Applications of the Quadratic Assignment Problem

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OSE SEMINAR, NOVEMBER 15, 2013
Introduced by Koopmans and Beckmann in 1957

- Cited by \( \approx 1500 \)
- Among the hardest combinatorial problems
- Real and test instances easily accessible (QAPLib - A quadratic assignment problem library)
- Instances of size \( N = 30 \) are still unsolved
Optimal assignment of factories to the cities marked in green
Distances between the cities and flows between the factories shown below

\[
A = \begin{bmatrix}
0 & 3 & 6 & 4 & 2 \\
3 & 0 & 2 & 3 & 3 \\
6 & 2 & 0 & 3 & 4 \\
4 & 3 & 3 & 0 & 1 \\
2 & 3 & 4 & 1 & 0
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
0 & 10 & 15 & 0 & 7 \\
10 & 0 & 5 & 6 & 0 \\
15 & 5 & 0 & 4 & 2 \\
0 & 6 & 4 & 0 & 5 \\
7 & 0 & 2 & 5 & 0
\end{bmatrix}
\]
- Optimal solution = 258
- Optimal permutation = [2 4 5 3 1]

\[
A = \begin{bmatrix}
0 & 3 & 6 & 4 & 2 \\
3 & 0 & 2 & 3 & 3 \\
6 & 2 & 0 & 3 & 4 \\
4 & 3 & 3 & 0 & 1 \\
2 & 3 & 4 & 1 & 0 \\
\end{bmatrix}
\]

\[
B_{24531} = \begin{bmatrix}
0 & 6 & 0 & 5 & 10 \\
6 & 0 & 5 & 4 & 0 \\
0 & 5 & 0 & 2 & 7 \\
5 & 4 & 2 & 0 & 15 \\
10 & 0 & 7 & 15 & 0 \\
\end{bmatrix}
\]
Hospital Layout - German university hospital, Klinikum Regensburg, built 1972

Optimality proven in the year 2000

[Krarup and Pruzan (1978)]
- Airport gate assignment
- Minimize total passenger movement
- Minimize total baggage movement
- [Haghani and Chen(1998)]
Applications—Three Examples in Electronics

- Steinberg wiring problem
- [Steinberg(1961)]
- Component placing on circuit boards
- [Rabak and Sichman(2003)]
- Minimizing the number of transistors needed on integrated circuits
- Burkard et al.(1993)
△ Optimal placing of letters on keyboards
△ Language specific
△ Burkard and Offermann (1977)

△ Optimal placing of letters on touchscreen devices
△ Only one or two fingers used
△ Dell’Amico et al. (2009)
Turbine runner in electricity generation

The weight of the blades can differ up to ±5%

Objective: To balance the turbine runner

[Laporte and Mercure (1988)]
Seating order at tonight’s dinner
Microarrays can have up to 1.3 million probes
Small subregions can be solved as QAPs
Objective: To reduce the risk of unintended illumination of probes
Bandwith minimization of a graph
Image processing
Economics
Molecular conformations in chemistry
Scheduling
Supply Chains
Manufacturing lines

In addition to all the above, many well known problems in combinatorial optimization can be written as QAPs, e.g.

- Traveling salesman problem
- Maximum cut problem
Three Objective Functions

- **Koopmanns-Beckmann**
  \[
  \min A \cdot XBX^T
  \]  

- **SDP**
  \[
  \min \text{tr}(AXBX^T)
  \]

- **DLR**
  \[
  \min XA \cdot BX
  \]
Koopmans Beckmann form

\[
\sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} \sum_{l=1}^{N} a_{ij} b_{kl} \cdot x_{ik} x_{jl}
\]

\[
\sum_{i=1}^{N} x_{ij} = 1, \quad j = 1, \ldots, N;
\]

\[
\sum_{j=1}^{N} x_{ij} = 1, \quad i = 1, \ldots, N;
\]

\[
x_{ij} \in \{0, 1\}, \quad i, j = 1, \ldots, N;
\]
Koopmans Beckmann form

\[ \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} \sum_{l=1}^{N} a_{ij} b_{kl} \cdot x_{ik} x_{jl} \]

\[ \sum_{i=1}^{N} x_{ij} = 1, \quad j = 1, \ldots, N; \]

\[ \sum_{j=1}^{N} x_{ij} = 1, \quad i = 1, \ldots, N; \]

\[ x_{ij} \in \{0, 1\}, \quad i, j = 1, \ldots, N; \]

▶ This formulation has \( N^2(N - 1)^2 \) bilinear terms.
\[
\min \sum_{i=1}^{N} \sum_{j=1}^{N} a'_{ij} b'_{ij}
\]

\[
a'_{ij} = \sum_{k=1}^{n} a_{kj} x_{ik} \quad \forall i,j
\]

\[
b'_{ij} = \sum_{k=1}^{n} b_{ik} x_{kj} \quad \forall i,j
\]

\[
A = \begin{bmatrix}
0 & 3 & 5 & 9 & 6 \\
3 & 0 & 2 & 6 & 9 \\
5 & 2 & 0 & 8 & 10 \\
9 & 6 & 8 & 0 & 2 \\
6 & 9 & 10 & 2 & 0 \\
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
0 & 4 & 3 & 7 & 7 \\
4 & 0 & 4 & 10 & 4 \\
3 & 4 & 0 & 2 & 3 \\
7 & 10 & 2 & 0 & 4 \\
7 & 4 & 3 & 4 & 0 \\
\end{bmatrix}
\]

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min $XA \cdot BX$

$$
\min \sum_{i=1}^{N} \sum_{j=1}^{N} a'_{ij} b'_{ij}
$$

$$
a'_{ij} = \sum_{k=1}^{n} a_{kj} x_{ik} \quad \forall i, j
$$

$$
b'_{ij} = \sum_{k=1}^{n} b_{ik} x_{kj} \quad \forall i, j
$$

$$
A = \begin{bmatrix}
0 & 3 & 5 & 9 & 6 \\
3 & 0 & 0 & 6 & 9 \\
5 & 2 & 0 & 8 & 10 \\
9 & 6 & 0 & 2 & 0 \\
6 & 9 & 10 & 2 & 0 \\
\end{bmatrix}
$$

$$
B = \begin{bmatrix}
0 & 4 & 3 & 7 & 7 \\
4 & 0 & 4 & 10 & 4 \\
3 & 4 & 0 & 2 & 3 \\
7 & 10 & 2 & 0 & 4 \\
7 & 4 & 3 & 4 & 0 \\
\end{bmatrix}
$$

$$
a'_{23} = 5x_{21} + 2x_{22} + 0x_{23} + 8x_{24} + 10x_{25}
$$

$$
b'_{23} = 4x_{13} + 0x_{23} + 4x_{33} + 10x_{43} + 4x_{53}
$$
Discrete Linear Reformulation (DLR)

\[
\begin{align*}
\min & \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{m=1}^{M_i} B_{ij}^m z_{ij}^m \\
\text{s.t.} & \quad M_i \sum_{k \in K_i^m} x_{kj} = a_{ij}' \\
& \quad z_{ij}^m \leq A_j \sum_{k \in K_i^m} x_{kj} \quad m = 1, ..., M_i \\
& \quad \sum_{m=1}^{M_i} z_{ij}^m = a_{ij}' \\
& \quad \forall i, j
\end{align*}
\]

Example for one bilinear term \(a'_{23} b'_{23}\)

\[
\begin{align*}
a'_{23} &= 5x_{21} + 2x_{22} + 0x_{23} + 8x_{24} + 10x_{25} \\
b'_{23} &= 4x_{13} + 0x_{23} + 4x_{33} + 10x_{43} + 4x_{53} \\
& \quad x_{13} + x_{23} + x_{33} + x_{43} + x_{53} = 1 \\
x_{21} + x_{22} + x_{23} + x_{24} + x_{25} = 1
\end{align*}
\]
Discrete Linear Reformulation (DLR)

\[ \min \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{m=1}^{M_i} B_{ij}^m z_{ij}^m \]

\[ z_{ij}^m \leq A_j \sum_{k \in K_i^m} x_{kj} \quad m = 1, \ldots, M_i \]

\[ \sum_{m=1}^{M_i} z_{ij}^m = a'_{ij} \quad \forall i, j \]

Example for one bilinear term \( a'_{23} b'_{23} \)

\[ a'_{23} = 5x_{21} + 2x_{22} + 0x_{23} + 8x_{24} + 10x_{25} \]

\[ b'_{23} = 4x_{13} + 0x_{23} + 4x_{33} + 10x_{43} + 4x_{53} \]

\[ x_{13} + x_{23} + x_{33} + x_{43} + x_{53} = 1 \]

\[ x_{21} + x_{22} + x_{23} + x_{24} + x_{25} = 1 \]

\[ 4z_{23}^1 + 10z_{23}^2 \]

\[ z_{23}^1 \leq 10(x_{13} + x_{33} + x_{53}) \]

\[ z_{23}^1 + z_{23}^2 = a'_{23} \]
Figure 1: Bilinear term $a'_{23}b'_{23}$ discretized in $b'_{23}$ (to the left) and in $a'_{23}$ (to the right)

- The size of the MILP problem is dependent on the number of unique elements per row.
- Tightness of the MILP problem is dependent on the differences between the elements in each row.
The size of the model is dependent on the number of unique elements per row.
The tightness of the model is dependent on the differences between the elements in each row.
A can be modified to any matrix $\tilde{A}$, where $\tilde{a}_{ij} + \tilde{a}_{ji} = a_{ij} + a_{ji}$.
By solving an LP a priori, we can decrease the model size, tighten the formulation and improve the lower bound.
The size of the model is dependent on the number of unique elements per row.

The tightness of the model is dependent on the differences between the elements in each row.

\( A \) can be modified to any matrix \( \tilde{A} \), where \( \tilde{a}_{ij} + \tilde{a}_{ji} = a_{ij} + a_{ji} \).

By solving an LP a priori, we can decrease the model size, tighten the formulation and improve the lower bound.

\[
A = \begin{bmatrix}
0 & 1 & 2 & 2 & 3 & 4 & 4 & 5 \\
1 & 0 & 1 & 1 & 2 & 3 & 3 & 4 \\
2 & 1 & 0 & 2 & 1 & 2 & 2 & 3 \\
2 & 1 & 2 & 0 & 1 & 2 & 2 & 3 \\
3 & 2 & 1 & 1 & 0 & 1 & 1 & 2 \\
4 & 3 & 2 & 2 & 1 & 0 & 2 & 3 \\
4 & 3 & 2 & 2 & 1 & 2 & 0 & 1 \\
5 & 4 & 3 & 3 & 2 & 3 & 1 & 0
\end{bmatrix}
\]

\[
\tilde{A} = \begin{bmatrix}
0 & 2 & 2 & 2 & 6 & 6 & 6 & 6 \\
0 & 0 & 0 & 0 & 4 & 4 & 4 & 4 \\
2 & 2 & 0 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 0 & 2 & 2 & 2 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & 2 & 2 & 2 & 0 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 0 & 2 & 2 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 0
\end{bmatrix}
\]

Does not break the symmetries in the problem.
<table>
<thead>
<tr>
<th>Instance</th>
<th>Size</th>
<th>opt</th>
<th>old LB</th>
<th>DLR</th>
<th>Time (minutes)</th>
</tr>
</thead>
<tbody>
<tr>
<td>esc32a</td>
<td>32</td>
<td>130</td>
<td>103</td>
<td>130</td>
<td>1964</td>
</tr>
<tr>
<td>esc32b</td>
<td>32</td>
<td>168</td>
<td>132</td>
<td>168</td>
<td>3500</td>
</tr>
<tr>
<td>esc32c</td>
<td>32</td>
<td>642</td>
<td>616</td>
<td>642</td>
<td>254</td>
</tr>
<tr>
<td>esc32d</td>
<td>32</td>
<td>200</td>
<td>191</td>
<td>200</td>
<td>10</td>
</tr>
<tr>
<td>esc64a</td>
<td>64</td>
<td>116</td>
<td>98</td>
<td>116</td>
<td>48</td>
</tr>
</tbody>
</table>

Table 1: Solution times for the instances esc32a, esc32b, esc32c, esc32d and esc64a from the QAPLIB to optimality using Gurobi 4.1 with default settings.

- Instances presented in 1990 and solved 2010 with our models
A few references

Ali Haghani and Min-Ching Chen.
Optimizing gate assignments at airport terminals.

Jakob Krarup and Peter Mark Pruzan.
Computer-aided layout design.

Gilbert Laporte and Helene Mercure.
Balancing hydraulic turbine runners: A quadratic assignment problem.

Axel Nyberg and Tapio Westerlund.
A new exact discrete linear reformulation of the quadratic assignment problem.

C.S Rabak and J.S Sichman.
Using a-teams to optimize automatic insertion of electronic components.

Leon Steinberg.
The backboard wiring problem: A placement algorithm.
Thank you for listening!

Questions?