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Stratified Gaussian Graphical Models

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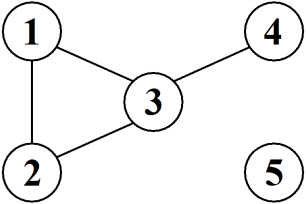
Introduction

- Multivariate Gaussian distribution.
- Gaussian graphical model.
  - Graphical illustration of the dependence structure among the variables in a multivariate Gaussian distribution.
- Multivariate Gaussian distributions constitute a very rigid family of distributions in regard to the dependence structure.
- Introduce context-specific independencies in order to accommodate a more diverse class of distributions and models.

Graphical Models



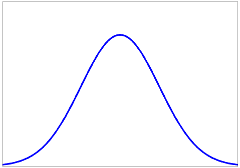
- Marginal independence

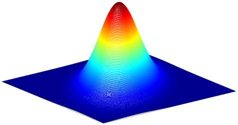
$$X_1 \perp X_5$$

- Conditional independence

$$X_1 \perp X_4 \mid X_3$$

Multivariate Gaussian Distribution



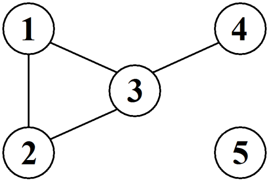


$$X \sim N(\mu, \Sigma)$$

$$f_{\mu,\Sigma}(x) = (2\pi)^{-d/2}|\Sigma|^{-1/2}e^{-1/2(x-\mu)^T\Sigma^{-1}(x-\mu)}$$

$$K = \Sigma^{-1}$$

Graphical Model Meets Multivariate Gaussian



$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{14} & 0 \\ \sigma_{12} & \sigma_{22} & \sigma_{23} & \sigma_{24} & 0 \\ \sigma_{13} & \sigma_{23} & \sigma_{33} & \sigma_{34} & 0 \\ \sigma_{14} & \sigma_{24} & \sigma_{34} & \sigma_{44} & 0 \\ 0 & 0 & 0 & 0 & \sigma_{55} \end{pmatrix}$$

$$K = \begin{pmatrix} k_{11} & k_{12} & k_{13} & 0 & 0 \\ k_{12} & k_{22} & k_{23} & 0 & 0 \\ k_{13} & k_{23} & k_{33} & k_{34} & 0 \\ 0 & 0 & k_{34} & k_{44} & 0 \\ 0 & 0 & 0 & 0 & k_{55} \end{pmatrix}$$

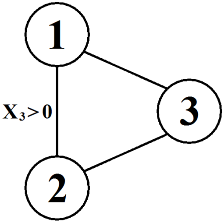
Fitting Σ to Data

- Consider a data matrix $\mathbf{X}$, consisting of n observations (rows) on d variables (columns).
- The maximum likelihood estimates for μ and Σ are calculate

$$\hat{\mu} = \frac{1}{n}\Sigma_{i=1}^{n} x_i \qquad \hat{\Sigma} = \frac{1}{n}\Sigma_{i=1}^{n}(x_i - \hat{\mu})(x_i - \hat{\mu})^T.$$

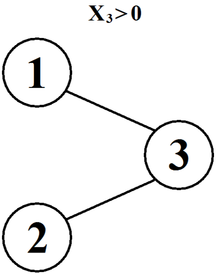
- Using iterative proportional fitting $\hat{\Sigma}$ can be transformed to fit any graph.

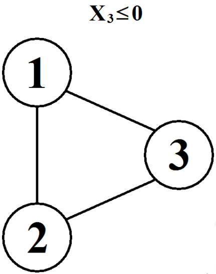
Context-Specific Independence

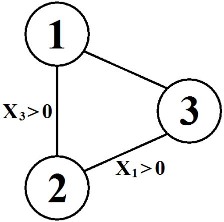


- Conditional independence that holds only in a subset of the outcome space.

$$X_1 \perp X_2 \mid X_3 > 0$$



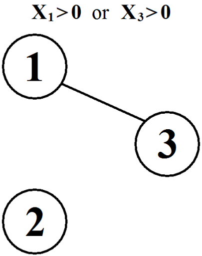


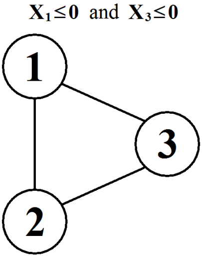


- Two context-specific independencies

$$X_1 \perp X_2 \mid X_3 > 0 \qquad X_3 \perp X_2 \mid X_1 > 0.$$

- For these to hold simultaneously X_2 has to be independent of both X_1 and X_3 once either $X_1 > 0$ or $X_3 > 0$.





- Transforming the strata to a discrete setting will allow for a coherent analysis of the dependence structure.

- This results in a set of conditions on the variables with each condition $c^{(i)}$ associated with a specific dependence structure in the form of a graph $G^{(i)}$.
  - The conditions form a partition of the outcome space of the variables.
  - The condition $c^{(i)}$ can be written in the form $a_j^{(i)} < X_j < b_j^{(i)}$, $j = 1, \ldots, d$.

- Using iterative proportional fitting the graph $G^{(i)}$ induces a specific covariance matrix $\Sigma^{(i)}$.

Density Function for Stratified Gaussian Graphical Models

- The density function of a distribution in a stratified Gaussian graphical model can be written

$$g_{\mu,\Sigma}(x) = \frac{1}{Z}\sum_{i=1}^{\rho} f_{\mu,\Sigma^{(i)}}(x) I_{c^{(i)}}(x)$$

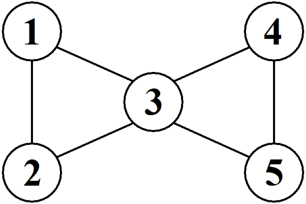
- The normalizing constant Z is calculated as

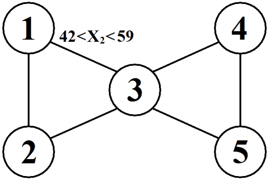
$$Z = \sum_{i=1}^{\rho} \int_{a_1^{(i)}}^{b_1^{(i)}} \dots \int_{a_d^{(i)}}^{b_d^{(i)}} f_{\mu,\Sigma^{(i)}}(x) dx_d \dots dx_1$$

- This distribution belongs to the curved exponential family.
  - Bayesian information criterion can be used to approximate the marginal likelihood of a model.

Math Mark Data

Variable	Label
Mechanics	1
Vectors	2
Algebra	3
Analysis	4
Statistics	5





$$K = \begin{pmatrix} 0.0053 & -0.0025 & -0.0029 & 0.0000 & -0.0001 \\ -0.0025 & 0.0105 & -0.0048 & -0.0008 & -0.0002 \\ -0.0028 & -0.0048 & 0.0273 & -0.0071 & -0.0048 \\ 0.0000 & -0.0008 & -0.0071 & 0.0100 & -0.0020 \\ -0.0001 & -0.0002 & -0.0048 & -0.0020 & 0.0065 \end{pmatrix}$$

Considering only the variables X_1, X_2, and X_3 we get these elements in the precision matrix.

$$k_{13}^{(1)} = -0.0078 \qquad k_{13}^{(2)} = 0.000015$$

References

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The end of the presentation

Thank you for listening!

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Questions?