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On the construction of finite Blaschke products with prescribed critical points

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OSE OPTIMIZATION AND SYSTEMS ENGINEERING

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Talk outline

- Motivation and problem description
- Examples of interpolation and construction of polynomials with prescribed critical points
- Blaschke products
- Formal problem statement
- Solution approach and problem formulation
- Illustrations and experiments
- A global optimization formulation
- Conclusions

- Given n distinct points $z_1, z_2, ..., z_n$ in the open complex unit disc $\mathbb{D}$. Find a Blaschke product (a special rational complex function) $B(z)$ that has zero derivative exactly at those points: $B'(z_i) = 0,\ i = 1, ..., n$.

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- Daniella Kraus and Oliver Roth in a survey paper from august 2011 ask: ″Is there any computationally efficient method for the construction of finite Blaschke products with prescribed critical points?″ ANSWER: **Now there is!**
- This problem has close connection to certain problems in differential geometry (Berger-Nirenberg problem) and for describing the zero sets of functions in Bergman spaces [2] [3].
- A method, based on circle packing and discrete analytic functions, exists for the construction of **discrete** finite Blaschke products [5]. NO method exists in the general case.

Definition

A rational function of the form

$$B(z) = \lambda \prod_{j=1}^{n} \frac{(z-\alpha_j)}{(1-\bar{\alpha}_j z)} \tag{1}$$

where $\lambda, \alpha_j \in \mathbb{C}$, $|\lambda| = 1$ and $|\alpha_j| < 1$ for $j = 1, ..., n$ is called a *Blaschke product* of degree n.

Definition

The function

$$\tilde{B}(z) = \frac{a_0 + a_1 z + ... + a_n z^n}{\bar{a}_n + \bar{a}_{n-1} z + ... + \bar{a}_0 z^n} \tag{2}$$

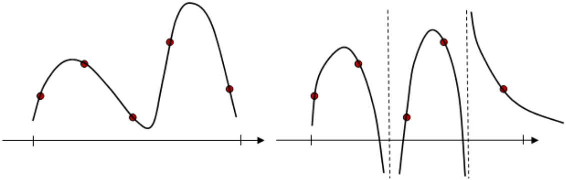
where $a_j \in \mathbb{C}$, $j = 1, ..., n$ is a Blaschke product iff all zeros of $\tilde{B}(z)$ are in $\mathbb{D}$. If $\tilde{B}(z)$ has at least one pole in $\mathbb{D}$ it is called a *Blaschke form*.

Notation: $B(z) = \frac{p(z)}{q(z)}$.

Example: Rational interpolation.

$$B(z_j) = c_j \quad \Leftrightarrow \quad \frac{a_0 + a_1 z_j + \dots + a_n z_j^n}{\bar{a}_n + \bar{a}_{n-1} z_j + \dots + \bar{a}_0 z_j^n} = c_j \quad j = 1, \dots, n$$

Interpolation leads to a (conjugate) *linear system* $p(z_j) - c_j q(z_j) = 0$ in coefficients $a_0, \dots, a_n$.

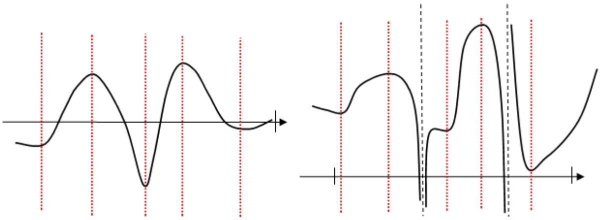


The left interpolant has no poles in (-1,1) and the right interpolant has two poles in (-1,1).

Example: Finding rational functions with prescribed critical points

$$B'(z_j) = 0 \quad \Leftrightarrow \quad p'(z_j)q(z_j) - p(z_j)q'(z_j) = 0 \quad j = 1, \ldots, n \qquad (3)$$

Prescribed critical points leads to a (conjugate) *quadratic system* in coefficients $a_0, \ldots, a_n$.



The left function has no poles in (-1,1) and the right has two poles in (-1,1).

Theoretical fact: Given n distinct critical points $z_1, \ldots, z_n$ in $\mathbb{D}$. There exists a unique (up to postcomposition with a conformal automorphism of $\mathbb{D}$) Blaschke product of degree $n+1$ with $B'(z_j) = 0$ for $j = 1, \ldots, n$.

Theoretical results in [1] (theorem 29.1) and [7] (with normalization $B(0) = 0$ and $B(1) = 1$.)

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For Blaschke products and forms it holds that $B'(z_j)=0 \Rightarrow B'(1/\bar{z}_j)=0$.
Vector of $2n$ critical points: $[z_1,...,z_n,1/\bar{z}_1,...,1/\bar{z}_n]^T$.
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Ansatz:

$$B(z)=\frac{a_1 z+...+a_n z^n+z^{n+1}}{1+\bar{a}_n z+...+\bar{a}_1 z^n}$$

$$\text{Critical points of } B(z) \quad \Leftrightarrow \quad B'(z_k)=p'(z_k)q(z_k)-p(z_k)q'(z_k)=0 \quad \Leftrightarrow$$

$$\underbrace{\sum_{i=1}^{n+1}\sum_{j=1}^{n}(i-j)a_i\bar{a}_{n-j+1}z_k^{i+j-1}}_{\text{quadratic in } a}+\underbrace{\sum_{i=1}^{n} i a_i z_k^{i-1}}_{\text{linear in } a}+\underbrace{(n+1)z_k^n}_{\text{constant}}=0 \quad \Leftrightarrow \quad Ax=b$$

where A is a matrix of size $2n\times(n^2+n)$, $b=-(n+1)[z_1^n,...,z_{2n}^n]^T$ and $x=x(a)$ is a vector of variables with **quadratic structure**.

Example

Analytical solution for $n = 1$.

Critical points: $z = [z_1,\ 1/\bar{z}_1]^T$ $(|z_1| < 1)$.

Ansatz:

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$$\Leftrightarrow \quad a_1 = \frac{-2z_1}{1 + |z_1|^2} \quad \Rightarrow \quad B(z) = \frac{\frac{-2z_1}{1+|z_1|^2} z + z^2}{1 + \frac{-2\bar{z}_1}{1+|z_1|^2} z}$$

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This is a Blaschke product since its zeros are $z = 0$ and $z = \frac{2z_1}{1+|z_1|^2}$ and both lie in $\mathbb{D}$.

Example

Problem formulation for $n = 3$: $B(z) = \frac{a_1 z + \ldots + a_3 z^3 + z^4}{1 + \bar{a}_3 z + \ldots + \bar{a}_1 z^3}$.

System $Ax = b$ looks like:

$$\begin{bmatrix} 1 & 2z_1 & 3z_1^2 & -z_1^2 & z_1^2 & -2z_1^3 & 2z_1^3 & z_1^4 & -z_1^4 & 3z_1^4 & 2z_1^5 & z_1^6 \\ \vdots & & & & & & \vdots & & & & & \vdots \\ 1 & 2z_6 & 3z_6^2 & -z_6^2 & z_6^2 & -2z_6^3 & 2z_6^3 & z_6^4 & -z_6^4 & 3z_6^4 & 2z_6^5 & z_6^6 \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_1 \bar{a}_2 \\ a_2 \bar{a}_3 \\ a_1 \bar{a}_1 \\ a_3 \bar{a}_3 \\ \bar{a}_2 a_3 \\ \bar{a}_1 a_2 \\ \bar{a}_3 \\ \bar{a}_2 \\ \bar{a}_1 \end{bmatrix} = -4 \cdot \begin{bmatrix} z_1^3 \\ z_2^3 \\ z_3^3 \\ z_4^3 \\ z_5^3 \\ z_6^3 \end{bmatrix}$$

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Crude solution approach:

1. Write $x = \alpha + d_1 v_1 + \ldots + d_t v_p$ where α is a particular solution to $Ax = b$ and $v_1, \ldots, v_p$ is a basis for the null space of A.
2. Impose necessary structural constraints on solution vector $x(a)$.

Example

$$x = \alpha + Bd \;\text{ with }\quad \alpha = \begin{bmatrix} 0.1130 + 0.0323i \\ -0.0472 + 0.4804i \\ 0.5713 + 0.6221i \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.5713 - 0.6221i \\ -0.0472 - 0.4804i \\ 0.1130 - 0.0323i \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0.1130 + 0.0323i \\ 0 & 0 & 0 & 0 & 0 & -0.0472 + 0.4804i \\ 1 & 0 & 0 & 0 & 0 & 0.4674 + 0.5090i \\ 3 & 1 & 0 & 0 & 0 & -0.1558 - 0.1697i \\ 0 & 1 & 0 & 0 & 0 & 0.1558 + 0.1697i \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0.1558 - 0.1697i \\ 0 & 0 & 0 & 1 & 3 & -0.1558 + 0.1697i \\ 0 & 0 & 0 & 0 & 1 & 0.4674 - 0.5090i \\ 0 & 0 & 0 & 0 & 0 & -0.0472 - 0.4804i \\ 0 & 0 & 0 & 0 & 0 & 0.1130 - 0.0323i \end{bmatrix}$$

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This system is now well-conditioned. Symmetry in α and $B \Rightarrow$ only half of $x = \alpha + Bd$ is necessary.

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_1\bar{a}_2 \\ a_2\bar{a}_3 \\ |a_1|^2 \\ |a_3|^2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 0.1130+0.0323i \\ -0.0472+0.4804i \\ 0.5713+0.6221i \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0.1130+0.0323i \\ 0 & 0 & 0 & -0.0472+0.4804i \\ 1 & 0 & 0 & 0.4674+0.5090i \\ 3 & 1 & 0 & -0.1558-0.1697i \\ 0 & 1 & 0 & 0.1558+0.1697i \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix}$$

$$x_4 = x_1\bar{x}_2 \quad x_5 = x_2\bar{x}_3$$

$$x_6 = |x_1|^2 \quad x_7 = |x_3|^2$$

$$x_1,\dots,x_7, d_1,\dots,d_4 \in \mathbb{C}$$

A quadratic model (QBP) for finding a Blaschke product/form with prescribed critical points.

$$
\begin{aligned}
x &= \alpha + Bd \\
x_k &= x_i \bar{x}_j \quad (i,j,k) \in \mathcal{T} \\
x_m &= |x_l|^2 \quad (l,m) \in \mathcal{S} \\
x &\in \mathbb{C}^{n_x} \\
d &\in \mathbb{C}^{n_d}
\end{aligned}
\tag{QBP}
$$

$$n_x = (n^2 + 2n - n \bmod 2)/2, \quad n_d = (n^2 - n \bmod 2)/2$$

- The number of complex constraints is $n^2 + 2n - n \bmod 2$.
- The system is square, i.e. the number of variables is equal to the number of constraints.
- Total number of real variables is $2n^2 + 2n$.

Detailed solution approach: Given n critical points in $\mathbb{D}$.

1. Compute a particular solution α to the system $Ax = b$.
2. Generate $p-1$ data independent (null space) vectors from the structure of A: $(v_1, \ldots, v_{p-1})$.
3. Compute the last nullspace vector v_p.
4. Store all vectors in matrix B.
5. Construct problem (QBP).
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Remarks:

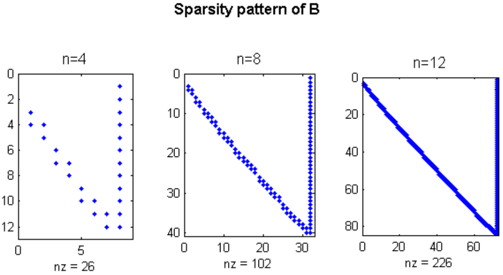
1) There is a particular solution of form $\alpha = (a_1, a_2, a_3, 0, 0, a_4, 0, \ldots., \bar{a}_4, 0, 0, \bar{a}_3, \bar{a}_2, \bar{a}_1)$. This solution can be computed in $\mathcal{O}(n \log n)$ time using the Fast Fourier Transform with proper normalization.

2) $p = n^2 - n$.

3) The last vector v_p is data dependent and can also be computed using FFT with another normalization.

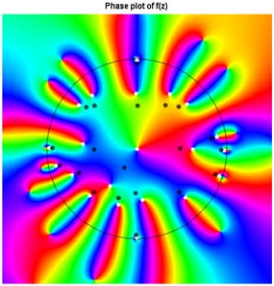
Properties of the linear system:

- The matrix $B = \left[v_1 \ v_2 \ldots v_{p-1} \ v_p\right]$ is very sparse.
- The set of linear equations $x = \alpha + Bd$ is sparse, i.e. each x_i depends on only a few d_j:s.



Example with 20 critical points in $\mathbb{D}$. (Matlab code for phase plot from [6])

black circle $\circ$ - prescribed critical point
black asterix $*$ - computed critical point
white circle - zero of $B(z)$



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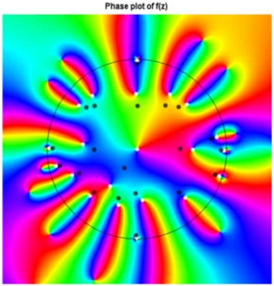
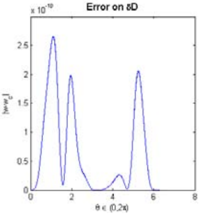
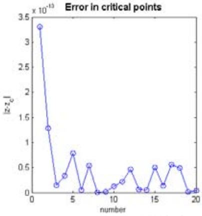
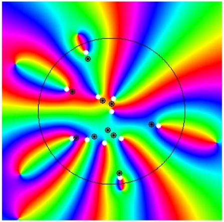






All problems are solved using fsolve method (trust-region dogleg) in matlab 2007b on a Dell laptop 2.26 GHz with 3.5 GB RAM running Windows XP SP2.

Many solutions exist. One Blaschke product and many meromorphic Blaschke forms.
$n = 10$: one Blashke product; two Blaschke forms with one zero outside of $\mathbb{D}$; one Blaschke form with two zeros outside of $\mathbb{D}$.



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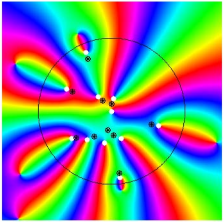

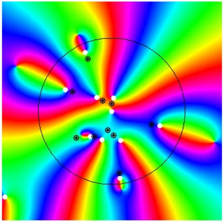

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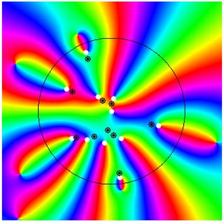

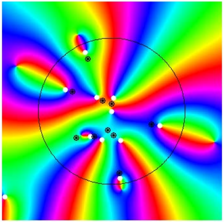

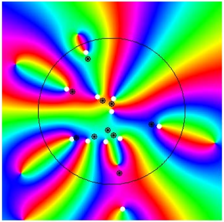

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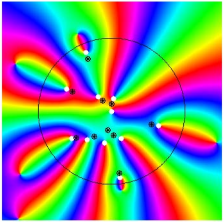

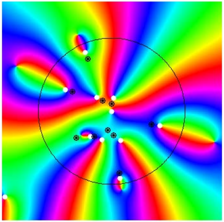

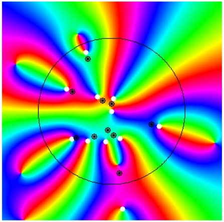

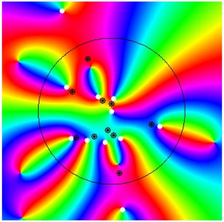

n=50, iters=123, fvals=6820, maxerr=$1.2 \cdot 10^{-4}$

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Starting point is set to $x_0 = \alpha$.

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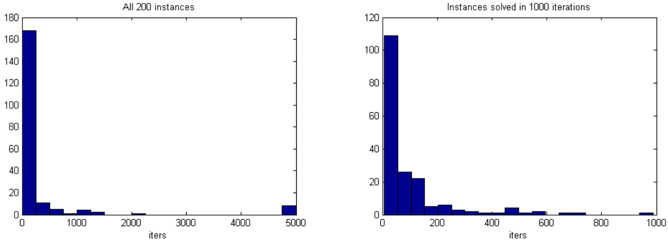

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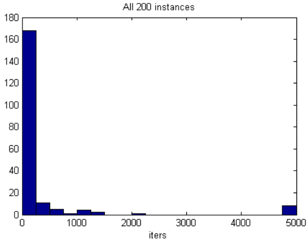
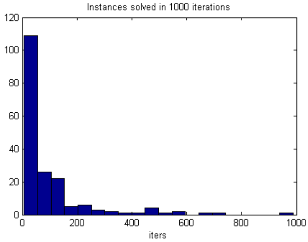
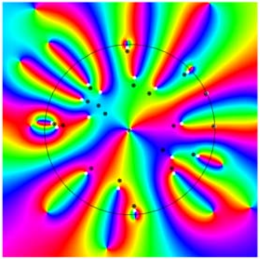
Starting point is set to $x_0 = \alpha$.

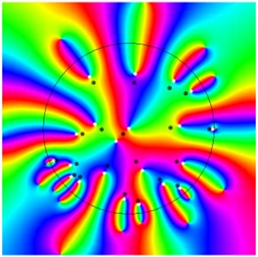




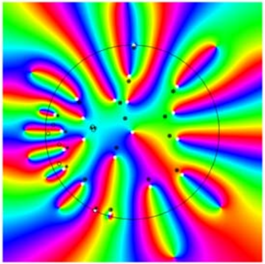
- median of iterations = 46
- median of cpu time = 1.7s
- most instances are solved quickly
- a few instances are hard (slow convergence)
- all 200 solutions are Blaschke products! (starting point $x_0 = \alpha$)



Fast convergence (20 iters)

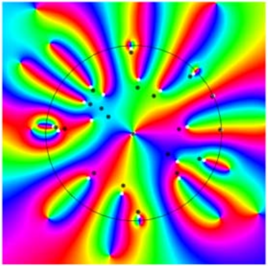


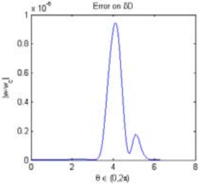
Medium convergence (1500 iters)

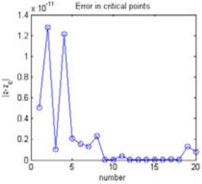


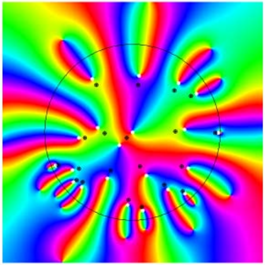
Slow convergence (>5000 iters)

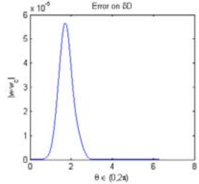
No obvious geometric reason for different rates of convergence.

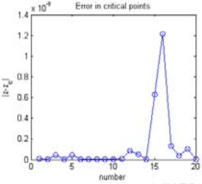



Error on ∂D
x 10^-6
|w-w_c|
θ ∈ [0,2π)


Error in critical points
x 10^-11
|z-z_c|
number




Error on ∂D
x 10^-5
|w-w_c|
θ ∈ [0,2π)


Error in critical points
x 10^-8
|z-z_c|
number

Hypothesis: Among all meromorphic solutions to (QBP) the Blaschke product corresponds to the solution with smallest value of $|a_1|$.

This leads to a global optimization model (QBP) for finding **the unique normalized Blaschke product** with prescribed critical points.

$$
\begin{aligned}
\text{minimize} \quad & |x_1| \\
x &= \alpha + Bd \\
x_k &= x_i \bar{x}_j \quad (i,j,k) \in \mathcal{T} \\
x_m &= |x_l|^2 \quad (l,m) \in \mathcal{S} \\
x &\in \mathbb{C}^{n_x} \\
d &\in \mathbb{C}^{n_d}
\end{aligned}
$$

Total number of real variables are $2n^2 + 2n$.

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- Very small basin of attraction for some of the Blaschke forms.

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Thank you for listening!

Questions?