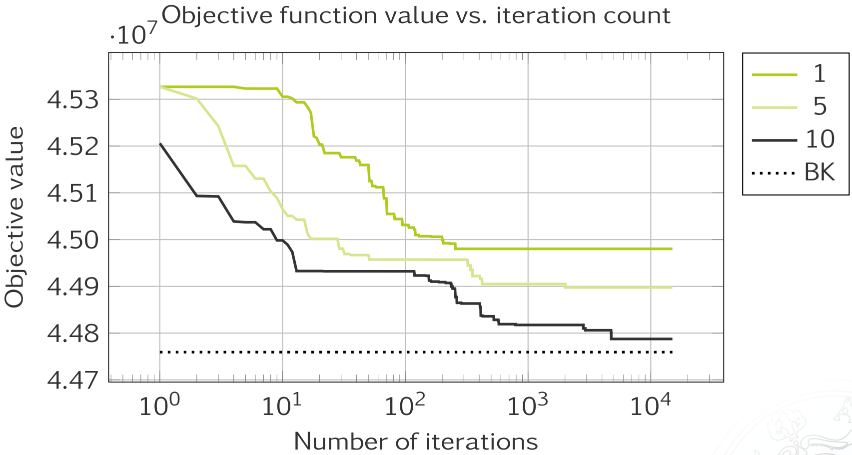
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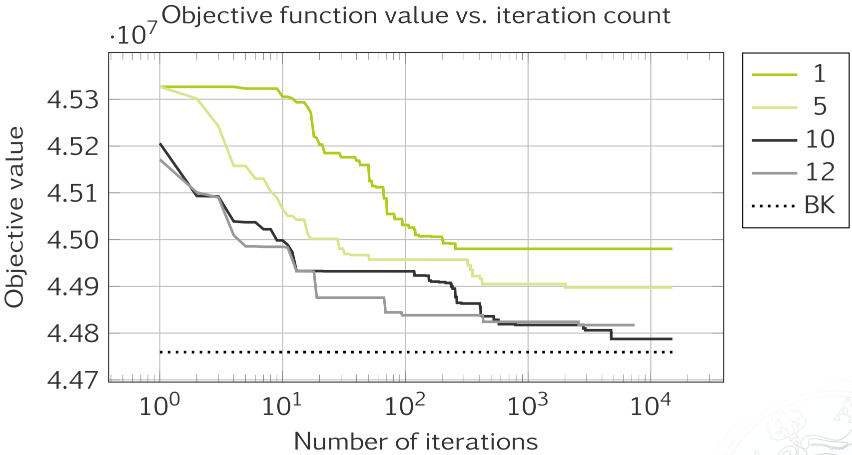
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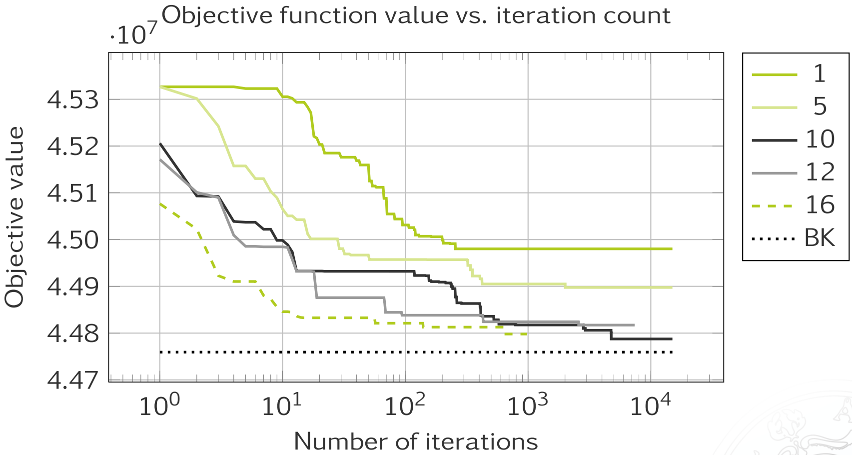
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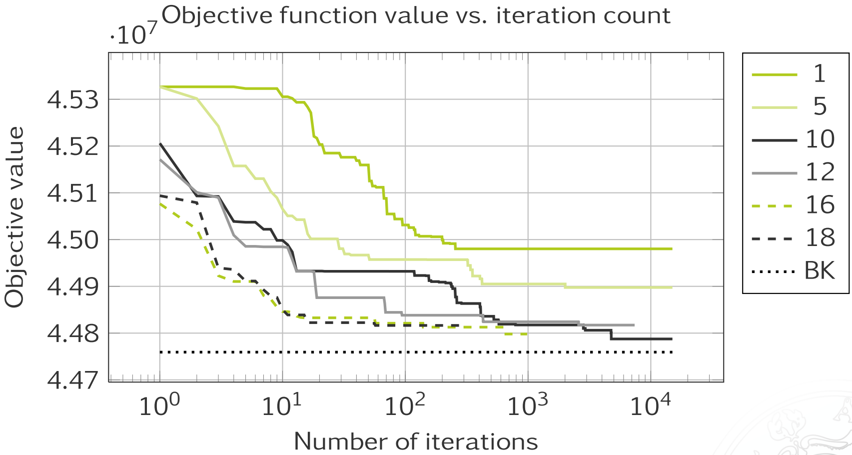
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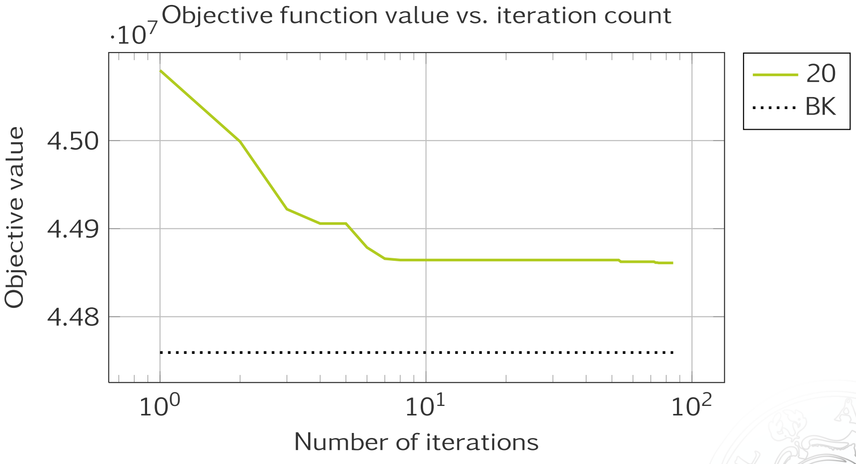
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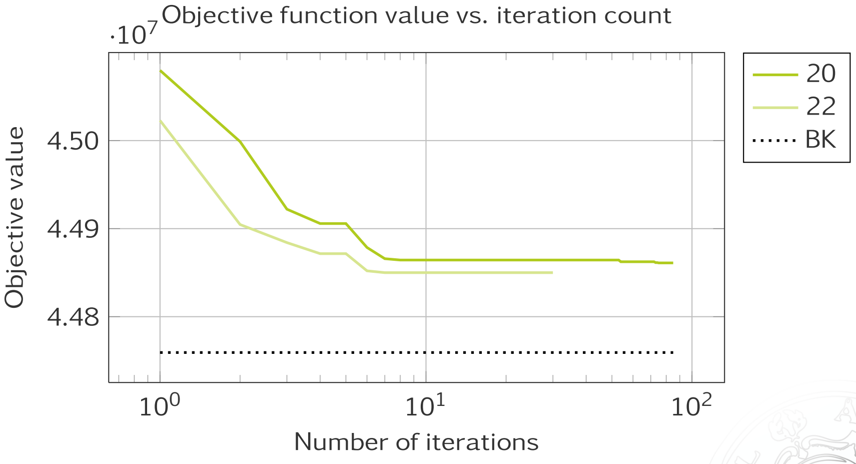
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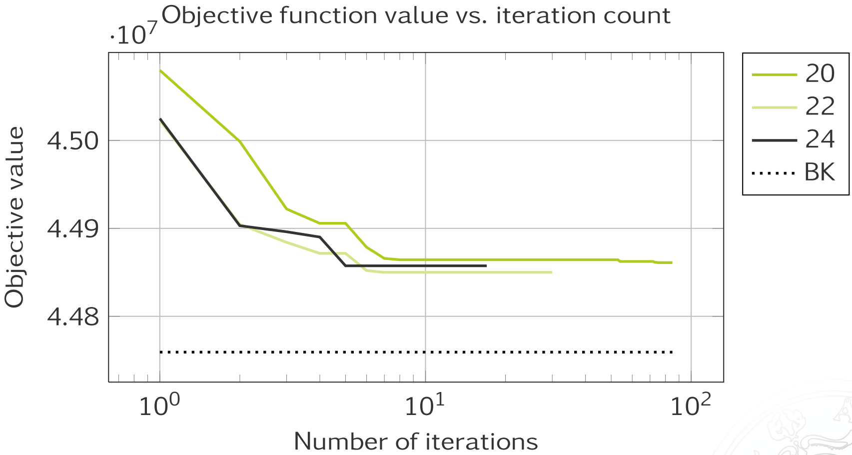
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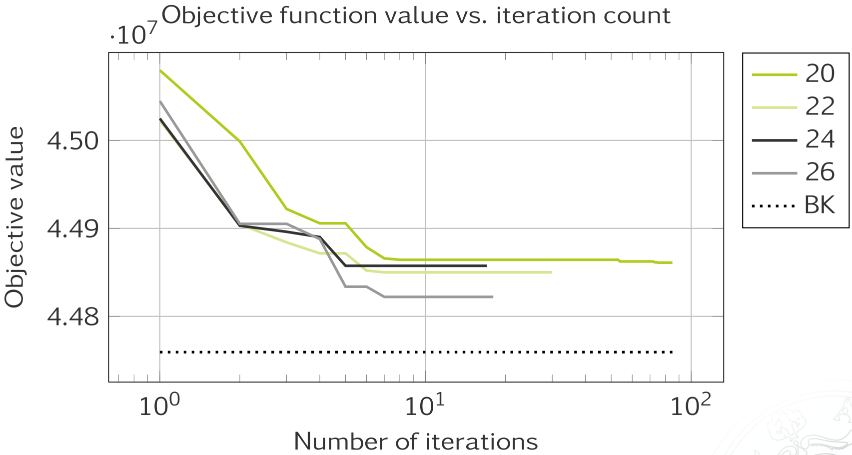
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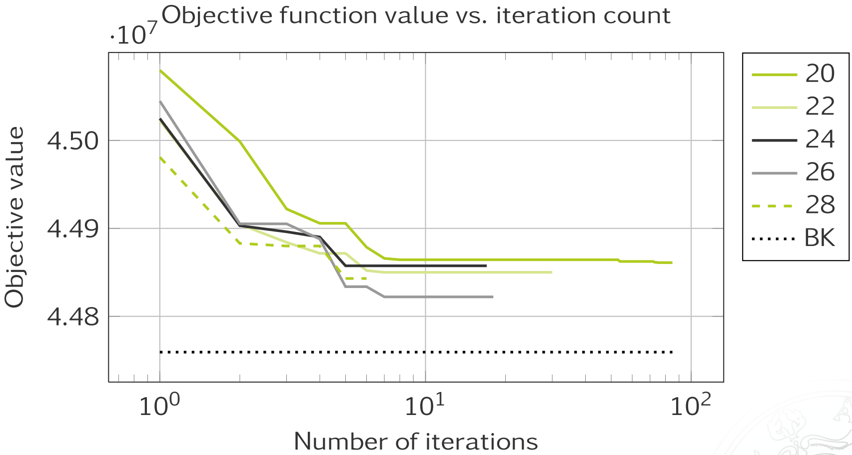
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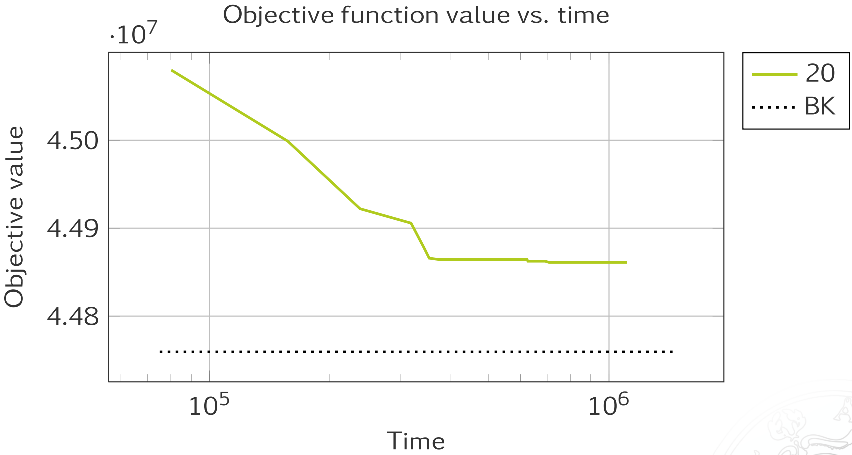
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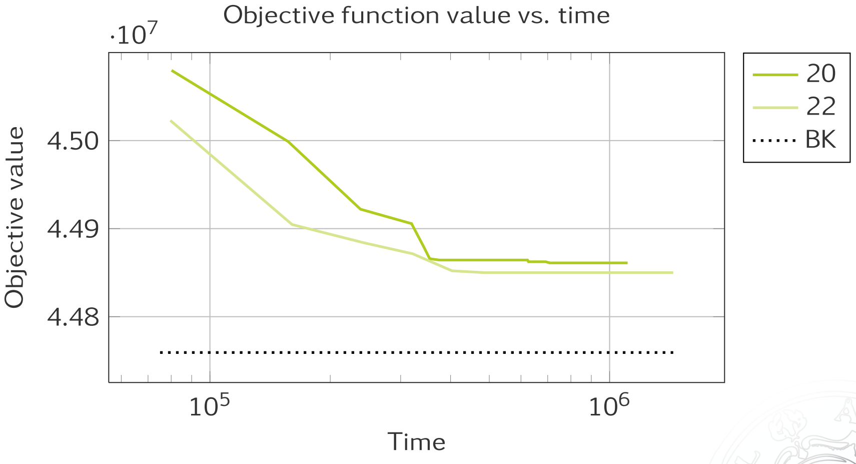
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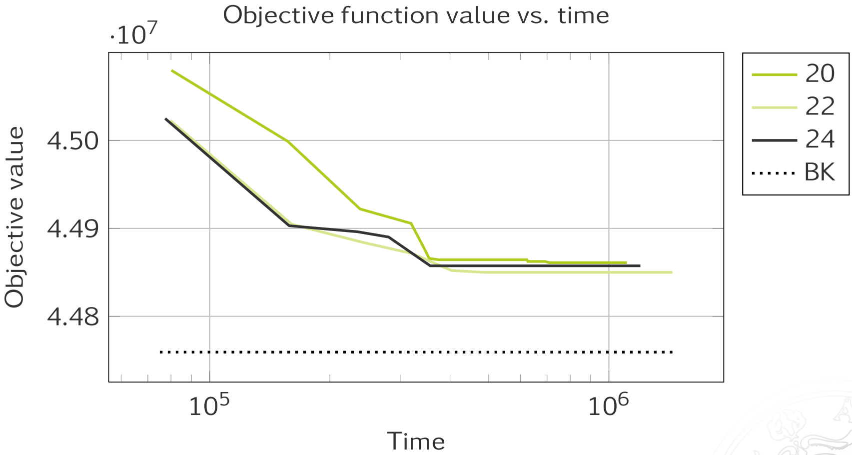

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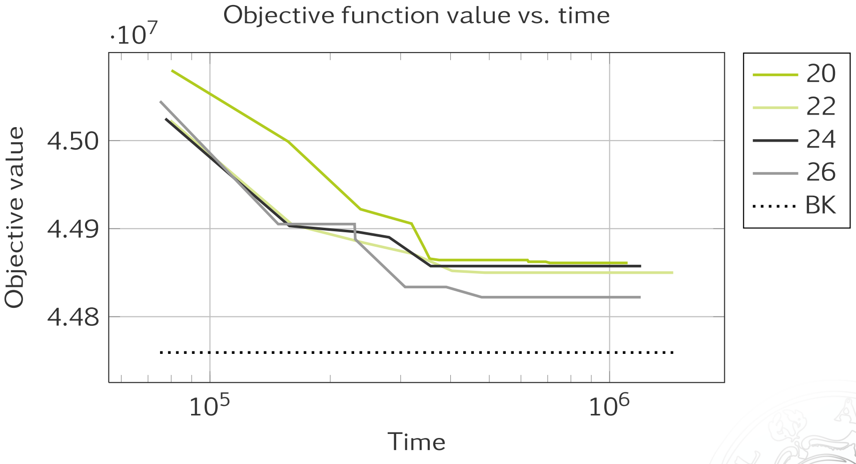
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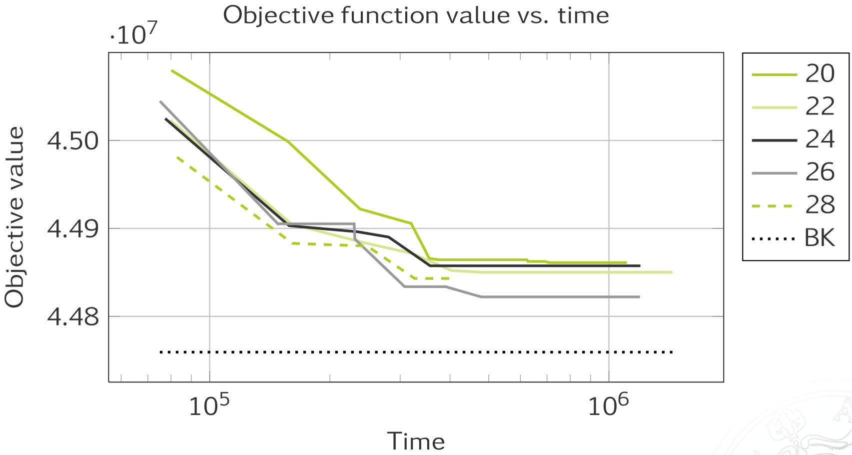
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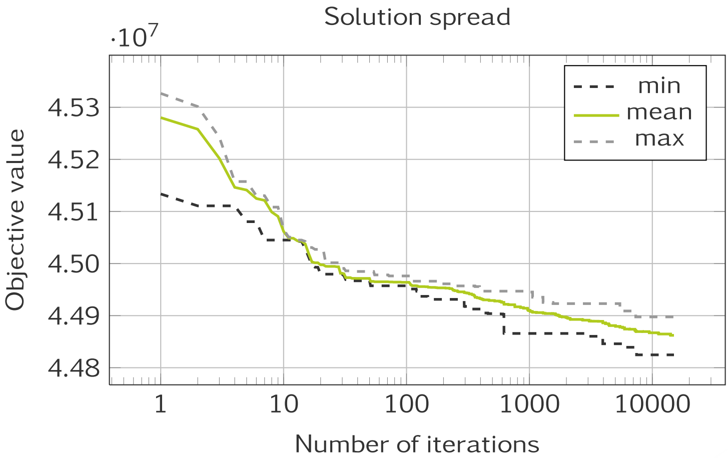

Figure 1: Spread of the solutions with $m = 5$

Some references

Alain Billionnet, Sourour Elloumi, and Marie-Christine Plateau.
Improving the performance of standard solvers for quadratic 0-1 programs by a tight convex reformulation: The qcr method.
Discrete Appl. Math., 157:1185–1197, March 2009.

R.E. Burkard, E. Cela, P.M. Pardalos, and L.S. Pitsoulis.
Handbook of Combinatorial Optimization, volume 3.
1998.

C. S. Edwards.
A branch and bound algorithm for the koopmans-beckmann quadratic assignment problem.
Combinatorial Optimization II, 13:35–52, 1980.

Tjalling C. Koopmans and Martin Beckmann.
Assignment problems and the location of economic activities.
Econometrica, 25(1):pp. 53–76, 1957.

Otto Nissfolk, Ray Pörn, Tapio Westerlund, and Fredrik Jansson.
A mixed integer quadratic reformulation of the quadratic assignment problem with rank-1 matrix.
In Iftekhar A. Karimi and Rajagopalan Srinivasan, editors, *11th International Symposium on Process Systems Engineering*, volume 31 of *Computer Aided Chemical Engineering*, pages 360 – 364. Elsevier, 2012.

É.D. Taillard.
Comparison of iterative searches for the quadratic assignment problem.
Location Science, 3(2):87 – 105, 1995.

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Thank you for listening!

Questions?