1.2.2011

V Bessel Functions

V.1 Repetition

In the last section we found two (or one) serial solution to Bessel's equation

(1) $x^2 y'' + xy' + (x^2 - n^2) y = 0$,

namely $J_n(x)$ and $J_{-n}(x)$, with

(2) $J_n(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,\Gamma(n+j+1)} \left(\frac{x}{2}\right)^{n+2j}$.

If $n = 0, 1, 2, \ldots$, then $J_{-n}(x) = (-1)^n J_n(x)$, so in this case we found only one (linearly independent) solution. The other solution is of the type $J_n(x)\, v(x)$, where $v(x)$ has a "complicated" singularity at the point $x = 0$ (it contains a logarithm).

V.2 Recursion formulas for Bessel Functions

2.1 Thm. The Bessel functions satisfy the following recursion formulas:

(3) $[x^n J_n(x)]' = x^n J_{n-1}(x)$

(4) $[x^{-n} J_n(x)]' = -x^{-n} J_{n+1}(x)$

(5) $J_n'(x) = -\frac{n}{x} J_n(x) + J_{n-1}(x)$

(6) $ = \frac{n}{x} J_n(x) - J_{n+1}(x)$

(7) $ = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]$.

(8) $J_{n-1}(x) - \frac{2n}{x} J_n(x) + J_{n+1}(x) = 0$.

Proof. $\boxed{(3):}$ Formula (2) gives

$$\left(\frac{x}{2}\right)^n J_n(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,\Gamma(j+n+1)} \left(\frac{x}{2}\right)^{2j+2n}.$$

Differentiate $\Rightarrow$

$$\frac{1}{2^n}[x^n J_n(x)]' = \sum_{j=0}^{\infty} \frac{(-1)^j\, 2(j+n)}{j!\,\Gamma(j+n+1)} \left(\frac{x}{2}\right)^{2j+2n-1} \cdot \frac{1}{2}$$

$$= \frac{1}{2^n} x^n \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,\Gamma(n+j)} \left(\frac{x}{2}\right)^{2j+n-1}$$

$$= \frac{1}{2^n} x^n J_{n-1}(x).$$

$\boxed{(4):}$ Similar proof. Multiply by $\left(\frac{x}{2}\right)^{-n}$ instead.

$\boxed{(5):}$ Carry out the differentiation in (3)

$\boxed{(6):}$ — " — (4)

$\boxed{(7):}$ (5) + (6) = (7)

$\boxed{(8):}$ (5) − (6) = (8). □

2.2 Corollary If we know J_n and J_{n-1}, then we can compute (recursively) J_{n+k} for all k from (8), $k = 0, \pm 1, \pm 2, \pm 3, \ldots$.

2.3 Lemma. $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$ and $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$. The other Bessel functions of order $n + \frac{1}{2}$, $n = 0, \pm 1, \pm 2, \ldots$ can be computed recursively from these.

Proof. Homework (compare the series for the Bessel function to the one for $\sin(x)$ or $\cos(x)$, or try to solve Bessel's differential equation with $n = \frac{1}{2}$ directly).

V. 3 Generating function ($n =$ integer)

The case $n =$ integer $= 0, 1, 2, \dots$ is very important. It appears when we solve different PDE's in cylindrical coordinates. When we add initial conditions (at time $= 0$) we need to expand the initial function and the general solution into a Fourier-Bessel series (cf. (= "jämför") sect. III. 1):

$$y(x) = \sum_{n=0}^{\infty} c_n J_n(x)$$

Alternatively, we can write this as (recall that $J_{-n}(x) = (-1)^n J_n(x)$) =

$$y(x) = \sum_{n=-\infty}^{\infty} \alpha_n J_n(x).$$

A very special and interesting case is when we take $\alpha_n = t^n$, where t is a given constant.

3.1 Defn. The generating function for the Bessel functions with integer indeces is

$$(1) \quad g(x,t) = \sum_{n=-\infty}^{\infty} J_n(x)\, t^n.$$

Note: Can also be written as

$$J_0(x) + (t - \tfrac{1}{t}) J_1(x) + (t^2 + \tfrac{1}{t^2}) J_2(x) + \cdots$$

by combining the index $+n$ with the index $-n$.

Note: As soon as we know $g(x,t)$ for all x and t, we can compute the functions $J_n(x)$: we expand $g(x,t)$ into a "Laurent series" of the type (1) with respect to t. Then the coefficients of this series are $J_n(x)$.

3.2 Problem: Give an explicit formula for $g(x,t)$ (as simple as possible) which does not contain any sums.

Solution:
$$\frac{\partial}{\partial t} g(x,t) = \sum n J_n t^{n-1}$$
$$= \frac{x}{2} \sum (J_{n-1} + J_{n+1}) t^{n-1} \quad \text{(change summation variable)}$$
$$= \frac{x}{2}\left[\sum J_n t^n + \frac{1}{t^2} \sum J_n t^n\right]$$
$$= \frac{x}{2}\left(1 + \frac{1}{t^2}\right) g(x,t) \quad \Rightarrow$$

(Think of x as a fixed number; denote $F(t) = g(x,t)$)

$$F'(t) = \frac{x}{2}\left(1 + \frac{1}{t^2}\right) F(t) \quad \text{(first order ODE)}$$

$$\Rightarrow \quad F(t) = \underset{\substack{\uparrow \\ \text{constant}}}{C}\, e^{\int \frac{x}{2}(1+\frac{1}{t^2})\,dt}$$

$$= C e^{\frac{x}{2}(t - \frac{1}{t})}$$

Here the constant C may depend on x, but not on t: $C = C(x)$. Substitute $t = 1$ in (1), to get

$$g(x,1) = C(x) = \sum_{n=-\infty}^{\infty} J_n(x)$$
$$= J_0(x) + 2\sum_{n=1}^{\infty} J_{2n}(x) =$$
(See Arfken)
$$= \text{-----------------------} = 1.$$

(we shall maybe see this later). Thus:

3.3 Thm. The generating function $g(x,t)$ of the integer order Bessel functions is given by

$$g(x,t) = e^{\frac{x}{2}(t - 1/t)}$$

From this function we get $J_n(x)$ as the n-th coefficient in the Laurent series with respect to t.

Conversion to Fourier series: Put

$$t = e^{i\varphi} = \cos\varphi + i\sin\varphi$$
$$\frac{1}{t} = e^{-i\varphi} = \cos\varphi - i\sin\varphi$$

$$\frac{x}{2}(t - 1/t) = i\,x\sin\varphi, \quad \text{and}$$

$$g(x, e^{i\varphi}) = e^{ix\sin\varphi} = \sum_{n=-\infty}^{\infty} J_n(x)\, e^{in\varphi}.$$

Thus, we get the following alternative to Thm 3.3

3.4 Thm. The integral order Bessel functions $J_n(x)$ are the (complex) Fourier coefficients with respect to t of the function $e^{ix\sin\varphi}$, i.e.,

$$(3) \quad \boxed{e^{ix\sin\varphi} = \sum_{n=-\infty}^{\infty} J_n(x)\, e^{in\varphi}}$$

There is a general rule how to compute Fourier coefficients (see the course next spring):

3.5 Thm: For integral n we have

$$(4) \quad J_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ix\sin\varphi}\, e^{-in\varphi}\, d\varphi.$$

From this we easily deduce:

3.6 Thm For integral (= heltal) n we have

$$(5) \quad J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\varphi - x\sin\varphi)\, d\varphi$$

$$(6) \quad \cos(x\sin\varphi) = J_0(x) + 2\sum_{n=1}^{\infty} J_{2n}(x)\cos(2n\varphi)$$

$$(7) \quad \sin(x\sin\varphi) = 2\sum_{n=1}^{\infty} J_{2n-1}(x)\sin((2n-1)\varphi).$$

Note: Taking $\varphi = 0$ in (6) we get the earlier needed formula

$$(*) \quad 1 = J_0(x) + 2\sum_{n=1}^{\infty} J_{2n}(x).$$

Is this a proof of $(*)$?

Proof of Thm 3.6.

$\boxed{(5):}$

$$J_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(x\sin\varphi - n\varphi)}\, d\varphi$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{\cos(x\sin\varphi - n\varphi)}_{\text{even}}\, d\varphi + \frac{i}{2\pi}\int_{-\pi}^{\pi} \underbrace{\sin(x\sin\varphi - n\varphi)}_{\text{odd}}\, d\varphi$$

$$= \frac{1}{\pi}\int_0^{\pi} \cos(n\varphi - x\sin\varphi)\, d\varphi.$$

$\boxed{(6) \text{ and } (7):}$ Split (3) into real and imaginary parts and use $J_{-n} = (-1)^n J_n$.

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V. 4. The positive zeros of the Bessel functions.

Note, Bessel functions are typically interesting only for values of $x \geq 0$ (because the physical meaning of x is a "distance to the origin").

How do Bessel functions behave as $x \to \infty$?

Picture says: like a power of x times a function which looks periodic (like a sinus or cosinus).

Is this true?

Yes, and this is a property which is shared by solutions of many different boundary value problems. (not just Bessel).

4.1 Sturm's Comparison Thm. Let $p(x) > 0$, and let $y_1 \not\equiv 0$ and $y_2 \not\equiv 0$ satisfy

$$\circledast \quad (p y_1')' + q_1 y_1 = 0 \qquad (q_1 = q_1(x))$$

$$\circledast\circledast \quad (p y_2')' + q_2 y_2 = 0. \qquad (q_2 = q_2(x))$$

If $q_2(x) < q_1(x)$ for all $x \in [a,b]$, $y_2(a) = y_2(b) = 0$, and $y_2(x) \neq 0$ for $x \in (a,b)$, then y_1 has at least one zero in the interval (a,b).

Interpretation: The zeros of y_1 lie "at least as close" as the zeros of y_2.
Or: Between any two zeros of y_2 there is a zero of y_1.

Proof. Assume $y_2 > 0$ (case $y_2 < 0$ analogous)

Counterassumption: $y_1(x) \neq 0$ for all $x \in (a,b)$.
If $y_1(x) < 0$, then we replace $y_1(x)$ by $-y_1(x)$, so we may assume that

$$y_1(x) > 0 \quad \text{for } x \in (a,b).$$

"Wronskian"

Define $H(x) = p(x)\left[y_1 y_2' - y_1' y_2\right] \Rightarrow$

$$H'(x) = p'\left[y_1 y_2' - y_1' y_2\right] + p\left[y_1 y_2'' - y_1'' y_2\right]$$

$$= y_1 (p y_2')' - y_2 (p y_1')' \quad (\text{by the DE})$$

$$= y_1 y_2 (q_1 - q_2) > 0.$$

Thus, $H(x)$ is an (strictly) increasing function. But

$$H(a) = p(a)\, y_1(a)\, y_2'(a) \geq 0 \quad \text{and}$$

$$H(b) = p(b)\, y_1(b)\, y_2'(b) \leq 0 \quad \Rightarrow \text{contradiction.}$$

The counterassumption is wrong!

4.2 Application Bessel's equation

$$(1) \quad x^2 y'' + x y' + (x^2 - n^2) y = 0.$$

We "simplify" it by changing the unknown:

$y = x^p u$	$x^2 - n^2$
$y' = p x^{p-1} u + x^p u'$	x
$y'' = p(p-1) x^{p-2} u + 2p\, x^{p-1} u' + x^p u''$	x^2
	Σ

$$0 = x^{p+2} u'' + (2p+1) x^{p+1} u' + (x^2 - n^2 + p^2) x^p u = 0.$$

Choose $\boxed{p = -\tfrac{1}{2}}$ to kill off the coefficient of u' $\Rightarrow$

$$(2) \quad y(x) = \frac{1}{\sqrt{x}} u(x), \quad \text{and}$$

$$(3) \quad u'' + \left(1 + \frac{\frac{1}{4} - n^2}{x^2}\right) u = 0$$

(reformulated Bessel's equation).

(Note: Case $|n| = \frac{1}{2}$ is special ?!)

We use this equation $\circledast$ in Thm. 4.1 ($p = 1$). As a second equation we take

$$(4) \quad u_2'' + \alpha^2 u_2 = 0,$$

where α = constant > 0. The solution of (4) can be written as $u_2(x) = \sin \alpha(x - c)$, where c is a constant (= phase shift).
The distance between two zeros is π/α.

Case 1. $0 \le |n| < 1/2$. Then $\frac{\frac14 - n^2}{x} > 0$,

We choose $\alpha = 1$, and find that

i) $\boxed{\text{Every interval of length } \pi \text{ contains at least one zero of } J_n(x), \ (0 \le |n| < 1/2)}$

But, as $x \to \infty$, we have $\frac{\frac14 - n^2}{x} \to 0$.

We can chose $\alpha^2 = (1+\varepsilon)^2$, where ε is arbitrarily small, and for large enough x, we have $1 + \frac{\frac14 - n^2}{x^2} < (1+\varepsilon)^2$. We interchange the two equations (*) and (**), and conclude that the distance of two zeros of $J_n(x)$ can be at most $\frac{\pi}{1+\varepsilon}$. (We adjust the constant c in (**) so that one zero of v_2 coincides with one zero of $J_n(x)$).

$\boxed{\text{Conclusion: The distance between the zeros is } < \pi, \text{ and it tends to } \pi \text{ as } x \to \infty \ (0 \le |n| < 1/2)}$

Case 2 $|n| = \frac12$. The distance between the zeros of $J_{-1/2}(x)$ and $J_{1/2}(x)$ are exactly π. Both of these functions are of the type

$$J_{\pm 1/2}(x) = \frac{c_1}{\sqrt{x}} \sin(x - c_2),$$

where c_1 and c_2 are suitable constants. (cf. p. 50).

2.2.2004

Case 3. ($|n| > \frac12$) A similar argument shows that the distance of consecutive (= "på varandra följande") zeros is $> \pi$, and it tends to π as $x \to \infty$.

3.9 Application Let $0 \le |n_1| < |n_2|$. Then, between any two zeros of J_{n_2} there is at least one zero of J_{n_1}.

Proof home work.

3.10 Application The zeros of J_n ($n \ge 0$) and J_{n+1} interlace each other ("kommer drvis"): Between any two zeros of J_{n+1} there is exactly one zero of J_n.

Proof. By Appl 3.9, J_n has at least one zero between any two zeros of J_{n+1}. Conversely, let $J_n(a) = J_n(b) = 0$. By the mean value theorem, there exists some point $c \in [a,b]$ such that

$$x^{-n} J_n(b) = x^{-n} J_n(a) + \frac{d}{dx}\left(x^{-n} J_n(x)\right)\Big|_{x=c} \cdot (b-a)$$

$$\Longleftrightarrow \quad \frac{d}{dx}\left(x^{-n} J_n(x)\right)\Big|_{x=c} = 0.$$

By formula (4) on p. 49,

$$\frac{d}{dx}\left(x^{-n} J_n(x)\right)\Big|_{x=c} = -c^{-n} J_{n+1}(c) = 0,$$

and thus $J_{n+1}(c) = 0$. (Thus, between any two zeros of J_n there exist exactly one zero of J_{n+1}.

[12] FRIEDRICH WILHELM BESSEL (1784—1846), German astronomer and mathematician, started out as an apprentice of a trade company, studying astronomy on his own in his spare time, later became an assistant at a small private observatory, and finally director of the new Königsberg observatory. His paper on the Bessel functions (dated 1824) appeared in 1826. Formulas are contained in Refs. [1], [6], [10], and the standard treatise [A9].

V.5 Neumann Functions

(Bessel functions of the second type)

<u>Note:</u> Mathematica: $J_n(x) = \text{BesselJ}[n,x]$

$N_n(x) = \text{BesselY}[n,x]$

<u>Recall:</u> For integral values of n we have found only one "nice" solution $J_n(x)$ of Bessel's equation

$$(1_\nu) \quad x^2y'' + xy' + (x^2-\nu^2)y = 0.$$

For nonintegral values of ν we have two solutions J_ν and $J_{-\nu}$. The other solution N_n that we tried to find looked messy (it had a "bad" expansion).

<u>Another approach to constructing N_n.</u> Try some combination of J_ν and $J_{-\nu}$, and let $\nu \to$ an integer. For example

$$N_\nu = \frac{\alpha(\nu) J_\nu - J_{-\nu}}{\beta(\nu)}$$

<u>Observations:</u>

– If we want to get something which is not a multiple we must take $\beta(\nu)=0$ when $\nu =$ integer. One such function would be $\beta(\nu) = \sin(\pi\nu)$.

[13]CARL NEUMANN (1832—1925), German mathematician and physicist, became a professor at Leipzig in 1868. His work on potential theory sparked the development in the field of integral equations by VITO VOLTERRA (1860—1940) of Rome, ERIC IVAR FREDHOLM (1866—1927) of Stockholm, whose famous 1901–1903 papers were a sensation to the mathematical world of his time, and DAVID HILBERT (1862—1943) of Göttingen (see the footnote in Sec. 7.15).

– In order for the limit to exist (with $\beta(n)=0$) we need $\alpha(\nu)J_\nu - J_{-\nu} \to 0$ as $\nu \to$ integer. We know that $J_{-n} = (-1)^n J_n$, so we need $\beta(n)$ to satisfy $\beta(n) = (-1)^n$. One suitable function is $\beta(n) = \cos(n\pi)$.

<u>5.1 Defn.</u> For $\nu \neq$ integer we define <u>Neumann's function</u> of order ν by

$$N_\nu(x) = \frac{\cos(\nu\pi) J_\nu(x) - J_{-\nu}(x)}{\sin(\nu\pi)}.$$

For $n =$ integer we define

$$N_n(x) = \lim_{\nu\to n} N_\nu(x).$$

<u>Note:</u> $\nu = n + \frac{1}{2}$, $n =$ integer, gives

$$N_\nu = \frac{-J_{-\nu}(x)}{(-1)^n} = (-1)^{n+1} J_{-\nu}.$$

<u>5.2 Lemma</u> For $n =$ integer we have

$$N_n(x) = \frac{1}{\pi}\frac{\partial}{\partial n}\left[J_n(x) - (-1)^n J_{-n}(x)\right].$$

<u>Proof.</u> Use L'Hospital's rule (exam).

<u>5.3 Lemma.</u> N_ν satisfies Bessel's equation for <u>all</u> ν (integral or not).

<u>Idea of proof:</u> True for $\nu \neq$ integer. Show that $J_\nu(x)$ is continuously differentiable with respect to <u>both</u> x <u>and</u> n. Plug the formula in Lemma 5.2 into Bessel's equation.

5.4 Lemma The general solution of (1_ν) is

$$y(x) = c_1 J_\nu(x) + c_2 N_\nu(x).$$

True, because J_ν and N_ν linearly independent solutions of (1_ν).

5.5 Lemma $|N_\nu(x)| \to \infty$ as $x \downarrow 0$.

Proof. See p. 48.

5.6 Lemma. If the solution y in Lemma 5.4 is bounded as $x \downarrow 0$, then $y(x) = c J_\nu(x)$, c = constant.

Proof: Lemmas 5.4 and 5.5.

5.7. Lemma $N_{-n}(x) = (-1)^n N_n(x)$. ($n$ integer)

Proof "A simple computation". ($= \frac{1}{2}$ page).

5.8 Lemma. N_ν satisfies the same recurrence formulas as J_ν (see p. 49).

Proof: Case $\nu \neq$ integer: "Straightforward" but long computation (plug the definition of N_ν into the formulas).

Case ν = integer: More difficult. Can be derived from the case $\nu \neq$ integer by a limit process, but it is a rather long computation.

5.9 Lemma: $N_n(x)$ has the series expansion (n = integer):

$$N_n(x) = \frac{2}{\pi} J_n(x)\left(\ln\frac{x}{2} + \gamma\right) + \frac{1}{\pi}\sum_{j=0}^{\infty} \frac{(-1)^{j-1}(h_j + h_{j+n})}{j!\,(j+n)!}\left(\frac{x}{2}\right)^{2j+n} - \frac{1}{\pi}\sum_{j=0}^{\infty}\frac{(n-j-1)!}{j!}\left(\frac{x}{2}\right)^{2j-n}$$

γ = Euler's constant

where $h_0 = 0$, $h_j = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{j}$.

Reference: Kreutzig, p. 239.

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V. 6 Hankel Functions
(Bessel functions of the third kind)

6.1 Defn.

$$H_\nu^{(1)}(x) = J_\nu(x) + i N_\nu(x)$$

$$H_\nu^{(2)}(x) = J_\nu(x) - i N_\nu(x).$$

Note: Not complex conjugates of each other if $x \neq$ real.

6.2 Lemma These satisfy the same recurrence relations (see p. 49) as J_n and N_n.

Proof easy ("obvious").

§ 7. The Generalized Bessel's Equation

Idea: How to recognize a "disguised" (= "förklädd") Bessel's equation?

Let w be a solution to Bessel's eq.

(1ν) $z^2 w'' + z w' + (z^2-\nu^2)w = 0.$ (see pages 27 and 31)

What happens if we replace $y(x)$ by

$$y(x) = x^{-\alpha} w(\gamma x^{\beta}),$$

where α, β, γ are constants? What is the differential equation satisfied by y?

$$z = \gamma x^{\beta}, \text{ i.e., } x = (z/\gamma)^{1/\beta},$$

and

$$w(z) = \left(\frac{z}{\gamma}\right)^{\alpha/\beta} y\left(\left(\frac{z}{\gamma}\right)^{1/\beta}\right)$$

$$w'(z) = \frac{\alpha}{\beta}\left(\frac{z}{\gamma}\right)^{\frac{\alpha}{\beta}-1}\frac{1}{\gamma}\, y\left(\left(\frac{z}{\gamma}\right)^{\frac{1}{\beta}}\right) + \left(\frac{z}{\gamma}\right)^{\frac{\alpha}{\beta}} y'\left(\left(\frac{z}{\gamma}\right)^{\frac{1}{\beta}}\right)\frac{1}{\beta}\left(\frac{z}{\gamma}\right)^{\frac{1}{\beta}-1}\frac{1}{\gamma}$$

$$w''(z) = \text{-------}$$

Multiply by the coefficients in (1ν), and add, and simplify (this is a place where the use of Mathematica is nice!)

(2) $$x^2y'' + (2\alpha+1)xy' + [\beta^2\gamma^2x^{2\beta} - (\beta^2\nu^2-\alpha^2)]y = 0.$$

This is of the type

$$x^2y'' + pxy' + [qx^{2\beta} - r]y,$$ where

- p determines α $\left(\alpha = \frac{p-1}{2}\right)$
- The power $x^{2\beta}$ gives us β
- q gives us γ^2 $(\gamma^2 = q/\beta^2)$
- r gives us ν^2 $\left(\nu^2 = \frac{1}{\beta^2}(\alpha^2+r)\right)$.

7.1 Defn. Equation (2) is called the generalized Bessel's equation with parameters $\nu, \alpha, \beta, \gamma$. These can be real or even complex.

7.2 Thm. The general solution to (2) is given by

$$y(x) = c_1\, x^{-\alpha} J_\nu(\gamma x^{\beta}) + c_2\, x^{-\alpha} N_\nu(\gamma x^{\beta})$$

(we can replace N_ν by $J_{-\nu}$ if $\nu \neq$ integer).

Proof. See the computation above.

7.3 Ex. The first "Bessel's equation" which we ran into on p. 27 was

(3) $$\rho^2 P'' + \rho P' + [(\pm k^2 + l^2)\rho^2 - m^2]P = 0.$$

(Compare this to (2): $(x \to \rho,\ y \to P)$. We take $\alpha = 0$, $\beta = 1$, $\gamma^2 = \pm k^2 + l^2$, $\nu = m$, and get

$$P(\rho) = c_1 J_m(\gamma x) + c_2 N_m(\gamma x).$$

Note: "Streched" versions of J_m and N_m. If $\pm k^2 - l^2 < 0$, then we take γ = pure imaginary.

7.4 Ex. $\alpha = 0$, $\beta = 1$, $\gamma = i$ gives

(4) $x^2 y'' + xy - (x^2 + \nu^2) y = 0$.

This is the modified Bessel's equ.
The solution is

$$y(x) = c_1 J_\nu (ix) + c_2 N_\nu (ix), \quad \text{or}$$

$$y(x) = c_1 J_\nu (ix) + c_2 J_{-\nu} (ix) \quad (\nu \neq \text{integer})$$

7.5 Defn. $I_\nu (x) = i^{-\nu} J_\nu (ix)$

$$K_\nu (x) = \frac{\pi}{2} \frac{I_{-\nu}(x) - I_\nu(x)}{\sin(\nu\pi)}$$

$$= \frac{\pi}{2} i^{\nu+1} \left[J_\nu (ix) + i N_\nu (ix) \right]$$

are the modified Bessel functions (also called hyperbolic Bessel functions).

7.6 Lemma. The function $I_\nu(x)$ has the

$$I_\nu (x) = \sum_{j=0}^{\infty} \frac{1}{j!\, \Gamma(j+\nu+1)} \left(\frac{x}{2}\right)^{2j+\nu}$$

(the factor $(-1)^j$ in the numerator is gone, due to the extra imaginary i).

The function $K_\nu(x)$ is more complicated (contains a logarithm when ν = integer).

7.7 Additional Bessel functions. See Arfken and Kreutzig!

V.8 Physical Applications

(p. 23) $\nabla^2 \psi \pm k^2 \psi = 0$

8.1 Ex. Helmholtz eq. in a cylinder with parameter $+k^2 > 0$ gives a solution in the radial direction of the type

$$P(\rho) = C_{m\gamma} J_m (\gamma\rho).$$

Here m and γ are separation constants. We assume that $P(0)$ is finite, thereby getting rid of the Neumann function $N_m(\gamma\rho)$.

8.2 Wave equation in a cylinder leads to a Helmholz equation with negative parameter $-k^2 < 0$. This turns the equation into a modified Bessel's eq, and

$$P(\rho) = C_{m\gamma} I_m (\gamma\rho),$$

where I_m is the modified (hyperbolic) Bessel function.

8.3 Helmholtz equation in spherical coordinates and a parameter $k^2 > 0$ gives us spherical Bessel's functions:

$$j_m(x) = \sqrt{\frac{\pi}{2x}}\, J_{m+\frac{1}{2}}(x)$$

$$n_m(x) = \sqrt{\frac{\pi}{2x}}\, N_{m+\frac{1}{2}}(x) = (-1)^{m+1} \sqrt{\frac{\pi}{2x}}\, J_{-m-\frac{1}{2}}(x)$$

8.4 Heat equation in spherical coordinates gives us modified sperical Bessel's functions

$$i_m(x) = j_m(ix)$$

$$k_m(x) = n_m(ix)$$

(cf. p. 22) (= ikölis)

8.5 Ex. Heat equation in a hollow cylinder (= a water pipe). Inside temperature $= 0°$ (or we could take $= 100°$). Outside isolated. The cylinder is infinitely long (to simplify things), and the heat is constant along the length of the cylinder. There is also rotational symmetry. Thus, in cylindrical coordinates,

$$\frac{\partial}{\partial z} u(\rho, \varphi, z, t) = 0$$

$$\frac{\partial}{\partial \varphi} u(\rho, \varphi, z, t) = 0,$$

so u depends <u>only</u> on ρ (= radial distance) and t (= time).

<u>Equation</u>: $u_t(\rho, t) = \nabla^2 u(\rho, t)$,

$$r_1 < \rho < r_2, \quad t > 0$$

<u>Boundary cond.</u>

$$u(r_1, t) = 0, \qquad \frac{\partial}{\partial \rho} u(r_2, t) = 0$$

inside temperature fixed outside isolated

<u>Initial state</u>: $u(\rho, 0) = u_0(\rho)$, $r_1 \le \rho \le r_2$

<u>Solution</u>: Separate variables: $u = R(\rho)T(t)$, continue as in Chapter III. Leads to

$$(1) \quad \frac{T'}{T} = \frac{R'' + \frac{1}{\rho}R'}{R} = -\lambda^2,$$

where $\lambda =$ constant. The time solution is

$$T(t) = T(0)\, e^{-\lambda^2 t}.$$

The radial equation is

$$R'' + \frac{1}{\rho}R' + \lambda^2 R = 0.$$

This is a generalized Bessel's eq. ($\alpha = 0$, $\beta = 1$, $\gamma = \lambda$, $\nu = 0$), with solution

$$R = a\, J_0(\lambda\rho) + b\, N_0(\lambda\rho).$$

<u>Boundary cond.</u> $R(r_1) = 0$, $R'(r_2) = 0$, i.e.

$$\begin{cases} J_0(\lambda r_1)\, a + N_0(\lambda r_1)\, b = 0 \\ \lambda J'(\lambda r_1)\, a + \lambda N_0'(\lambda r_2)\, b = 0 \end{cases}$$

The solution $a = b = 0$ gives us a trivial solution. No good. To have <u>nonzero</u> solutions we must have

$$\begin{vmatrix} J_0(\lambda r_1) & N_0(\lambda r_1) \\ J_0'(\lambda r_2) & N_0'(\lambda r_2) \end{vmatrix} = 0.$$

This equation will, in general, have ∞ many solutions. Call these solutions α_k, $k = 1, 2, 3, 4$ ($0 < \alpha_1 < \alpha_2 < \alpha_3 \cdots$). Then we get solutions of the type

$$R_k(\rho) = a_k J_0(\alpha_k \rho) + b_k N_0(\alpha_k \rho), \text{ and}$$

$$u(\rho, t) = \sum_{k=1}^{N} \left[a_k J_0(\alpha_k \rho) + b_k N_0(\alpha_k \rho)\right] e^{-\alpha_k^2 t}.$$

$$R_k(r_1) = 0 \Rightarrow a_k J_0(\alpha_k r_1) + b_k N_0(\alpha_k r_1) = 0 \Rightarrow b_k = -\frac{J_0(\lambda r_1)}{N_0(\lambda r_1)} a_k \quad (\text{if } \neq 0)$$

<u>Initial value</u>: $u(\rho, 0) = u_0(\rho)$ given.

To satisfy this we must let $N \to \infty$ and require that

Here we can use formula for b_k given above

$$u_0 = \sum_{k=1}^{\infty} a_k J_0(\alpha_k \rho) + b_k N_0(\alpha_k \rho),$$

$$r_1 < \rho < r_2.$$

Is this possible? (Yes! see later theory).