Solving a Challenging Quadratic 3D Assignment Problem

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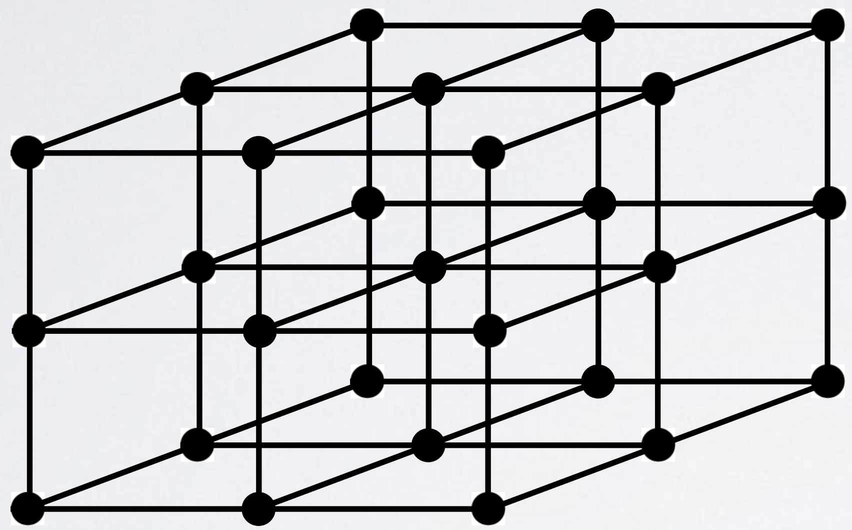
Quadratic 3D Assignment Problem



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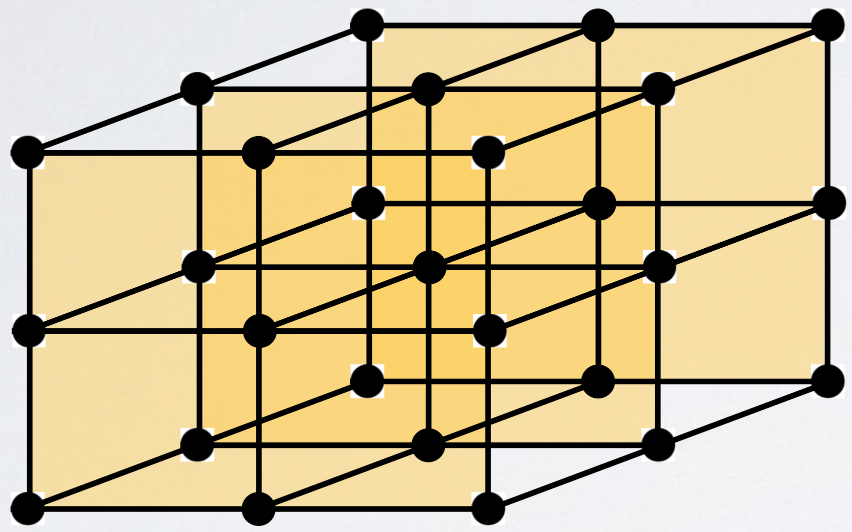

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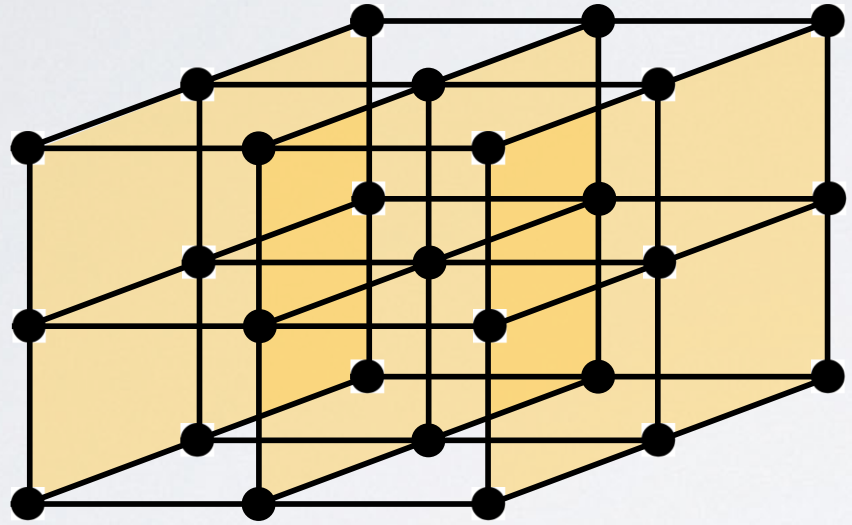

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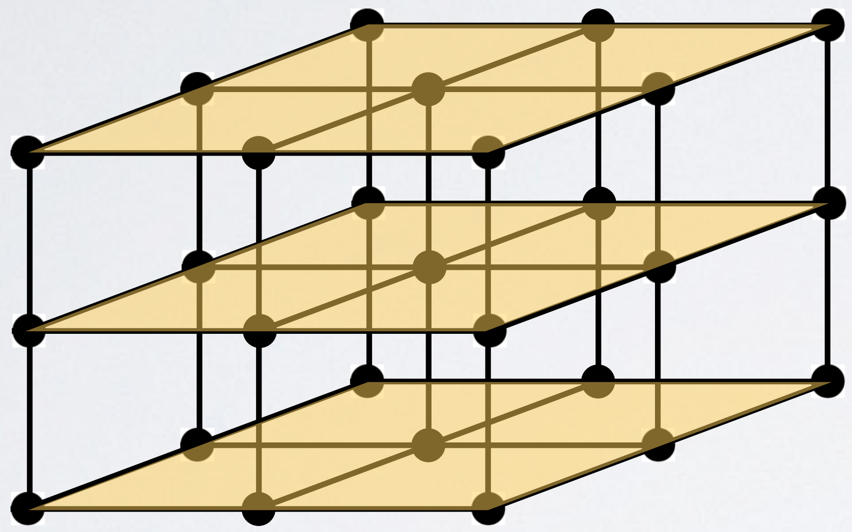

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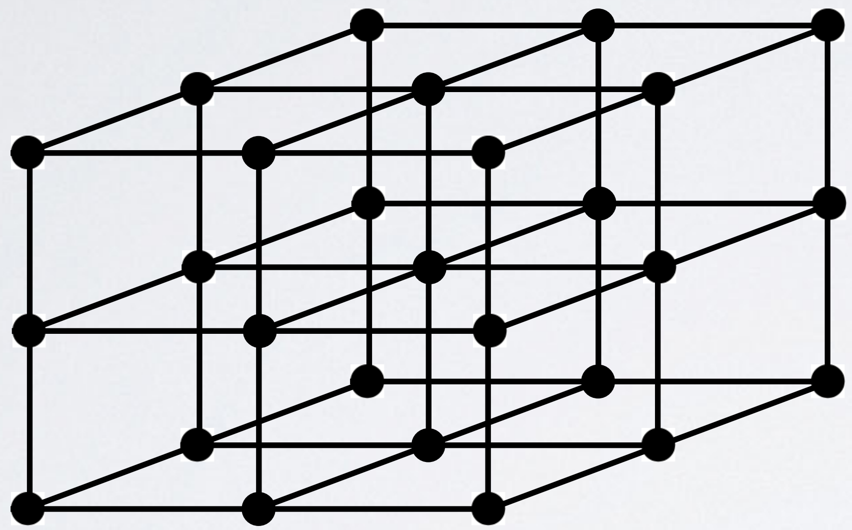

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$$c_{ijkpqr}x_{ijk}x_{pqr}$$

Our Instance

16-PSK digital communication retransmission protocol

$$c_{ijkpqr}$$

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16-PSK digital communication retransmission protocol

$$c_{ijkpqr}$$

- extremely dense objective: >12M coefficients
- high dynamism 3.6×10^{12}
- symmetric: group of order 49.152

```
Parallel mode: deterministic, using up to 16 threads.
Root relaxation solution time = 43.45 sec. (17943.71 ticks)

        Nodes                                         Cuts/
   Node  Left     Objective  IInf  Best Integer    Best Bound    ItCnt     Gap

*     0+    0                      207392.0000        0.0000     4538  100.00%
      0     0     9832.0000    50   207392.0000     9832.0000     4538   95.26%
      0     0     9832.0000    50   207392.0000      Cuts: 12     5450   95.26%
      0     0     9832.0000    55   207392.0000      Cuts: 25     5569   95.26%
Heuristic still looking.
      0     2     9832.0000    51   207392.0000     9832.0000     5569   95.26%
Elapsed time = 186.17 sec. (166732.53 ticks, tree = 0.01 MB, solutions = 1)
      1     3    10639.0000    39   207392.0000     9832.0000    11529   95.26%
…
…
…
 25953439 25481647    16202.0920    34   207392.0000    11819.7613 3.89e+08   94.30%
 25958606 25486740        cutoff         207392.0000    11819.8664 3.89e+08   94.30%
 25965184 25493181    27448.9838    33   207392.0000    11820.0129 3.89e+08   94.30%
 25973355 25501284    21883.6636    33   207392.0000    11820.0279 3.89e+08   94.30%
Elapsed time = 188682.61 sec. (70111125.75 ticks, tree = 85733.89 MB, solutions = 1)
Nodefile size = 85605.64 MB (8364.42 MB after compression)
 25982488 25510197    23406.9047    37   207392.0000    11820.2149 3.89e+08   94.30%
 25992703 25520262   111140.2318    24   207392.0000    11820.4710 3.89e+08   94.30%
 26004478 25531838    24797.8353    29   207392.0000    11820.5560 3.89e+08   94.30%
 26013616 25540843   114000.7561    32   207392.0000    11820.8606 3.89e+08   94.30%
 26021413 25548479    25846.1198    35   207392.0000    11821.0349 3.90e+08   94.30%
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Extremely Challenging for MIP solvers

1. lightweight MIP model

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2. cutting planes

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3. symmetry handling

Lightweight MIP model

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} w_{ijk}$$

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$$w_{ijk} \geq \sum_{p=1}^{n} \sum_{q=1}^{n} \sum_{r=1}^{n} c_{ijkpqr} x_{pqr} - M(1 - x_{ijk})$$

$$x_{ijk} \in \{0, 1\}$$

$$w_{ijk} \geq 0$$

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$\Theta(n^3)$ *variables and constraints*

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superweak dual bound

Cutting Planes I

$$w_{ijk} \geq L_{ijk} x_{ijk}$$

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computed by solving a **LINEAR** 3D assignment problem

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- NP-hard in theory, quite cheap in practice
- can exploit additional constraints (if available) both global and local

Cutting Planes II

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computed by solving a MIP

- increase consistency between pairs of artificial variables
- not significantly harder than family I
- need to be conservative with separation

Symmetry Handling

binary variables can be partitioned into 6 orbits

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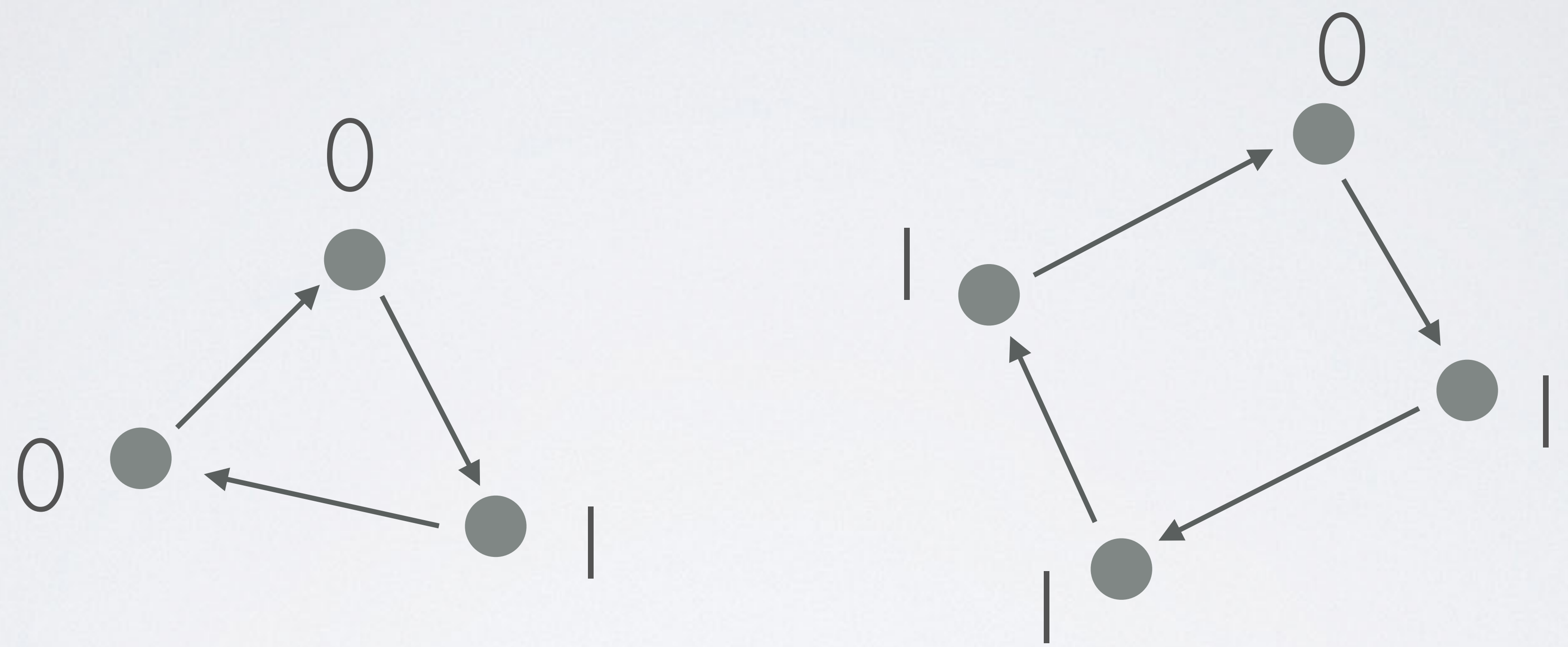

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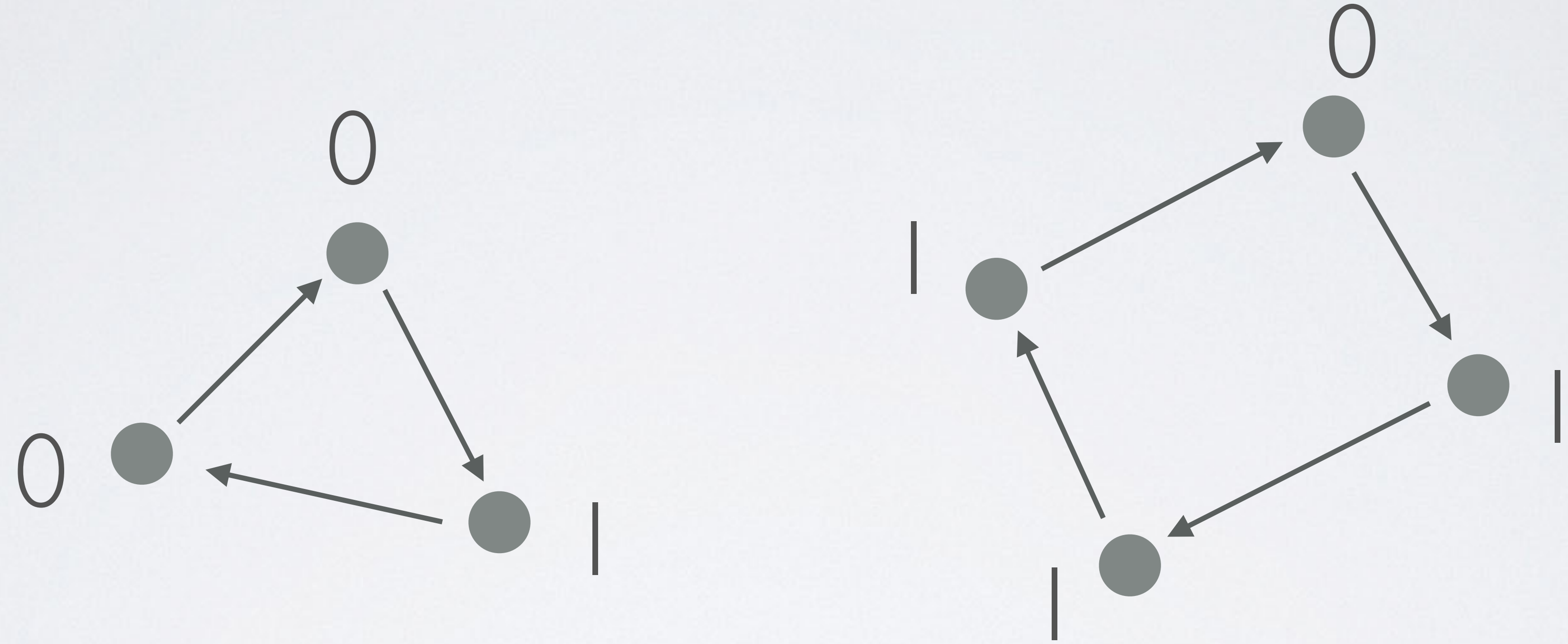

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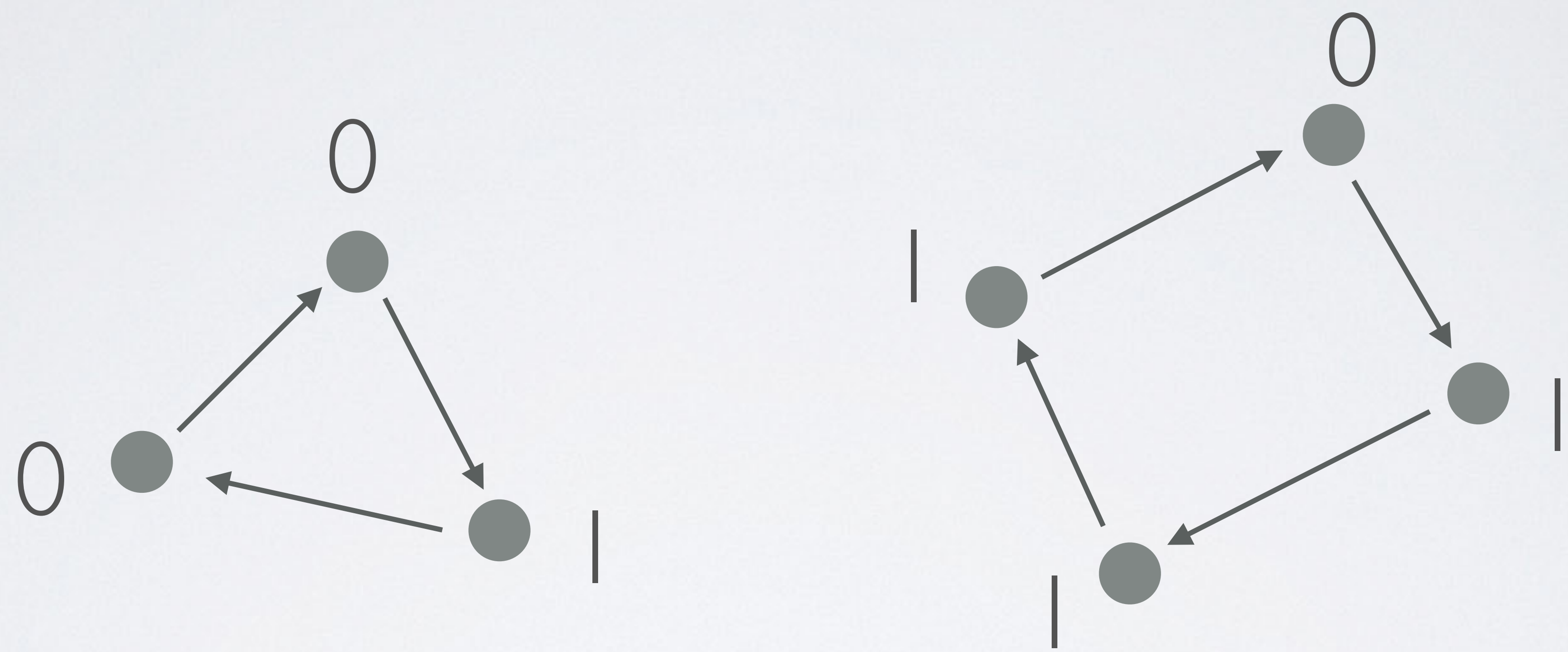

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aggregated variables as first level decisions

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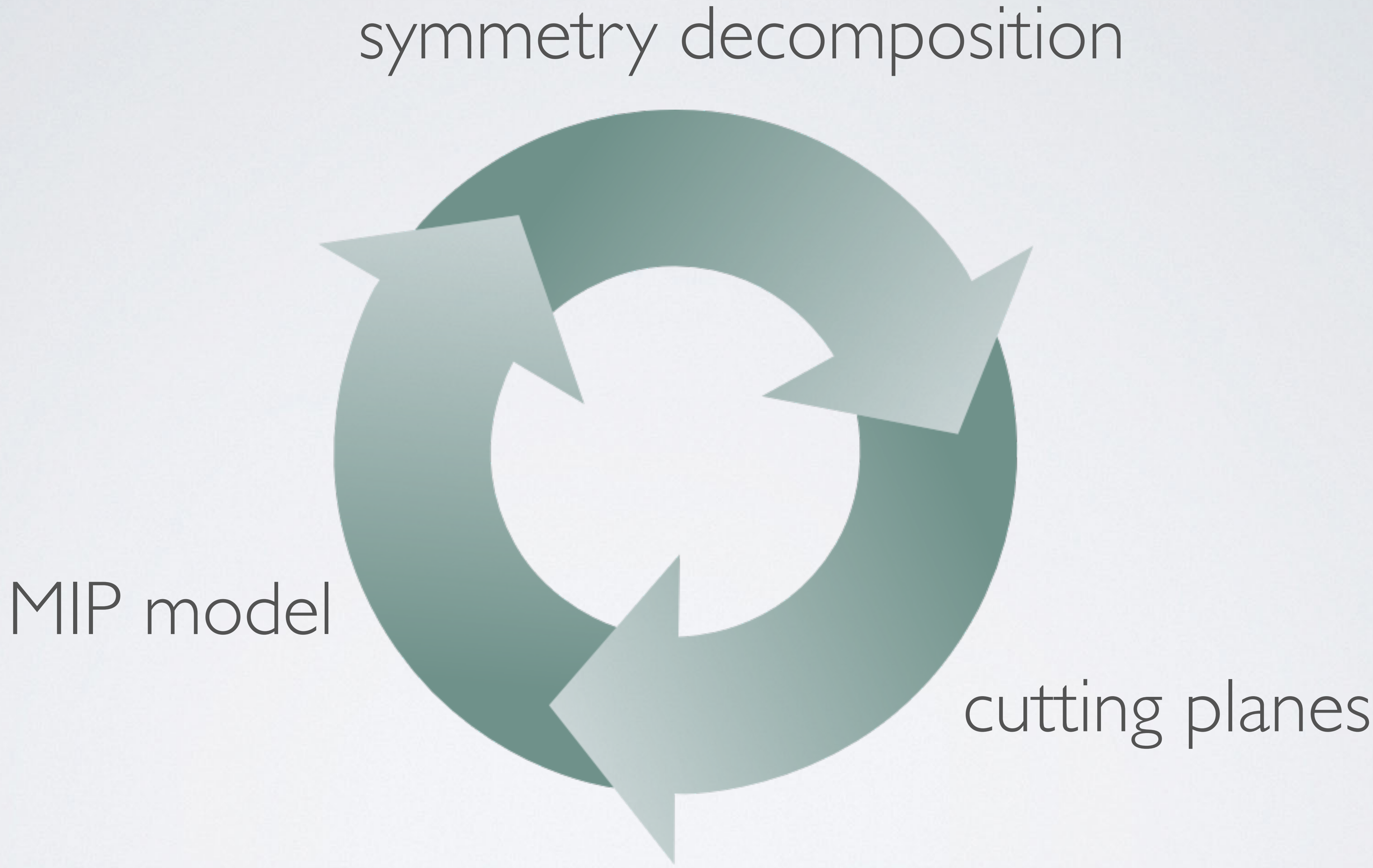
isomorphism pruning within sub-MIPs… exploit symmetry twice!!!


symmetry decomposition
MIP model
cutting planes

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Computational Results

subproblems	count	time	nodes

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less than one week on a desktop PC!

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- solved biggest Q3AP instance to date
- had fun :-)
- developed (extended) techniques that can be used for other Q3APs and (more importantly) QAPs and beyond…

Thanks for your attention!

Questions?