OSE SEMINAR 2011

MILP FORMULATIONS FOR THE QUADRATIC ASSIGNMENT PROBLEM

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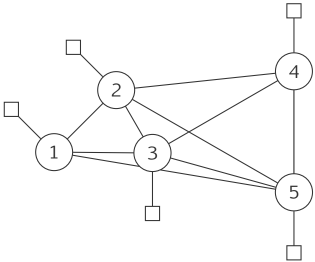

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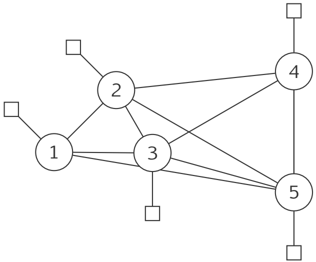

$$\mathbf{A} = \begin{bmatrix} 0 & 3 & 2 & 0 & 1 \\ 3 & 0 & 4 & 2 & 0 \\ 2 & 4 & 0 & 1 & 5 \\ 0 & 2 & 1 & 0 & 0 \\ 1 & 0 & 5 & 0 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 & 10 & 15 & 0 & 3 \\ 10 & 0 & 5 & 6 & 0 \\ 15 & 5 & 0 & 4 & 2 \\ 0 & 6 & 4 & 0 & 1 \\ 3 & 0 & 2 & 1 & 0 \end{bmatrix}$$

Koopmans Beckmann form

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{l=1}^{N} a_{ij} b_{kl} \cdot x_{ik} x_{jl}$$

$$\sum_{i=1}^{N} x_{ij} = 1, \qquad j = 1, ..., N;$$

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$$x_{ij} \in \{0, 1\}, \qquad i, j = 1, ..., N;$$

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- This formulation has $N^2(N-1)^2$ bilinear terms.

Matrix form

$$\mathbf{A} \bullet \mathbf{X}^T \mathbf{B} \mathbf{X} = \mathbf{X}^T \mathbf{A} \mathbf{X} \bullet \mathbf{B} = \mathbf{X} \mathbf{A} \bullet \mathbf{B} \mathbf{X}$$

- $\mathbf{X}$ is a permutation matrix ($\mathbf{X}\mathbf{X}^T = \mathbf{I}$)
- $\bullet$ is the element wise, scalar product of the matrices

MINLP form

$$\min \sum_{i=1}^{N} \sum_{j=1}^{N} a'_{ij} b'_{ij}$$

$$a'_{ij} = \sum_{k=1}^{n} a_{kj} x_{ik} \qquad \forall i,j$$

$$b'_{ij} = \sum_{k=1}^{n} b_{ik} x_{kj} \qquad \forall i,j$$

0	3	5	9	6
3	0	2	6	9
5	2	0	8	10
9	6	8	0	2
6	9	10	2	0

0	4	3	7	7
4	0	4	10	4
3	4	0	2	3
7	10	2	0	4
7	4	3	4	0

MINLP form

$$\min \sum_{i=1}^{N} \sum_{j=1}^{N} a'_{ij} b'_{ij}$$

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3	0	**2**	6	9
5	2	**0**	8	10
9	6	**8**	0	2
6	9	**10**	2	0

0	4	3	7	7
4	**0**	**4**	**10**	**4**
3	4	0	2	3
7	10	2	0	4
7	4	3	4	0

$$a'_{23} = 5x_{21} + 2x_{22} + 0x_{23} + 8x_{24} + 10x_{25}$$

$$b'_{23} = 4x_{13} + 0x_{23} + 4x_{33} + 10x_{43} + 4x_{53}$$

Discrete Linear Reformulation (DLR)

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{m=1}^{M_i} B_i^m z_{ij}^m$$

$$\left.\begin{array}{ll} z_{ij}^m \leq \overline{A}_j \sum_{k \in \mathbf{K}_i^m} x_{kj} & m = 1, \ldots, M_i \\ \sum_{m=1}^{M_i} z_{ij}^m = a'_{ij} & \end{array}\right\} \forall i, j$$

Example for one bilinear term $a'_{23} b'_{23}$

$$a'_{23} = 5x_{21} + 2x_{22} + 0x_{23} + 8x_{24} + 10x_{25}$$

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$$x_{13} + x_{23} + x_{33} + x_{43} + x_{53} = 1$$

$$x_{21} + x_{22} + x_{23} + x_{24} + x_{25} = 1$$

$$0z_{23}^0 + 4z_{23}^1 + 10z_{23}^2$$

$$z_{23}^0 \leq 10x_{23}$$

$$z_{23}^1 \leq 10(x_{13} + x_{33} + x_{53})$$

$$z_{23}^2 \leq 10x_{43}$$

$$z_{23}^0 + z_{23}^1 + z_{23}^2 = a'_{23}$$

Discrete Linear Reformulation (DLR)

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$$4z_{23}^1 + 10z_{23}^2$$

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$$z_{23}^2 \leq 10(x_{43})$$

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Discrete Linear Reformulation (DLR)

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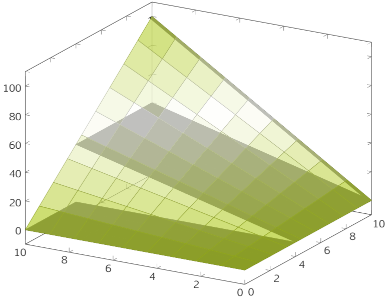

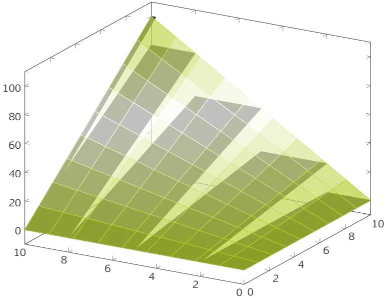

Figure 1: Bilinear term $a'_{23}b'_{23}$ discretized in b'_{23} (to the left) and in a'_{23} (to the right)

DLR version 2

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} B_i^1 a'_{ij} + \sum_{m=2}^{M_i} (B_i^m - B_i^{(m-1)})(z_{ij}^m)$$

$$z_{ij}^m \geq a'_{ij} + \overline{A}_j \Big(\sum_{\substack{m' \geq m \\ k \in \mathbf{K}_i^{m'}}} x_{kj} \Big) - \overline{A}_j$$

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$$4a'_{23} + 6z_{23}^1$$

$$z_{23}^1 \geq a'_{23} + 10x_{43} - 10$$

DLR Version 3

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} \overline{A}_j b'_{ij}$$

$$+ \sum_{m=1}^{M_i} (B_i^m - B_i^{(m-1)})(z_{ij}^m - (\overline{A}_j - \underline{A}_j) \sum_{\substack{m' \geq m \\ k \in \mathbf{K}_i^{m'}}} x_{kj})$$

$$z_{ij}^m \geq a'_{ij} + (\overline{A}_j - \underline{A}_j)(\sum_{\substack{m' \geq m \\ k \in \mathbf{K}_i^{m'}}} x_{kj}) - \overline{A}_j$$

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DLR Version 3

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} \overline{A}_j b'_{ij}$$

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$$z_{ij}^m \geq a'_{ij} + (\overline{A}_j - \underline{A}_j)(\sum_{\substack{m' \geq m \\ k \in \mathbf{K}_i^{m'}}} x_{kj}) - \overline{A}_j$$

Example for one bilinear term $a'_{23} b'_{23}$

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$$x_{21} + x_{22} + x_{23} + x_{24} + x_{25} = 1$$

$$10b'_{23} + 4(z_{23}^1 - 6(x_{13} + x_{33} + x_{43} + x_{53})) + 6(z_{23}^1 - 6x_{43})$$

$$z_{23}^1 \geq a'_{23} + 6(x_{13} + x_{33} + x_{43} + x_{53}) - 10$$

$$z_{23}^2 \geq a'_{23} + 6x_{43} - 10$$

Matrix Modifications

When $\mathbf{B}$ is symmetric (i.e. $\mathbf{B} = \mathbf{B}^T$)

$$A \bullet X^T B X = A^T \bullet X^T B^T X = A^T \bullet X^T B X$$

$$A = A_1 + A_2$$

$$\ldots = A_1 \bullet X^T B X + A_2 \bullet X^T B X = (A_1 + A_2^T) \bullet X^T B X$$

Matrix Modifications

When $\mathbf{B}$ is symmetric (i.e. $\mathbf{B} = \mathbf{B}^T$)

$$\mathbf{A} \bullet \mathbf{X}^T\mathbf{B}\mathbf{X} = \mathbf{A}^T \bullet \mathbf{X}^T\mathbf{B}^T\mathbf{X} = \mathbf{A}^T \bullet \mathbf{X}^T\mathbf{B}\mathbf{X}$$

$$\mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2$$
$$\ldots = \mathbf{A}_1 \bullet \mathbf{X}^T\mathbf{B}\mathbf{X} + \mathbf{A}_2 \bullet \mathbf{X}^T\mathbf{B}\mathbf{X} = (\mathbf{A}_\mathbf{1} + \mathbf{A}_\mathbf{2}^\mathbf{T}) \bullet \mathbf{X}^T\mathbf{B}\mathbf{X}$$

- $\mathbf{A}$ can be modified to any matrix $\tilde{\mathbf{A}}$, where $\tilde{a}_{ij} + \tilde{a}_{ji} = a_{ij} + a_{ji}$.
- Size of MILP problem dependent on number of unique elements per row.
- Tightness of MILP problem dependent on difference between elements in row.

$$\min \sum_i \overline{y}_i - \underline{y}_i$$

$$\overline{y}_i \geq y_{ij} \quad \forall i,j$$

$$\underline{y}_i \leq y_{ij} \quad \forall i,j$$

$$y_{ij} + y_{ji} = a_{ij} + a_{ji}$$

- The size of the MILP problem is dependent on the number of unique elements per row.
- Tightness of the MILP problem is dependent on the differences between the elements in each row.

$$\min \sum_{i,j,k} \Delta_{ijk}$$

$$\Delta_{ijk} \geq y_{ij} - y_{ik}$$

$$\Delta_{ijk} \geq y_{ik} - y_{ij}$$

$$y_{ij} + y_{ji} = a_{ij} + a_{ji}$$

$$\min \sum_{i,j,k} \Delta_{ijk}$$

$$\Delta_{ijk} \geq y_{ij} - y_{ik}$$

$$\Delta_{ijk} \geq y_{ik} - y_{ij}$$

$$y_{ij} + y_{ji} = a_{ij} + a_{ji}$$

0	1	2	2	3	4	4	5
1	0	1	1	2	3	3	4
2	1	0	2	1	2	2	3
2	1	2	0	1	2	2	3
3	2	1	1	0	1	1	2
4	3	2	2	1	0	2	3
4	3	2	2	1	2	0	1
5	4	3	3	2	3	1	0

0	2	2	2	6	6	6	6
0	0	0	0	4	4	4	4
2	2	0	2	2	2	2	2
2	2	2	0	2	2	2	2
0	0	0	0	0	0	0	0
2	2	2	2	2	0	2	2
2	2	2	2	2	2	0	2
4	4	4	4	4	4	0	0

Table 1: **A** before and after modification

Instance	BKS	old LB	DLR	Time(minutes)
esc32a	130	103	130	1964
esc32b	168	132	168	3500
esc32c	642	616	642	254
esc32d	200	191	200	10
esc64a	116	98	116	272

Table 2: Solution times when solving the instances esc32a, esc32b, esc32c, esc32d and esc64a from the QAPLIB to global optimality

- Previously unsolved instances presented in 1990.
- Nug30 ($n = 30$) solved in 2001 (in 7 days) using 1000 computers in parallel.

Some references

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Thank you for listening!

Questions?