OSE SEMINAR 2011

A Mixed Integer Quadratic Reformulation of the Quadratic Assignment Problem with Rank-1 Matrix

Otto Nissfolk

CENTER OF EXCELLENCE IN
OPTIMIZATION AND SYSTEMS ENGINEERING
ÅBO AKADEMI UNIVERSITY

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$$\min_{x \in X} \quad \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} f_{ik} d_{jl} x_{ij} x_{kl} + \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

$$\begin{aligned} X = \{x \mid \sum_{j=1}^{n} x_{ij} &= 1 \quad i \in N \\ \sum_{i=1}^{n} x_{ij} &= 1 \quad j \in N \\ x_{ij} &\in \{0,1\} \quad i,j \in M\} \end{aligned}$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} f_{ik} d_{jl} x_{ij} x_{kl} = \text{trace}(\mathbf{DXFX}^T)$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} f_{ik} d_{jl} x_{ij} x_{kl} = \mathrm{trace}(\mathbf{DXFX}^T)$$

$$\mathbf{F}=\mathbf{qq}^T$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} f_{ik}\, d_{jl}\, x_{ij}\, x_{kl} = \mathrm{trace}(\mathbf{DXFX}^T)$$

$$\mathbf{F}=\mathbf{qq}^T$$

$$= \mathrm{trace}(\mathbf{DXqq}^T\mathbf{X}^T)$$
$$= \mathrm{trace}(\mathbf{q}^T\mathbf{X}^T\mathbf{DXq})$$
$$= \mathrm{trace}(\mathbf{Xq}^T\mathbf{DXq})$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} f_{ik} d_{jl} x_{ij} x_{kl} = \text{trace}(\mathbf{DXFX}^T)$$

$$\mathbf{F}\text{=}\mathbf{qq}^T$$

$$\begin{aligned} &= \text{trace}(\mathbf{DXqq}^T\mathbf{X}^T) \\ &= \text{trace}(\mathbf{q}^T\mathbf{X}^T\mathbf{DXq}) \\ &= \text{trace}(\mathbf{Xq}^T\mathbf{DXq}) \end{aligned}$$

$$= \text{trace}(\mathbf{y}^T\mathbf{Dy}) = \mathbf{y}^T\mathbf{Dy}$$

$$\min_{x \in X, y \in \mathbb{R}^n} \quad \mathbf{y}^T \mathbf{D} \mathbf{y}$$

subject to

$$y_i = \sum_{j=1}^{n} x_{ij} q_j \quad \forall\, i$$

$$\sum_{i=1}^{n} y_i = \sum_{j=1}^{n} q_j$$

$$\min_{x \in X,\, y, z \in \mathbb{R}^n} \quad \mathbf{y}^T(\mathbf{D} + \mathrm{Diag}(\mathbf{u}))\mathbf{y} - \mathbf{u}^T\mathbf{z}$$

subject to

$$y_i = \sum_{j=1}^{n} x_{ij} q_j \quad \forall\, i$$

$$z_i = \sum_{j=1}^{n} x_{ij} q_j^2 \quad \forall\, i$$

$$\sum_{i=1}^{n} y_i = \sum_{j=1}^{n} q_j$$

$$\min_{y,z \in \mathbb{R}^n} \quad \mathbf{y}^T(\mathbf{D} + \mathrm{Diag}(\mathbf{u}))\mathbf{y} - \mathbf{u}^T\mathbf{z}$$

subject to

$$y_i = \sum_{j=1}^{m} x_{ij} v_j \quad \forall\, i$$

$$z_i = \sum_{j=1}^{m} x_{ij} v_j^2 \quad \forall\, i$$

$$\sum_{i=1}^{n} y_i = \sum_{j=1}^{m} f_j v_j$$

$$f_j = \sum_{i=1}^{n} x_{ij} \quad \forall\, j$$

$$\sum_{j=1}^{m} x_{ij} = 1 \quad \forall\, i$$

$$\min_{y,z\in\mathbb{R}^n} \quad \mathbf{y}^T(\mathbf{D}+\mathrm{Diag}(\mathbf{u}))\mathbf{y}-\mathbf{u}^T\mathbf{z}$$

subject to

$$y_i = \sum_{j=1}^{m} x_{ij} v_j \quad \forall\, i$$

$$z_i = \sum_{j=1}^{m} x_{ij} v_j^2 \quad \forall\, i$$

$$\sum_{i=1}^{n} y_i = \sum_{j=1}^{m} f_j v_j$$

$$f_j = \sum_{i=1}^{n} x_{ij} \quad \forall\, j$$

$$\sum_{j=1}^{m} x_{ij} = 1 \quad \forall\, i$$

$$\mathbf{q} = \begin{pmatrix} 0.0000 \\ 0.0371 \\ 0.7655 \\ 0.7655 \\ 0.7655 \\ 0.7672 \\ 1.3284 \\ 1.3284 \\ 1.3284 \\ 1.3284 \\ 1.3284 \\ 1.3284 \\ 1.3284 \\ 1.3924 \\ 1.3924 \\ 1.4512 \end{pmatrix}$$

$$\min_{y,z \in \mathbb{R}^n} \quad \mathbf{y}^T(\mathbf{D} + \mathrm{Diag}(\mathbf{u}))\mathbf{y} - \mathbf{u}^T\mathbf{z}$$

subject to

$$y_i = \sum_{j=1}^{m} x_{ij} v_j \quad \forall\, i$$

$$z_i = \sum_{j=1}^{m} x_{ij} v_j^2 \quad \forall\, i$$

$$\sum_{i=1}^{n} y_i = \sum_{j=1}^{m} f_j v_j$$

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$$\mathbf{q} = \begin{pmatrix} 0.0000 \\ 0.0371 \\ 0.7655 \\ 0.7655 \\ 0.7655 \\ 0.7672 \\ 1.3284 \\ 1.3284 \\ 1.3284 \\ 1.3284 \\ 1.3284 \\ 1.3284 \\ 1.3284 \\ 1.3924 \\ 1.3924 \\ 1.4512 \end{pmatrix}$$

$$\mathbf{v} = \begin{pmatrix} 0.0000 \\ 0.0371 \\ 0.7655 \\ 0.7672 \\ 1.3284 \\ 1.3924 \\ 1.4512 \end{pmatrix}$$

$$\mathbf{f} = \begin{pmatrix} 1 \\ 1 \\ 3 \\ 1 \\ 7 \\ 2 \\ 1 \end{pmatrix}$$

- XYL - Xia-Yuan linearization
- GLL - Gilmore-Lawler linearization

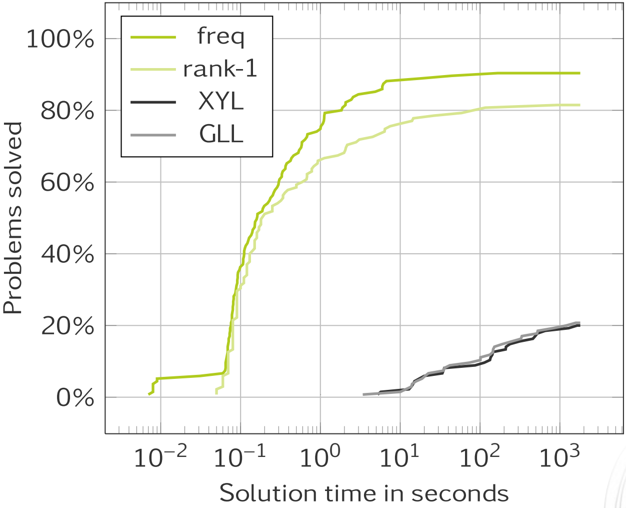
$$T_{rstu} = \max_{v,w \in \{-1,0,1\}} \frac{1}{(r-t+nv)^2 + (s-u+nw)^2}$$

$$f_{ij} = \begin{cases} 1 & \text{if } i \leq m \text{ and } j \leq m \\ 0 & \text{otherwise} \end{cases}$$

$$d_{ij} = d_{n(r-1)+s\; n(t-1)+u} = T_{rstu}$$

where (r, s) are the coordinates for i and (t, u) are the coordinates for j

QAPLIB-Results



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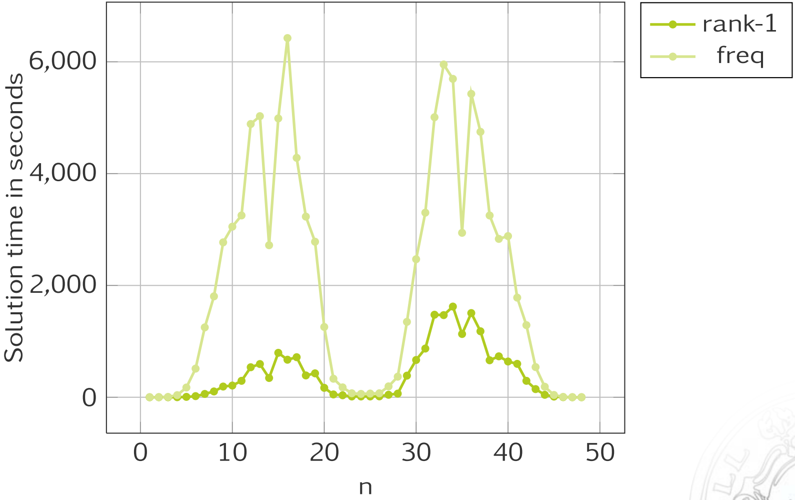
rank-1
freq
6,000
4,000
2,000
0
Solution time in seconds
0
10
20
30
40
50
n

Problem	Freq Time	m	Rank-1 Time	XYL Time	GLL Time
chr12a	0.26	12	0.133	5.694	15.794
chr12b	0.125	12	0.09	14.52	10.005
chr12c	0.163	12	0.103	12.879	11.751
chr15a	0.15	15	0.188	236.978	189.874
chr15b	0.237	15	0.25	1288.27	824.66
chr15c	0.253	15	0.25	15.069	13.797
chr22a	2.061	22	1.977	459.071	333.687
chr22b	2.437	22	2.09	211.309	132.858
esc16a	0.422	6	82.164	209.268	100.967
esc16d	4.864	11	10.113	34.77	22.518
esc16e	1.074	6	6.431	17.309	20.495
esc16g	5.963	9	7.465	33.72	18.801
esc16i	2.988	10	5.354	135.822	41.76
esc16j	1.888	7	3.066	13.804	14.081
esc32e	0.584	3	0.778	142.505	145.589
esc32g	2.57	5	4.513	144.472	151.165
had12	0.08	12	0.069	522.193	325.841
lipa20a	0.031	1	0.065	19.943	35.759
lipa30a	0.008	2	0.046	112.821	145.24
lipa40a	0.007	1	0.054	490.771	528.238
lipa50a	0.008	2	0.06	1647.234	1578.647
scr12	0.899	12	0.931	35.596	34.849
tai10a	0.074	10	0.057	86.777	73.501
tai10b	0.069	10	0.054	5.221	3.393
tai12a	0.078	12	0.065	654.933	517.752
tai12b	0.076	12	0.063	133.037	101.315

Table 1: Problems solved within timelimit of 1800 s

Problem	Freq Time	m	Rank-1 Time	XYL Time	GLL Time
chr12a	0.26	12	0.133	5.694	15.794
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lipa20a	0.031	1	0.065	19.943	35.759
lipa30a	0.008	2	0.046	112.821	145.24
lipa40a	0.007	1	0.054	490.771	528.238
lipa50a	0.008	2	0.06	1647.234	1578.647
scr12	0.899	12	0.931	35.596	34.849
tai10a	0.074	10	0.057	86.777	73.501
tai10b	0.069	10	0.054	5.221	3.393
tai12a	0.078	12	0.065	654.933	517.752
tai12b	0.076	12	0.063	133.037	101.315

Table 1: Problems solved within timelimit of 1800 s

Problem	Time	m
esc16b	1.84	7
esc16c	166.46	12
esc16h	0.15	5
sko100a	0.30	18
sko100c	0.54	18
sko100d	0.46	18
sko100e	0.30	18
sko100f	0.33	18
tho150	5.96	45
wil100	0.64	18

Table 2: Problems only solved by freq

Some references

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The end of the presentation

Thank you for listening!

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Questions?