Bdry: $u(R,t)=0 \Rightarrow J_0(\lambda R)=0$, so

$\lambda R =$ a zero of J_0. Call these zeros $\alpha_1, \alpha_2, \dots, \alpha_k, \dots$ Then $\lambda_k = \frac{\alpha_k}{R}$.

By adding solutions we get a "general" solution

$$u(\rho,t) = \sum_{k=1}^{\infty} c_k J_0(\lambda_k \rho) e^{-\lambda_k^2 t}.$$

Initial cond: $u(\rho,0) = T_0$ (= constant) $\Rightarrow$

$$\sum_{k=1}^{\infty} c_k J_0(\lambda_k \rho) = T_0 \quad \text{(for all } \rho).$$

Problem. Find the coefficients c_k.

Solution. We know that $\{J_0(\lambda_k\rho)\}$ is orthogonal w.r.t. the inner prod.

$$\langle f,g\rangle = \int_0^R \rho f(\rho) g(\rho)\, d\rho. \quad \text{(see Sect. V.8)}$$

According to the theory in Section VI.3, the "correct" coefficients are

$$c_k = \frac{\langle J_0(\lambda_k\rho), T_0\rangle}{\|J_0(\lambda_k\rho)\|^2}, \quad \text{i.e.,}$$

$$c_k = \frac{\int_0^R \rho J_0(\lambda_k\rho) T_0\, d\rho}{\int_0^R \rho |J_0(\lambda_k\rho)|^2\, d\rho}.$$

We computed in Section V.8:

$$\int_0^R \rho\, [J_0(\lambda_k\rho)]^2 d\rho = \frac{R^2}{2}[J_1(\alpha_k)]^2.$$

In this particular case we can also compute the numerator "exactly" (usually only numerically):

$$\int_0^R \rho J_0(\lambda_k\rho)\, d\rho \quad (\lambda_k\rho = t,\ \lambda_k d\rho = dt)$$

$$= \frac{1}{\lambda_k^2}\int_0^{\alpha_k} t J_0(t)\, dt \quad \text{(see page 49)}$$

$$= \frac{1}{\lambda_k^2} \Big| _0^{\alpha_k} t J_1(t)$$

$$= \frac{\alpha_k}{\lambda_k^2} J_1(\alpha_k) = \frac{\alpha_k R^2}{\alpha_k^2} J_1(\alpha_k) = \frac{R^2}{\alpha_k} J_1(\alpha_k).$$

Thus: We get the following "formal" solution:

$$\boxed{u(\rho,t) = \sum_{k=1}^{\infty} c_k J_0\left(\frac{\alpha_k\rho}{R}\right) e^{-\left(\frac{\alpha_k}{R}\right)^2 t}, \quad \text{where } c_k = \frac{2T_0}{\alpha_k J_1(\alpha_k)}}$$

We also know from Section VI.3, that if the sequence $\{J_0(\lambda_k\rho)\}$ is complete, then

$$\sum_{k=1}^{N} c_k J_0\left(\frac{\alpha_k\rho}{R}\right) \to T_0$$

in the norm which is determined by the inner product

$$\langle f,g\rangle = \int_0^R \rho f(\rho) g(\rho)\, d\rho.$$

Lemma The sequence $\{J_0(\frac{\alpha_k\rho}{R})\}$ is complete, and we do get a solution in this way in a well-defined sense.

Proof: Completeness: Later (maybe)
Convergence: Too difficult.

More generally: For all $\nu \geq 0$ the sequence $\{J_\nu(\frac{\alpha_k\rho}{R})\}_{k=1}^{\infty}$ is complete.

VI. 6 Hankel Transformations

(= Fourier-Bessel series)

Those series that we saw in the preceding section have a name:

<u>6.1 Defn.</u> Let $\nu \geq 0$, and let α_k, $k = 1, 2, \ldots$ be the zeros (=nollställen) of J_ν in $(0,\infty)$. Then the series $\{c_k\}_{k=1}^\infty$, where

$$c_k = \frac{2}{R^2[J_{\nu+1}(\alpha_k)]^2}\langle \phi, J_\nu\left(\frac{\alpha_k \rho}{R}\right)\rangle$$

$$= \frac{2}{R^2[J_{\nu+1}(\alpha_k)]^2}\int_0^R \phi(\rho) J_\nu\left(\frac{\alpha_k\rho}{R}\right)\rho\, d\rho$$

is called the <u>Hankel transform</u> of ϕ of order ν.

As we saw in Section VI.5 we get

<u>6.2 Thm.</u> Let $\{c_k\}_{k=1}^\infty$ be the Hankel transform of ϕ of order ν. Then

$$\phi(\rho) = \lim_{N\to\infty}\sum_{k=1}^N c_k J_\nu\left(\frac{\alpha_k\rho}{R}\right),$$

where the limit exists in the norm induced by the inner product

$$\langle \phi, g\rangle = \int_0^R \phi(\rho) g(\rho)\rho\, d\rho. \qquad \circledast$$

Moreover,

$$\|f\|^2 = \int_0^R |\phi(\rho)|^2 \rho\, d\rho = \sum_{k=1}^\infty |c_k|^2 \frac{R^2}{2}[J_{\nu+1}(\alpha_k)]^2.$$

In other words, if we define the approximation ϕ_N of ϕ by

$$\phi_N(\rho) = \sum_{k=1}^N c_k J_\nu\left(\frac{\alpha_k\rho}{R}\right),$$

then

$$\lim_{N\to\infty}\int_0^R |\phi(\rho) - \phi_N(\rho)|^2 \rho\, d\rho = 0.$$

<u>Note:</u> We do <u>not</u> claim that $\phi_N(\rho) \to \phi(\rho)$ at <u>all points</u> as $N\to\infty$, only "on the average". The problem of in which point we do have convergence has been studied, and reasonably good results do exist, but we do not have time to present them (approximately the same results as for the usual Fourier series.)

<u>Proof.</u> We know from Section V.9 (See thm 9.2 on p. 72) that the sequence $\{J_n\left(\frac{\alpha_k\rho}{R}\right)\}_{k=1}^\infty$ is orthogonal with respect to the inner product $\circledast$. However, it is <u>not</u> orthonormal. Instead $\|J_n\left(\frac{\alpha_k\rho}{R}\right)\| = \beta_k$, where

$$\beta_k = \frac{R}{2}|J_{\nu+1}(\alpha_k)|.$$

Put $\varphi_k(\rho) = \frac{1}{\beta_k}J_\nu\left(\frac{\alpha_k\rho}{R}\right)$. Then $\{\varphi_k\}$ is orthonormalized. If we accept the fact that this sequence is complete, the Thm 3.12 on p. 85 gives

$$\phi(\rho) = \sum_{k=1}^\infty \langle\phi,\varphi_k\rangle\varphi_k$$

$$= \sum_{k=1}^\infty \langle\phi, \frac{1}{\beta_k}J_\nu\left(\frac{\alpha_k\rho}{R}\right)\rangle\frac{1}{\beta_k}J_\nu\left(\frac{\alpha_k\rho}{R}\right)$$

$$= \sum_{k=1}^\infty \underbrace{\frac{1}{\beta_k^2}\langle\phi, J_\nu\left(\frac{\alpha_k\rho}{R}\right)\rangle}_{c_k} J_\nu\left(\frac{\alpha_k\rho}{R}\right).$$

Parseval's identity now becomes

$$\|\phi\|^2 = \sum_{k=1}^{\infty} |\langle\phi, \varphi_k\rangle|^2 = \sum_{k=1}^{\infty} \left|\langle\phi, \tfrac{1}{\beta_k} J_\nu(\tfrac{\alpha_k \rho}{R})\rangle\right|^2$$

$$= \sum_{k=1}^{\infty} \frac{1}{\beta_k^2} \Big|\underbrace{\langle\phi, J_\nu(\tfrac{\alpha_k\rho}{R})\rangle}_{=(\beta_k^2 c_k)^2}\Big|^2$$

$$= \sum_{k=1}^{\infty} \beta_k^2 |c_k|^2 . \qquad \square$$

VI.7 Gramm-Schmidt Orthogonolization

7.1 Problem: We have an arbitrary (not orthogonal) sequence $\{\varphi_n\}_{n=0}^{\infty}$, and want to expand ϕ into

$$\phi = \sum_{n=0}^{\infty} c_n \varphi_n$$

What to do?

7.2 Solution. We replace the original sequence φ_n by a new orthogonal sequence ψ_n, and use the old theory.

More details: i) Choose $\psi_0 = \varphi_0$

ii) Choose $\psi_1 = \varphi_1 + \alpha_{1,0}\varphi_0$, adjust α_1 so that $\psi_1 \perp \psi_0$.

iii) Choose $\psi_2 = \varphi_2 + \alpha_{2,1}\psi_1 + \alpha_{2,0}\psi_0$,

and adjust $\alpha_{2,1}$ and $\alpha_{2,0}$ so that $\psi_2 \perp \psi_1$ and $\psi_2 \perp \psi_0$,

iv) Go on forever.

Solution formulas. i) OK

ii) $\psi_1 \perp \psi_0$ means that $\langle\psi_1, \psi_0\rangle = 0 \iff$

$$\langle\varphi_1 + \alpha_{1,0}\psi_0, \psi_0\rangle = 0 \iff$$

$$\alpha_{1,0}\langle\psi_0, \psi_0\rangle + \langle\varphi_1, \psi_0\rangle = 0, \quad \text{and}$$

$$\alpha_{1,0} = -\frac{\langle\varphi_1, \psi_0\rangle}{\|\psi_0\|^2}$$

iii) A similar computation to the one above (take the inner product of ψ_2 with ψ_1 and ψ_0, and note that $\psi_1 \perp \psi_0$) gives

$$\alpha_{2,1} = -\frac{\langle\varphi_2, \psi_1\rangle}{\|\psi_1\|^2},$$

$$\alpha_{2,0} = -\frac{\langle\varphi_2, \psi_0\rangle}{\|\psi_0\|^2},$$

Thm 7.3 We can construct a orthogonal sequence $\{\psi_n\}_{n=0}^{\infty}$ from the sequence $\{\varphi_n\}_{n=0}^{\infty}$ in the following way: Put

$$\psi_0 = \varphi_0, \quad \text{and}$$

$$(1) \quad \psi_n = \varphi_n - \sum_{k=0}^{n-1} \frac{\langle\varphi_n, \psi_k\rangle}{\|\psi_k\|^2}\psi_k .$$

(If $\psi_n = 0$ for some ψ_n, then we throw away both φ_n and ψ_n, and continue with the next φ_n.)

We can improve this further, and also normalize the functions ψ_n so that $\|\psi_n\| = 1$. We do this by combining Thm. 7.3 with Thm. 3.4 on page 84, and get the following result:

Thm 7.4. From each sequence $\{\varphi_n\}_{n=0}^{\infty}$ we can construct an orthonormal sequence ψ_n as follows: Take

$$\psi_0 = \frac{\varphi_0}{\|\varphi_0\|},$$

and for $n \geq 1$, choose

$$(2)\quad \begin{cases} \tilde{\varphi}_n = \varphi_n - \sum_{k=0}^{n-1} \langle \varphi_n, \psi_k \rangle \psi_k, \\ \psi_n = \frac{\tilde{\varphi}_n}{\|\tilde{\varphi}_n\|} \end{cases}$$

(If some $\tilde{\varphi}_n = 0$, then we throw away φ_n and continue with φ_{n+1}.)

Recall Defn. 3.5 on p. 81:

7.5 Defn. ϕ is a linear combination of $\{\varphi_0, \dots, \varphi_{n-1}\}$ if it can be written as

$$\phi = \sum_{k=0}^{n-1} \alpha_k \varphi_k$$

for some constants α_k. The sequence $\{\varphi_n\}_{n=1}^{\infty}$ linearly independent if no φ_n is a linear combination of $\{\varphi_0, \varphi_1, \dots, \varphi_{n-1}\}$, i.e., independently of how we choose $\alpha_0, \dots \alpha_{n-1}$, we always have $\varphi_n \neq \sum_{k=0}^{n-1} \alpha_k \varphi_k$.

7.6 Lemma. The sequence $\{\varphi_n\}_{n=0}^{\infty}$ is linearly independent if and only if none of the functions ψ_n in Thms. 7.3 or 7.4 is the zero function. When this is the case it is possible to find constants $\alpha_{k,n}$ and $\beta_{k,n}$ so that (over)

$$(3)\quad \psi_n = \sum_{k=0}^{n} \alpha_{k,n} \varphi_k$$

$$(4)\quad \varphi_n = \sum_{k=0}^{n} \beta_{k,n} \psi_k,$$

i.e., ψ_n is a linear combination of $\varphi_0, \dots, \varphi_n$, and φ_n is a linear combination of $\psi_0, \dots, \psi_n$.

Proof: To prove (3) we we iterate formula (1) and (2) (substitute the previously obtained formulas for $\psi_0, \psi_1, \dots, \psi_{n-1}$ in terms of $\varphi_0, \varphi_1, \dots, \varphi_{n-1}$). From the same formulas we can also solve for φ_n in terms of $\psi_0, \dots, \psi_n$. Observe that, by Thm 3.3, p. 81, we have

$$\beta_{k,n} = \frac{\langle \varphi_n, \psi_k \rangle}{\|\psi_k\|^2} \quad (\text{Thm. } 7.3) \text{ or}$$

$$\beta_{k,n} = \langle \varphi_n, \psi_k \rangle \quad (\text{Thm } 7.4).$$

Proof that all $\psi_n \neq 0$ if the sequence is independent: Suppose that $\psi_0, \psi_1, \dots, \psi_{n-1} \neq 0$ but $\psi_n = 0$. Then

$$0 = \psi_n = \varphi_n - \sum_{k=0}^{n-1} \langle \varphi_n, \psi_k \rangle \psi_k$$

$$(\text{use } (3)) = \varphi_n - \sum_{k=0}^{n-1} \beta_{k,n} \varphi_k.$$

This means that φ_n is a linear combination of $\varphi_0, \dots, \varphi_{n-1}$, so $\{\varphi_n\}_{n=0}^{\infty}$ is not independent in this case.

The proof in the opposite direction is similar (but substantially longer).

7.7 Back to Problem 7.1. Assume that $\{\varphi_n\}_{n=0}^{\infty}$ is linearly independent (otherwise we remove the dependent ones). By the theory in Section VI.3, if $\{\psi_n\}$ is complete (in which case we also call the original sequence complete or total), then

$$\phi = \lim_{N\to\infty} \sum_{n=0}^{\infty} \langle \phi, \psi_n \rangle \psi_n$$

(here we assume ψ_n to be orthonormal. If we here substitute $\psi_n = \sum_{k=0}^{n} \alpha_{k,n} \varphi_n$, then we get a formula of the type

$$\phi = \lim_{N\to\infty} \sum_{n=0}^{N} \gamma_{N,n} \varphi_n ,$$

where $\gamma_{N,n}$ are suitable constants (which can be compiled from the preceding formulas).

7.8 Ex. Construct an orthonormal sequence of polynomials P_n of degree n as follows: The inner product is $\langle \phi, g \rangle = \int_{-1}^{1} f(x) g(x)\, dx$, and we want

i) P_n has degree $= n$

ii) $P_n \perp P_m$ for $n \neq m$

iii) $\|P_n\| = 1$ for all n

$\Big\}$ orthonormal

Solution: Start with the sequence $\varphi_n = x^n$, and orthogonalize:

i) $\psi_0 = \varphi_0$, $\|\psi_0\|^2 = \int_{-1}^{1} |\overbrace{\varphi_0(x)}^{\equiv 1}|^2 dx = 2$, so

$$\boxed{P_0(x) = \tfrac{1}{\sqrt{2}} \psi_0(x) = \tfrac{1}{\sqrt{2}}}$$

ii) $\psi_1(x) = \overbrace{x}^{=\varphi_1} - \langle \varphi_1, P_0 \rangle P_0(x)$ — odd function

$= x - \int_{-1}^{1} x \cdot \frac{1}{\sqrt{2}} \cdot dx \frac{1}{\sqrt{2}} = x$

$\|\psi_2(x)\|^2 = \int_{-1}^{1} |\psi_1(x)|^2 dx = 2\int_{0}^{1} x^3 dx = \frac{2}{3}$,

$$\boxed{P_1(x) = \sqrt{\tfrac{3}{2}}\, x}$$

iii) $\psi_2(x) = \overbrace{\varphi_2}^{=x^2} - \langle \overbrace{\varphi_2}^{=x^2}, P_0 \rangle P_0 - \langle \varphi_2, P_1 \rangle P_1$

$= x^2 - \underbrace{\int_{-1}^{1} x^2 \underbrace{\tfrac{1}{\sqrt{2}}}_{P_0} dx}\ \underbrace{\tfrac{1}{\sqrt{2}}}_{P_0} - \int_{-1}^{1} x^2 \underbrace{\sqrt{\tfrac{3}{2}}x}_{P_1} dx\ \underbrace{\sqrt{\tfrac{3}{2}}x}_{P_1}$ odd

$= x^2 - \int_{0}^{1} x^2 dx$ even $= x^2 - \frac{1}{3}$

$\|\psi_2(x)\|^2 = \int_{-1}^{1} (x^2 - \tfrac{1}{3})^2 dx = \cdots = \frac{8}{45}$, and

$$\boxed{P_2(x) = \tfrac{3}{2}\sqrt{\tfrac{5}{2}}\,(x^2 - \tfrac{1}{3}) = \sqrt{\tfrac{5}{8}}\,(3x^2 - 1)}$$

etc.

Note: These polynomials have a name: They are normalized versions of the Legendre polynomials. More about these in the next chapter.

VII Legendre Polynoms

VII.1 Serial solution of Legendre's Differential Equation

When we solved the heat and wave equations in spherical coordinates by separating the variables we got the following equation in the latitude variable $x = \cos(\theta)$ (θ = polar angle)

$$(1-x^2)y'' - 2xy' + \left[\lambda - \frac{m^2}{1-x^2}\right] y = 0,$$

where λ and m are separation constants. The case $\boxed{m=0}$ gives Legendre's equation

$$(1) \qquad (1-x^2)y'' - 2xy' + \lambda y = 0$$

Try a series solution:

$$\begin{array}{rcl|l} y &=& \sum_{j=0}^{\infty} a_j x^{k+j} & \lambda \\ y' &=& \sum_{j=0}^{\infty} (k+j) a_j x^{k+j-1} & -2x \\ y'' &=& \sum_{j=0}^{\infty} (k+j)(k+j-1) a_j x^{k+j-2} & 1-x^2 \\ \hline & & & \Sigma \end{array}$$

$$0 = \sum_{j=0}^{\infty} \left[\lambda - 2(k+j) + (k+j)(k+j-1)\right] a_j x^{k+j} + \sum_{j=0}^{\infty} (k+j)(k+j-1) a_j x^{k+j-2} \qquad (j-2 \to j)$$

$$= \sum_{j=0}^{\infty} [\quad] a_j x^{k+j} + \sum_{j=-2}^{\infty} (k+j+2)(k+j+1) a_{j+2} x^{k+j}.$$

The power $j=-2$ gives the index equation:

$j=-2$: $\quad k(k-1)a_0 = 0 \Rightarrow \boxed{k=0}$ or $\boxed{k=1}$

$j=-1$: $\quad (k+1)k\, a_1 = 0 \Rightarrow a_1 = 0$ if $k=1$; a_1 = undetermined if $k=0$.

$j \geq 0$: $$a_{j+2} = -\frac{(k+j)(k+j+1)-\lambda}{(k+j+1)(k+j+2)} a_j \qquad (*)$$

Without loss of generality, let $a_1 = 0$ in both cases. (Subtract a multiple of the solution with $k=1$ from the solution with $k=0$ to make $a_1 = 0$ also there).

This proves:

1.1. Thm The functions $\sum_{j=0}^{\infty} a_{2j} x^{2j}$ $\boxed{\text{even}}$ and $\sum_{j=0}^{\infty} b_{2j+1} x^{2j+1}$ $\boxed{\text{odd}}$ are two linearly independent solutions of Legendre's differential equation, where $a_0 = 1 = b_1$, and the other coefficients are computed recursively from the formula $(*)$ above.

There are two especially interesting cases:

Case A: $\lambda = n(n+1)$, with n = an even integer. Then the case $k=0$ gives $a_{n+2} = 0$, and all successive $a_{n+4}, a_{n+6}, \ldots$ are also zero. In this case the even solution is a polynomial (of order n):

$$\sum_{j=0}^{n/2} a_{2j} x^{2j}.$$

Case B: $\lambda = n(n+1)$ with n odd. Then the same phenomenon shows up in the odd solution corresponding to $k=1$ and we get an odd polynomial (order n)

$$\sum_{j=0}^{(n-1)/2} b_{2j+1} x^{2j+1}$$

Case C. The third possibility $n(n+1)$ is not an integer is the least interesting one. It can be shown by studying the series more closely that the solutions have singularities (poles) at $x = \pm 1$ in this case. This means that the solution to our original equation blows up at the north and south poles. Often this cannot be allowed for physical reasons (the solution should be differentiable here, too) and we put a

(**) $\boxed{\text{separation condition: } \lambda = n(n+1), \text{ with } n = 0, 1, 2, \ldots}$

1.2 Thm. A) If $\lambda = n(n+1)$ where n is either even or odd, then the corresponding even or odd, respectively, solution in Thm. 1.1 is a polynomial, and the other is not a polynomial.

B) If λ is not of the type $\lambda = n(n+1)$, where n = integer, then neither of the two solutions in Thm 1.1 is a polynomial.

C) Only the polynomial solutions are differentiable at $x = \pm 1$. The absolute value of the other solutions tend to ∞ as $x \to \pm 1$.

1.3 Cor. If we require the solution to be differentiable at $x = \pm 1$, then necessarily

$$\lambda = n(n+1) \quad (n = 0, 1, 2, 3, \ldots),$$

and we get only one solution of this type. This gives us the separation condition (**)

1.4 Explicit formulae: With a suitable choice of a_0 and b_0 we get the following formula (proof extra home work):

(Even) $$P_{2m}(x) = \frac{(-1)^m}{2^{2m}} \sum_{j=0}^{m} (-1)^j \frac{(2m+2j)!\, x^{2j}}{(m-j)!\,(m+j)!\,(2j)!}$$

(Odd) $$P_{2m+1}(x) = \frac{(-1)^m}{2^{2m}} \sum_{j=0}^{m} \frac{(2m+2j+1)!\, x^{2j+1}}{(m-j)!\,(m+j)!\,(2j+1)!}.$$

Note: Normalized so that

$$\left.\begin{aligned} P_n(1) &= 1 \\ P_n(-1) &= (-1)^n \end{aligned}\right\} \text{ for all } n.$$

1.5 Alternative formula. Replace j by $k = m-j$, and put $n = 2m$ or $n = 2m+1$ depending on the case (= the degree of P_n). Then both for even and odd n we get

$$P_n(x) = \sum_{k \le n/2} (-1)^k \frac{(2n-2k)!\, x^{n-2k}}{2^n\, k!\, (n-k)!\, (n-2k)!}$$

(Arfken, p. 696, Kreyszig p. 212).

1.6 Note There is a simpler way to compute P_n. See next section.

1.7 Note: The most important discovery was the correct (physical) separation condition

$$\lambda = n(n+1), \quad n = 0, 1, 2, \ldots$$