V.19 Orthogonality of Bessel Functions

Idea: To be able to treat series of the type

$$w(\rho) = \sum_{k=1}^{\infty} a_k J_0(\alpha_k \rho) + b_k N_0(\alpha_k \rho)$$

in an efficient way we need the functions $J_0(\alpha_k \rho)$ and $N_0(\alpha_k \rho)$ to be "orthogonal" in some sense. This would make it look like a Fourier series!

We look at one of the most common case of the generalized Bessel's eq, namely

(1) $x^2 y'' + x y' + (\lambda^2 x^2 - n^2) y = 0$

Take $\alpha = 0$, $\beta = 1$, $\gamma = \lambda$, $\nu = n$ to get

$$y(x) = c_1 J_n(\lambda x) + c_2 N_n(\lambda x).$$

Let us manipulate (divide by x)
(the general rule will be given later):

$$(x y')' + \left(\lambda^2 x - \frac{n^2}{x}\right) y = 0.$$

If we $u(x) = J_n(\lambda x)$, $v(x) = J_n(\mu x)$, then

$$\begin{array}{l|l} (x u')' + \left(\lambda^2 x - \frac{n}{x}\right) u = 0 & v \\ (x v')' + \left(\mu^2 x - \frac{n}{x}\right) v = 0 & -u \\ \hline & \Sigma \end{array}$$

$$v(xu')' - u(xv')' + (\lambda^2 - \mu^2) x u v = 0$$

Recall: The physical interpretation was:

$x = 0$ = the z-axis
$x = R$ = at distance R from z-axis.

If we "average" (= integrate) over the bar, then we get an interesting identity:

$$(\mu^2 - \lambda^2) \int_0^R x u(x) v(x)\, dx$$

$$= \int_0^R [v(xu')' - u(xv')']\, dx \quad \text{(integrate by parts)}$$

$$= \Big|_0^R (v x u' - u x v')$$

$$- \int_0^R \underbrace{(v' x u' - u' x v')}_{=0}.$$

Thus, for $\mu \neq \lambda$ we get

$$\int_0^R x J_n(\lambda x) J_n(\mu x)\, dx$$

$$= \frac{R}{\mu^2 - \lambda^2} \left[\lambda J_n'(\lambda R) J_n(\mu R) - \mu J_n'(\mu R) J_n(\lambda R)\right]$$

We have proved:

9.1 Thm. The functions $J_n(\lambda x)$ and $J_n(\mu x)$ are orthogonal over the interval $(0, R)$ with respect to the weight function x, i.e.,

$$\int_0^R x J_n(\lambda x) J_n(\mu x)\, dx = 0$$

at least in the following cases (take $\lambda \neq \mu$)

(over)

i) $J_n(\lambda R)=0$ and $J_n(\mu R)=0$

ii) $J_n'(\lambda R)=0$ and $J_n'(\mu R)=0$.

Note: This is not true for Neumann functions. We get problems with the substitution term at $x=0$. The Neumann functions come into play only when we have a hollow cylinder.

Case $\boxed{\lambda=\mu}$. Put $u(x)=J_n(\lambda x)$. Then

$$(xu')'+\left(\lambda x^2-\frac{n^2}{x}\right)u=0.$$

Multiply by xu':

$$\frac{d}{dx}\frac{1}{2}(xu')^2+(\lambda^2x^2-n)\frac{d}{dx}\frac{1}{2}u^2=0$$

Integrate (by parts)

$$\frac{1}{2}\int_0^R(xu')^2+\frac{1}{2}\int_0^R(\lambda^2x^2-n)\,u^2$$

$$-\frac{1}{2}\int_0^R\lambda^2\cdot 2x\cdot u^2\,dx=0 \Rightarrow$$

$$\lambda^2\int_0^R xu^2(x)\,dx=\frac{1}{2}R^2[u'(R)]^2+\frac{1}{2}(\lambda^2R^2-n^2)[u(R)]^2+\underbrace{\frac{1}{2}n^2[u(0)]^2}_{=0};$$

($u(0)=J_n(0)=0$ if $n>0$)
if $n=0 \Rightarrow n^2[u(0)]^2=0$ also)

This leads to:

9.2 Thm. i) Denote the zeros of $J_n(x)$ by $0<\alpha_1<\alpha_2<\alpha_3<\ldots$ (n is fixed). Then the functions

$$J_n\left(\frac{\alpha_k x}{R}\right),\quad k=1,2,3,\ldots$$

are orthogonal with respect to the "inner product" $\int_0^R x\,f(x)\,g(x)\,dx$, and their corresponding norms are

$$\left\|J_n\left(\frac{\alpha_k x}{R}\right)\right\|^2=\int_0^R x\left[J_n\left(\frac{\alpha_k x}{R}\right)\right]^2dx$$

$$=\frac{R^2}{2}\left[J_{n+1}(\alpha_k)\right]^2.$$

ii) Denote the zeros of $J_n'(x)$ by $0<\beta_1<\beta_2<\ldots$. Then the functions $J_n\left(\frac{\beta_k x}{R}\right)$, $k=1,2,3,\ldots$, are orthogonal over the interval $(0,R)$ with respect to the same inner product, and their norms are

$$\int_0^R x\,J_n\left(\frac{\beta_k x}{R}\right)^2dx=\frac{R^2}{2}\left(1-\frac{n^2}{\beta_k^2}\right)\left[J_n(\beta_k)\right]^2.$$

Note: This is the reason for all those old volumes of tables of Bessel functions and the zeros of these functions!

Proof. i) We saw above:

$$\|u\|^2=\frac{R^2}{2\lambda^2}[u'(R)]^2=\frac{R^2}{2\lambda^2}\lambda^2\left[J_n'(\lambda R)\right]^2$$

$$=\frac{R^2}{2}\left[J_n'(\alpha_k)\right]^2=\frac{R^2}{2}\Big[\underbrace{\frac{n}{\alpha_k}J_n(\alpha_k)}_{=0}-J_{n+1}(\alpha_k)\Big]^2$$

$$=\frac{R^2}{2}\left[J_{n+1}(\alpha_k)\right]^2.$$

ii) $\|u\|^2=\frac{1}{2}\left(R^2-\frac{n^2}{\lambda^2}\right)[u(R)]^2=\frac{1}{2}\left[R^2-\frac{R^2n^2}{\beta_k^2}\right]\left[J_n(\beta_k)\right]^2$ □

VI Orthogonal Function Sequences

VI.1 Introduction

We have encountered severa orthogonal function sequences

(1) $\boxed{\varphi_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, 3}$

are orthogonal with respect to the "inner product"

$$\langle f, g\rangle = \int_0^L f(x)\, g(x)\, dx$$

in the sense that if $n \neq m$, then

$$\langle \varphi_n, \varphi_m\rangle = \int_0^L \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L}\, dx = 0.$$

This was the sequence we got in the first separation of variables (see pp. 17–21)

(2) $\boxed{\varphi_n(x) = \cos\left(\frac{(n+1/2)\pi x}{L}\right), \quad n = 0, 1, 2, 3, \ldots}$

are orthogonal in the same sense. These where the functions in homework 9.

Proofs of 1) and 2) will come later.

(3) $\varphi_k(\rho) = J_n\left(\frac{\alpha_k \rho}{R}\right), \quad k = 0, 1, 2, \ldots$

(n = fixed) is orthogonal with respect to the inner product

$$\langle f, g\rangle = \int_0^R f(\rho)\, g(\rho)\, \rho\, d\rho .$$

See Thm. 8.1, page 70. Here α_k are the zeros of J_n.

(4) Same example, but change $\alpha_k \to \beta_k$, where β_k = zeros of $J_n'(x)$.

General tasks (= "uppgift"). Suppose that we have an orthogonal sequence of functions φ_k. Try to expand a given function $f(x)$ in the form

$$f(x) = \sum_{k=0}^{\infty} c_k \varphi_k(x)$$

(a "generalized Fourier series").

VI.2 Inner Product Spaces

In this section we treat functions like vectors in a function space (= space whose elements are functions). Call this space $\mathcal{F}$. The usual vector space rules are still valid (as we shall see).

2.1 Ex. $\mathcal{F} = C[0,1]$ = "all continuous functions on $[0,1]$", real values.

2.2 Ex. $\mathcal{F} = \mathcal{L}^2(0,1)$ = "all real valued functions on $(0,1)$ which satisfy

$$\int_0^1 |f(x)|^2 dx < \infty$$

and are "Lebesgue" measurable", (we shall in general ignore the "Lebesgue measurable").

2.3 Addition and Multiplication. i) We multiply a function $f \in \mathcal{F}$ with a constant $\alpha \in \mathbb{R}$ in the natural way: The function "αf" is the one whose value at x is $\alpha f(x)$.

ii) The function $f+g$ (where $f \in \mathcal{F}$, $g \in \mathcal{F}$) is the function which takes the value $f(x)+g(x)$ at a point x (in the common domain of f and g).

iii) The product ϕg is the function which at the point x takes the value $\phi(x)\,g(x)$.

All of these are "trivial" or "obvious" operations. The inner product is more complicated.

2.4 Defn. Let w be a (continuous) function on an interval (a,b), satisfying $w(x)>0$ for $x \in (a,b)$. (Could be zero or infinite at $x=a$ and $x=b$).

i) In the complex-valued case we define the inner product of two continuous functions ϕ and g defined on (a,b) by

$$\langle \phi, g\rangle = \int_a^b w(x)\,\phi(x)\,\overline{g(x)}\,dx \quad \leftarrow \text{compl. conjugate}$$

(Note: $e^{in\varphi}$ is complex)

ii) In the real-valued case we define

$$\langle \phi, g\rangle = \int_a^b w(x)\,\phi(x)\,g(x)\,dx$$

(since $\overline{g(x)} = g(x)$ if g is real).

2.5 Ex. In Examples 2.1 ($\sin \frac{n\pi x}{L}$) and Ex. 2.2 ($\cos \frac{(n+\frac{1}{2})\pi x}{L}$) we take $w(x)\equiv 1$ and define

$$\langle \phi, g\rangle = \int_0^L \phi(x)\,g(x)\,dx.$$

2.6 Ex In connection with Bessel's functions we use $w(x)=x$, and define

$$\langle \phi, g\rangle = \int_0^R x\,\phi(x)\,g(x)\,dx.$$

More examples later.

2.7 Lemma The inner product introduced above has the following properties:

i) Linear in the first argument:

$$\langle \alpha\phi + \beta g, h\rangle = \alpha\langle \phi, h\rangle + \beta\langle g, h\rangle.$$

ii) Symmetric (real case) or conjugate symmetric (complex case):

$$\langle g, \phi\rangle = \langle \phi, g\rangle \qquad \text{(real case)}$$

$$\langle g, \phi\rangle = \overline{\langle \phi, g\rangle} \qquad \text{(complex case)}$$

iii) Strict positive: $\langle \phi, \phi\rangle > 0$, except when $\phi(x)\equiv 0$ (which gives $\langle \phi,\phi\rangle = 0$).

iv) Linear or conjugate linear in the second argument:

$$\langle \phi, \alpha g + \beta h\rangle = \alpha\langle \phi, g\rangle + \beta\langle \phi, h\rangle \qquad \text{(real case)}$$

$$\langle \phi, \alpha g + \beta h\rangle = \bar\alpha\langle \phi, g\rangle + \bar\beta\langle \phi, h\rangle \qquad \text{(complex case)}$$

Proof easy: Just use definition. For example iii):

$$\langle \phi, \phi\rangle = \int_a^b w(x)\,\phi(x)\,\overline{\phi(x)}\,dx = \int_a^b w(x)\,|\phi(x)|^2\,dx$$

> 0, except if $\phi(x)\equiv 0$ which gives a zero (recall that $w(x)>0$ everywhere in (a,b)).

Note: These are the same rules as in $\mathbb{R}^n$ or $\mathbb{C}^n$, with $\langle x,y\rangle = \sum_{k=1}^n x_k y_k$ or $\langle x,y\rangle = \sum_{k=1}^n x_k \bar{y}_k$!

Using this inner product we define orthogonality:

2.8 Defn. The functions $\phi \in \mathcal{F}$ and $g \in \mathcal{F}$ are orthogonal if $\langle \phi, g \rangle = 0$, i.e., if

$$\int_a^b w(x)\, \phi(x)\, \overline{g(x)}\, dx = 0 \qquad \text{(complex case).}$$

2.9 Note: The zero function $\phi(x) \equiv 0$ is orthogonal to every other function.

As in $\mathbb{R}^n$ and $\mathbb{C}^n$, we use the inner product to define a norm (= a distance from zero).

2.10 Defn. The norm of $\phi \in C(a,b)$ is given by

$$\|f\| = \sqrt{\langle \phi, \phi \rangle} = \left[\int_a^b w(x) |\phi(x)|^2 dx \right]^{1/2}$$

(absolute values not needed in real case)

2.11 Lemma. $\|f\| = 0 \iff \phi(x) \equiv 0.$

Proof: Lemma 2.7(iii).

By using the norm we can define convergence:

2.12 Defn. i) We say that ϕ_n converges to zero, and write $\phi_n \to 0$, if $\|f_n\| \to 0$ as $n \to \infty$, i.e.,

$$\left[\int_a^b w(x) |\phi_n(x)|^2 dx \right]^{1/2} \to 0 \quad \text{as } n \to \infty.$$

ii) We say that ϕ_n converges to ϕ if $\|\phi_n - \phi\| \to 0$ as $n \to \infty$, i.e.,

$$\left[\int_a^b w(x) \, |\phi_n(x) - \phi(x)|^2 dx \right]^{1/2} \to 0 \quad \text{as } n \to \infty.$$

iii) We say that $\{\phi_n\}_{n=1}^{\infty}$ is a Cauchy sequence if

$$\lim_{\substack{n\to\infty \\ m\to\infty}} \|\phi_n - \phi_m\| = 0, \quad \text{i.e.,}$$

for all $\varepsilon > 0$ there is a $N > 0$ so that

$$\left[\int_a^b w(x) \, |f_n(x) - \phi_m(x)|^2 dx \right]^{1/2} \le \varepsilon$$

if $n \ge N$ and $m \ge N$.

Note: In probability theory this is called "convergence in mean square". We need not have $\phi_n(x) \to \phi(x)$ at every point x, but "on the average" $\phi_n(x) \to \phi(x)$.

In control theory this is the "energy norm". Same name used in physics, too.

What we have said so far applies to all possible function spaces (where the functions are defined on (a,b) and take real or complex values). The next property is more special:

2.13 Defn. The function space $\mathcal{F}$ is complete if every Cauchy sequence in $\mathcal{F}$ converges (to some limit in $\mathcal{F}$). Thus, there is some $\phi \in \mathcal{F}$ so that $\|\phi_n - \phi\| \to 0$ as $n \to \infty$.

Note: Crucial requirement: The limit function ϕ should belong to the same space $\mathcal{F}$ as the original sequence.

From "Analysis" $\mathbb{R}$, $\mathbb{C}$, $\mathbb{R}^n$, $\mathbb{C}^n$ are complete, but the set of rational numbers is not complete.

2.14 Defn. A complete space with an inner product is a Hilbert space.

2.15 Agreement In the rest of this course we use the function space $\mathcal{F} = C(a,b)$ = set of all continuous functions on (a,b) (they can be discontinuous at the end points).

2.16 Pseudotheorem You are allowed to use all the standard rules about vectors in $\mathbb{R}^n$ and $\mathbb{C}^n$ also in the new setting, except those properties which are based on completeness and compactness.

Forbidden rules:

- ~~"Every Cauchy sequence converges"~~
- ~~"Every bounded infinite set has a cluster point"~~
- ~~"A set is compact if it is closed and bounded"~~
- ~~"Every increasing sequence which is bounded from above converges"~~

"All" other rules are valid. More about this in Analysis II.

VI.3 Orthogonal Function Sequences

3.1 Defn. A function sequence $\{\varphi_n\}$ is orthogonal if for all $n \neq m$,

$$\varphi_n \perp \varphi_m \iff \langle \varphi_n, \varphi_m \rangle = 0 \iff \int_a^b w(x)\, \varphi_n(x)\, \overline{\varphi_m(x)}\, dx = 0.$$

It is orthonormal if, in addition, for all n

$$\|\varphi_n\| = 1 \iff \langle \varphi_n, \varphi_n \rangle = 1 \iff \int_a^b w(x) |\varphi_n(x)|^2 dx = 1.$$

We return to the old Task on p. 74:

How to expand a given function ϕ into a series $$\phi(x) = \sum_{n=1}^{\infty} \alpha_n \varphi_n(x) \;?$$

We start with a simpler task:

3.2 Task. Suppose that $\phi(x) = \sum_{n=1}^{N} \alpha_n \varphi_n(x)$. Find the coefficients α_n!

Solution. Multiply by $\overline{\varphi_m}(x)\, w(x)$, and integrate over $\int_a^b$ to get

$$\int_a^b w(x)\, \phi(x)\, \overline{\varphi_m}(x)\, dx = \int_a^b w(x) \Big(\sum_{n=1}^{N} \alpha_n \varphi_n(x) \Big) \overline{\varphi_m}(x)\, dx$$

or equivalently, using the inner product:

$$\langle \phi, \varphi_m \rangle = \langle \sum_{n=1}^{N} \alpha_n \varphi_n, \varphi_m \rangle \quad \text{(linearity)}$$
$$= \sum_{n=1}^{N} \alpha_n \langle \varphi_n, \varphi_m \rangle \quad (\varphi_n \perp \varphi_m)$$
$$= \alpha_m \langle \varphi_m, \varphi_m \rangle$$
$$= \alpha_m \|\varphi_m\|^2, \quad \text{so}$$
$$\alpha_m = \frac{\langle \phi, \varphi_m \rangle}{\|\varphi_m\|^2}$$

(as in $\mathbb{R}^n$ or $\mathbb{C}^n$!)

<u>3.3 Thm.</u> If $\phi = \sum_{n=1}^{N} \alpha_n \varphi_n$, and $\{\varphi_n\}_{n=1}^{N}$ is orthogonal, then
$$\alpha_m = \frac{\langle \phi, \varphi_m \rangle}{\|\varphi_m\|^2}, \quad m = 1, 2, \ldots N.$$

If the sequence is ortho<u>normal</u>, then
$$\alpha_m = \langle \phi, \varphi_m \rangle, \quad m = 1, 2, \ldots, N.$$

We can simplify this by orthonormalizing $\{\varphi_n\}$:

<u>3.4 Thm</u> (orthonormalization). If $\{\varphi_n\}_{n=1}^{\infty}$ is <u>orthogonal</u> and every $\varphi_n \not\equiv 0$, then $\{\psi_n\}_{n=1}^{\infty}$ is ortho<u>normal</u>, where
$$\psi_n = \frac{\varphi_n}{\|\varphi_n\|}.$$

Proof easy.

<u>3.5 Defn.</u> Every function ϕ of the type $\phi = \sum_{n=1}^{N} \alpha_n \varphi_n$ is called a <u>linear combination</u> of $\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$.

<u>3.6 Corollary.</u> If ϕ is a linear combination of $\varphi_1, \ldots, \varphi_N$, and if $\{\varphi_n\}$ is orthonormal, then
$$\phi = \sum_{n=1}^{N} \langle \phi, \varphi_n \rangle \varphi_n$$

<u>Solution to the Task 3.2.</u> Obviously, the natural candidate for α_n in the expansion
$$\phi(x) = \sum_{n=1}^{\infty} \alpha_n \varphi_n$$
would be $\alpha_n = \dfrac{\langle \phi, \varphi_n \rangle}{\|\varphi_n\|^2}$ (general orthogonal case)

or $\alpha_n = \langle \phi, \varphi_n \rangle$ (orthonormal case).

<u>3.7 Problem:</u> Is it always true that
$$\phi(x) = \sum_{n=1}^{\infty} \langle \phi, \varphi_n \rangle \varphi_n$$
if $\{\varphi_n\}$ is orthonormal ?

<u>Answer:</u> $\boxed{\text{No}}$.
<u>Reason:</u> "Some direction may be missing". For example, if we try to expand (cf. Example (1) on p. 73)
$$\sin \frac{\pi x}{L} = \sum_{k=1}^{\infty} \alpha_{2k} \sin \frac{2k\pi x}{L}, \quad (n = 2k)$$
then the above formula gives $\alpha_{2k} = 0$ for all k, and
$$\sin \frac{\pi x}{L} \equiv 0, \quad \text{a contradiction.}$$
The problem here was that "there were not enough functions in the sequence $\varphi_k = \sin \frac{2\pi k x}{L}$." (Some important ones were missing).

3.8 Defn. The orthogonal function sequence $\{\varphi_n\}_{n=1}^{\infty}$ is complete (= "fullständig") ("nothing is missing") if, for every $\phi \in \mathcal{F}$ we have

$$\phi = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{\langle \phi, \varphi_n\rangle}{\|\varphi_n\|^2}\varphi_n,$$

(or $\phi = \lim_{N\to\infty} \sum_{n=1}^{N} \langle \phi, \varphi_n\rangle \varphi_n$ in the orthonormal case.)

3.9 Corollary T.f.c.a.e (= the following conditions are equivalent) for an orthonormal sequence $\{\varphi_n\}_{n=1}^{\infty}$.

i) $\{\varphi_n\}$ is complete

ii) $\phi = \lim_{N\to\infty} \sum_{n=1}^{N} \langle \phi, \varphi_n\rangle \varphi_n$ for all $\phi \in \mathcal{F}$

iii) $\lim_{N\to\infty} \int_a^b w(x) \left| \phi(x) - \sum_{n=1}^{N} \alpha_n \varphi_n(x)\right|^2 dx = 0$,

where $\alpha_n = \langle \phi, \varphi_n\rangle = \int_a^b w(x)\,\phi(x)\,\overline{\varphi_n(x)}\,dx$.

Proof: Directly from definition!

3.10 Problem (Difficult): How do we know that a sequence $\{\varphi_n\}$ is complete?

Answer A: Use Sturm–Liouville theory (later)

Answer B: All the sequences that we encounter in this course by separating variables are complete! (This is a special case of A.)

3.11 Bessel's inequality. Let $\phi \in \mathcal{F}$, and let $\{\varphi_n\}$ be an orthonormal sequence. Define the N-th approximation ϕ_N of ϕ by

$$\phi_N = \sum_{n=1}^{N} \langle \phi, \varphi_n\rangle \varphi_n.$$

Then

i) $\phi - \phi_N \perp \phi_N$ (error orthogonal to appr.)

ii) $\|\phi_N\|^2 = \sum_{n=1}^{N} |\langle \phi, \varphi_n\rangle|^2$ (Pythagoras)

iii) $\|\phi\|^2 = \|\phi_N\|^2 + \|\phi - \phi_N\|^2$ (— " —)

iv) $\sum_{n=1}^{N} |\langle \phi, \varphi_n\rangle|^2 \le \|\phi\|^2$ (Bessel's ineq.)

Proof. i) This means that $\langle \phi - \phi_N, \phi_N\rangle = 0$, or equivalently,

$$\circledast \quad \langle \phi, \phi_N\rangle = \langle \phi_N, \phi_N\rangle$$

R.h.s. (= right hand side) is

$$\|\phi_N\|^2 = \langle \phi_N, \phi_N\rangle = \left\langle \sum_{n=1}^{N} \langle \phi, \varphi_n\rangle \varphi_n, \sum_{m=1}^{N} \langle \phi, \varphi_m\rangle \varphi_m \right\rangle$$

(use linearity and conjugate linearity)

$$= \sum_{n=1}^{N} \sum_{m=1}^{N} \langle \phi, \varphi_n\rangle \overline{\langle \phi, \varphi_m\rangle} \langle \varphi_n, \varphi_m\rangle \qquad \text{"} \delta_n^m = \begin{cases} 1, & n=m \\ 0, & n\ne m \end{cases}$$

$$= \sum_{n=1}^{N} \langle \phi, \varphi_n\rangle \overline{\langle \phi, \varphi_n\rangle} =$$

$$= \sum_{n=1}^{N} |\langle \phi, \varphi_n\rangle|^2.$$ This proves (ii).

The l.h.s. of $\circledast$ is $\langle \phi, \phi_N\rangle = \langle \phi, \sum_{n=1}^{N} \langle \phi, \varphi_n\rangle \varphi_n\rangle$

$$= \sum_{n=1}^{N} \overline{\langle \phi, \varphi_n\rangle} \langle \phi, \varphi_n\rangle = \sum_{n=1}^{N} |\langle \phi, \varphi_n\rangle|^2.$$ This proves (i) and (ii), and gives

$$\langle f, f_N\rangle = \langle f_N, f_N\rangle = \sum_{n=1}^{N} |\langle f, \varphi_n\rangle|^2.$$

iii) $\|f - f_N\|^2 = \langle f - f_N, f - f_N\rangle$

$$= \langle f, f\rangle - \langle f_N, f\rangle \underbrace{- \langle f, f_N\rangle + \langle f_N, f_N\rangle}_{=0}$$

$\| \; \langle f, f_N\rangle = \langle f_N, f_N\rangle$

$$= \langle f, f\rangle - \langle f_N, f_N\rangle$$

$$= \|f\|^2 - \|f_N\|^2.$$

iv) Follows from ii) and iii). ∎

3.12 Parseval's Identity. T.f.c.a.e. for every orthonormal sequence $\{\varphi_n\}$:

i) $\{\varphi_n\}_{n=1}^{\infty}$ is complete

ii) $f = \lim_{N\to\infty} \sum_{n=1}^{N} \langle f, \varphi_n\rangle \varphi_n$ for all $f \in \mathcal{F}$

iii) $\|f\|^2 = \sum_{n=1}^{\infty} |\langle f, \varphi_n\rangle|^2$ (Parseval's identity)

Note: The series of positive numbers

$$\sum_{n=1}^{\infty} |\langle f, \varphi_n\rangle|^2$$

always converges: The partial sums are bounded from above by $\|f\|^2$!

Proof: (i) ⇔ (ii) is the definition.

(ii) ⇔ (iii) because of iii) in Thm 3.11: we have $\|f - f_N\| \to 0 \iff \|f_N\|^2 \to \|f\|^2$, and

$\|f_N\|^2 = \sum_{n=1}^{N} |\langle f, \varphi_n\rangle|^2$, so this is

equivalent to $\|f\|^2 = \lim_{N\to\infty} \sum_{n=1}^{N} |\langle f, \varphi_n\rangle|^2 = \sum_{n=1}^{\infty} |\langle f, \varphi_n\rangle|^2$.

3.13 Schwartz inequality

$$|\langle f, g\rangle| \le \|f\| \, \|g\|, \quad \text{i.e.,}$$

$$\left| \int_a^b w(x) f(x) \overline{g(x)} \, dx \right|^2 \le \int_a^b w(x) |f(x)|^2 dx \int_a^b w(x) |g(x)|^2 dx,$$

Proof: Copy the proof from multidimensional analysis, or from the matrix case.

3.14 Improved Schwartz inequality. If we have equality, and if $f \not\equiv 0$, then g is a multiple of f, i.e.,

$$g(x) = \alpha f(x) \quad \text{for some } \alpha \in \mathbb{C}.$$

Proof: As above.

VI.4 Examples

4.1 The Fourier series The sequence $\{e^{in\varphi}\}_{n=-\infty}^{\infty}$ is orthogonal and complete w.r.t. (=with respect to) the inner product

(*) $$\int_{-\pi}^{\pi} f(\varphi) \overline{g(\varphi)} \, d\varphi.$$

(Note: complex case).

Proof: $\int_{-\pi}^{\pi} e^{in\varphi} \overline{e^{im\varphi}} \, d\varphi$

$$= \int_{-\pi}^{\pi} e^{in\varphi} e^{-im\varphi} \, d\varphi = \int_{-\pi}^{\pi} e^{i(n-m)\varphi} d\varphi$$

(if $n \ne m$) $$= \Big|_{-\pi}^{\pi} \frac{e^{i(n-m)\varphi}}{i(n-m)} = \frac{1}{i(n-m)} \left[e^{i(n-m)\pi} - e^{-i(n-m)\pi} \right]$$

$$= \frac{1}{i(n-m)} e^{-i(n-m)\pi} \left[\underbrace{e^{2i(n-m)\pi}}_{2\pi\text{-periodic}} - 1 \right] = 0.$$

Also note that the norm of $e^{in\varphi}$ is

$$\|e^{in\varphi}\|^2 = \int_{-\pi}^{\pi} e^{in\varphi}\,\overline{e^{in\varphi}}\,d\varphi = \int_{-\pi}^{\pi} 1\cdot d\varphi = 2\pi,$$

so $\|e^{in\varphi}\| = \sqrt{2\pi}$.

(For this reason many divide (+) by $\sqrt{2\pi}$, and use the inner product

$$\frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} f(\varphi)\overline{g(\varphi)}\,d\varphi \quad \text{instead.}$$

Some divide by 2π.)

Completeness: Difficult. One way is to show that these functions are the eigenfunctions of a certain Sturm–Liouville problem (more about this later).

4.2 The sinus series.

$\{\sin(n\varphi)\}_{n=1}^{\infty}$ is orthogonal both with respect to the inner product

$$(*)\quad \langle f,g\rangle = \int_{0}^{\pi} f(\varphi)\overline{g(\varphi)}\,d\varphi, \quad \text{and}$$

$$(**)\quad \langle f,g\rangle = \int_{-\pi}^{\pi} f(\varphi)\overline{g(\varphi)}\,d\varphi.$$

It is complete with respect to $(*)$, but not w.r.t. $(**)$.

Proof. Orthogonality w.r.t. $(**)$

$$\int_{-\pi}^{\pi} \underbrace{\underbrace{\sin(n\varphi)}_{\text{odd}}\,\underbrace{\sin(m\varphi)}_{\text{odd}}}_{\text{even}}\,d\varphi = 0 \quad \text{if } n\neq m.$$

$$= 2\int_{0}^{\pi} \sin(n\varphi)\sin(m\varphi)\,d\varphi.$$

Thus: orthogonal w.r.t. $(*)$ $\Leftrightarrow$ orthogonal w.r.t. $(**)$.

We can compute the above integrals as follows:

$$\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \quad \Big|\; -1\\ \cos(\alpha-\beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta \quad \Big|\; +1\\ &\qquad\qquad\qquad\qquad\qquad\qquad\quad \Sigma\end{aligned}$$

$$\sin\alpha\sin\beta = \tfrac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right] \Rightarrow$$

$$\int_{0}^{\pi} \sin(n\varphi)\sin(m\varphi)\,d\varphi = \frac{1}{2}\int_{0}^{\pi}\left[\cos(n-m)\varphi - \cos(n+m)\varphi\right]d\varphi$$

$$= \frac{1}{2}\Big/_{0}^{\pi} \frac{\sin(n-m)\varphi}{n-m} - \frac{1}{2}\Big/_{0}^{\pi} \frac{\sin(n+m)\varphi}{n+m} = 0.$$

Completeness w.r.t. $(*)$: This is the set of functions which arise when we separate variables in the heat equation (see pp. 17–21). Therefore it is complete (more later).

Non-completeness w.r.t. $(**)$: Take $f(\varphi)\equiv 1$.

Then $\langle f, \sin n\varphi\rangle = \int_{-\pi}^{\pi} 1\cdot \sin n\varphi\,d\varphi = 0$,

so $\sum_{n=1}^{\infty} |\langle f, \sin n\varphi\rangle|^2 = 0$, but

$$\|f\|^2 = \int_{-\pi}^{\pi} 1\,d\varphi = 1$$

By thm 3.12, it is not complete.

4.3 The cosinus series.

$\{\cos(n\varphi)\}_{n=0}^{\infty}$ is orthogonal with respect to both (note $n=0$) inner products $(*)$ and $(**)$ on p. 87. It is complete w.r.t. $(*)$, but not w.r.t. $(**)$.

Proof: Similar to Ex. 4.2.

4.4 Mixed sines-cosines series: If we combine examples 4.2 and 4.3 and we both $\{\sin n\varphi\}_{n=1}^{\infty}$ and $\{\cos n\varphi\}_{n=0}^{\infty}$, then we get a complete orthogonal sequence w.r.t.

$$(*1) \quad \int_{-\pi}^{\pi} f(\varphi)\,\overline{g(\varphi)}\,d\varphi.$$

Proof: "Straight forward".

VI.5 Heat equation in a cylinder

We study the heat equation in a massive cylinder, which is infinitely long and have rotational symmetry. This is the same as Example 8.3, p. 53, but the cylinder is now massive and we have different boundary conditions. See also section IV.3

Because of symmetry, we take directly

$$u = u(\underset{\text{radial}}{\rho}, \underset{\text{time}}{t})$$

Equ: $\quad u_t = \nabla^2 u \quad , \quad 0 \le \rho \le R, \quad t > 0$

Bdry: $\quad u(R,t) = 0 \quad t > 0 \quad$ (fixed temperature)

Initial: $\quad u(\rho,0) = T_0 \neq 0 \quad , \quad 0 \le \rho < R$

Separate variables: $\quad u = R(\rho)\,T(t) \Rightarrow$

$$\frac{T'}{T} = \frac{R'' + \frac{1}{\rho}R'}{R} = -\lambda^2,$$

(Laplacian in cylindrical coordinates: See p. 24)

Time: $\quad T(t) = T(0)\,e^{-\lambda^2 t},$

Space: $\quad R'' + \frac{1}{\rho}R' + \lambda^2 R = 0, \quad$ i.e.,

$$R = a\,J_0(\lambda\rho) + b\,N(\lambda\rho).$$

$\rho = 0$ is the center of the cylinder, temperature is finite $\Rightarrow b = 0$.

Bdry: $\quad u(R,t) = 0 \Rightarrow J_0(\lambda R) = 0$, so

$\lambda R =$ a zero of J_0. Call these zeros $\alpha_1, \alpha_2, \dots, \alpha_k, \dots$ Then $\lambda_k = \frac{\alpha_k}{R}$.

By adding solutions we get a "general" solution

$$u(\rho,t) = \sum_{k=1}^{\infty} c_k\,J_0(\lambda_k\rho)\,e^{-\lambda_k^2 t}.$$

Initial cond: $\quad u(\rho,0) = T_0 \quad$ (= constant) $\Rightarrow$

$$\sum_{k=1}^{\infty} c_k\,J_0(\lambda_k\rho) = T_0 \quad \text{(for all } \rho\text{)}.$$

Problem. Find the coefficients c_k.

Solution. We know that $\{J_0(\lambda_k\rho)\}$ is orthogonal w.r.t. the inner prod.

$$\langle f, g\rangle = \int_0^R \rho\,f(\rho)\,g(\rho)\,d\rho. \quad \text{(See Sect. V.8)}$$

According to the theory in Section VI.3, the "correct" coefficients are

$$c_k = \frac{\langle J_0(\lambda_k\rho), T_0\rangle}{\|J_0(\lambda_k\rho)\|^2}, \quad \text{i.e.,}$$

$$c_k = \frac{\int_0^R \rho\,J_0(\lambda_k\rho)\,T_0\,d\rho}{\int_0^R \rho\,|J_0(\lambda_k\rho)|^2\,d\rho}.$$

We computed in Section V.8:

$$\int_0^R \rho\,[J_0(\lambda_k\rho)]^2\,d\rho = \frac{R^2}{2}[J_1(\alpha_k)]^2.$$

In this particular case we can also compute the numerator "exactly" (usually only numerically):