IV Serial solutions of ODEs

IV.1 Singular points of ODEs

Many equations of physical interest are of second order and linear:

$$(1) \quad y'' + Py' + Qy = 0,$$

where $y = y(x)$, $P = P(x)$, $Q = Q(x)$ are analytic functions, possibly with some isolated points where they are singular (have poles). This particular equation is, in addition, homogeneous (the right hand side is $=0$).

1.1 Classification of points. A point $x_0 \in \mathbb{R}$ is an ordinary or non-singular point of (1) (= "vanlig, icke singulär") iff P and Q are analytic at x_0. Otherwise x_0 is a singular or special point.

1.2 Note. If the equation is written as

$$(2) \quad A(x)y'' + B(x)y' + C(x)y = 0,$$

then we first divide by $A(x)$:

$$P(x) = \frac{B(x)}{A(x)}, \quad Q(x) = \frac{C(x)}{A(x)}.$$

Thus, the zeros (= "nollställen") of A are singular points.

1.3 Note Very often $x_0 = 0$ is a singular point. For example when x = distance to the origin (Bessel).

1.4. Note "Special points" create "special functions".

1.5 Classification of the point at ∞. To study the point $x_0 = \infty$ we make a change of variables: $x = 1/z$.

$$x = \frac{1}{z}, \quad y(x) = Y(z) = Y(1/x)$$

$$\frac{dy}{dx} = \frac{d}{dx} Y\left(\frac{1}{x}\right) = Y'\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(Y'\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)\right) = Y''\left(\frac{1}{x}\right)\frac{1}{x^4} + \frac{2}{x^3}Y'\left(\frac{1}{x}\right)$$

Multiply by P and Q, add, put $\frac{1}{x} = z$:

$$y''(x) + P(x)y'(x) + Q(x)y(x) = 0 \iff$$

$$z^4 Y''(z) + 2z^3 Y'(z) + P\left(\frac{1}{z}\right)\left[-z^2 Y'(z)\right] + Q\left(\frac{1}{z}\right)Y(z) = 0$$

$$\iff (3) \quad Y''(z) + \underbrace{\frac{2z - P\left(\frac{1}{z}\right)}{z^2}}_{\text{"new function } P\text{"}} Y'(z) + \underbrace{\frac{Q\left(\frac{1}{z}\right)}{z^4}}_{\text{"new function } Q\text{"}} Y(z) = 0$$

We call $x = \infty$ a ordinary or singular point of (1) if $z = 0$ is a ordinary or singular point of (3). Thus:

1.6 Lemma. $x = \infty$ is an ordinary point of (1) iff (= if and only if) both

$$\tilde{P}(z) = \frac{2z - P(1/z)}{z^2} \quad \text{and}$$

$$\tilde{Q}(z) = \frac{Q(1/z)}{z^4}$$

are analytic at zero.

1.7 Ex. Bessel's equation

(4) $$x^2y'' + xy' + (x^2-n^2)y = 0$$

$$P(x) = \frac{x}{x^2} = \frac{1}{x}; \quad Q(x) = \frac{x^2-n^2}{x^2} = 1 - \frac{n^2}{x^2}.$$

Thus, $x_0 = 0$ is the only finite (= "ändlig", dvs $x_0 \neq \infty$) singular point.

$$\tilde{P}(z) = \frac{2z - \frac{1}{1/z}}{z^2} = \frac{1}{z}$$

$$\tilde{Q}(z) = \frac{1-n^2z^2}{z^4}$$

$\Rightarrow$ $x_0 = \infty$ is also a special point.

1.8 Note $x_0 = \infty$ is "almost always" a special point.

1.9. Further classification A singular point $x_0 \in \mathbb{R}$ of (1) is regular or weak if both $(x-x_0)P(x)$ and $(x-x_0)^2Q(x)$ are analytic at x_0. Otherwise x_0 is an irregular or essential singular (special) point.

1.10 Note. Essential singular points are "bad", and the solution of (1) behaves very badly at such points. In the sequel we treat only regular singular points.

1.11 Lemma Infinity is a regular singular point of (1) iff

$$\frac{2z - P(1/z)}{z} \quad \text{and} \quad \frac{Q(1/z)}{z^2}$$

are analytic at zero.

Proof: Regular iff $(z-0)\tilde{P}(z)$ and $(z-0)^2\tilde{Q}(z)$ are analytic at zero.

1.12 Ex. $x_0 = 0$ is a regular special point of Bessel's equation (4), but ∞ is an essential singular point.

IV. 2 Serial solutions of ODEs

2.1 Idea: Pick an "arbitrary" point $x_0 \in \mathbb{R}$, and look for a solution of

(1) $$y'' + Py' + Qy = 0$$

of the form

(2) $$y(x) = \sum_{j=0}^{\infty} a_j (x-x_0)^j,$$

or more generally, of the form

(3) $$y(x) = (x-x_0)^k \sum_{j=0}^{\infty} a_j (x-x_0)^j = \sum_{j=0}^{\infty} a_j (x-x_0)^{k+j}.$$

Based on:

2.2 Thm. The function $(x-x_0)^{-k}y(x)$ is analytic at the point x_0 iff this function can be written as an absolutely converging "power series" of the form (3).

Proof: Course on analytic functions

2.3 Ordinary points: If x_0 is an ordinary point, then we know (course on diff. eqs) that y is analytic at x_0, and we may take $k=0$ in (3).

2.4 Special points. Need, in general, $k \neq 0$. Usually the method does not work in essential special points. (Because y has an essential singularity, not a pole).

2.5 Examples of analytic solutions: See Arfken, pp. 484-486.

To treat this type of serial solutions we need two theorems from analytic function theory:

2.6 Thm. It is permitted to differentiate an (absolutely) converging power series term by term. That is, if y is given by (3), then

$$y'(x) = \sum_{j=0}^{\infty} (k+j)\, a_j\, (x-x_0)^{k+j-1}.$$

This expansion is valid for $|x-x_0| < R$, where R is the radius of convergence of the series in (3).

2.7 Thm. Two (analytic) functions

$$y(x) = \sum_{j=0}^{\infty} a_j\, (x-x_0)^{k+j}$$

and

$$z(x) = \sum_{j=0}^{\infty} b_j\, (x-x_0)^{k+j}$$

are identical if and only if

$$a_j = b_j, \quad j = 0, 1, 2, 3 \ldots$$

Proof: Course on analytic functions.

2.8 Example: Bessel's equation

$$(4) \quad x^2 y'' + x y' + (x^2-n^2)\, y = 0.$$

$x_0 = 0$ is a regular special point. Use this x_0 in the expansion in (3): (over)

We normalize n so that $n \geq 0$ (if n real), $\mathrm{Re}\, n \geq 0$ (if n complex). Possible since $n^2 = (-n)^2$.

By Theorems 2.6, if the series in (3) converges absolutely, then

$$\begin{array}{ll|l}
y = & \sum_{j=0}^{\infty} a_j\, x^{(k+j)} & x^2-n^2 \\
y' = & \sum_{j=0}^{\infty} (k+j)\, a_j\, x^{(k+j-1)} & x \\
y'' = & \sum_{j=0}^{\infty} (k+j)(k+j-1)\, a_j\, x^{k+j-2} & x^2 \\
\hline
 & & \Sigma
\end{array}$$

$$\begin{aligned}
0 &= x^2 y'' + x y' + (x^2-n^2)\, y \\
&= \sum_{j=0}^{\infty} (k+j)(k+j-1)\, a_j\, x^{k+j} \\
&\quad + \sum_{j=0}^{\infty} (k+j)\, a_j\, x^{k+j} \\
&\quad + \underbrace{\sum_{j=0}^{\infty} a_j\, x^{k+j+2}}_{\text{change summation}} - \sum_{j=0}^{\infty} n^2 a_j\, x^{k+j} = 0
\end{aligned}$$

index $k+j+2 = k+l$, i.e., $j = l-2$

$$\sum_{j=0}^{\infty} a_j\, x^{k+j+2} = \sum_{l=2}^{\infty} a_{l-2}\, x^{k+l}. \qquad (\text{rename } l \to j)$$

$$= \sum_{j=2}^{\infty} a_{j-2}\, x^{k+j}.$$

Thus,

$$\sum_{j=0}^{\infty} \Big[\underbrace{(k+j)(k+j-1) + (k+j)}_{(k+j)^2} - n^2\Big]\, a_j\, x^{k+j} + \sum_{j=2}^{\infty} a_{j-2}\, x^{k+j} = 0 = \sum_{j=0}^{\infty} \underbrace{0}_{= b_j} \cdot x^{k+j}$$

By Thm 2.7, all the coefficients on the left hand side must be zero!

i) $j=0$: $(k^2-n^2)\,a_0=0$

ii) $j=1$: $((k+1)^2-n^2)\,a_1=0$

iii) $j\geq 2$: $((k+j)^2-n^2)\,a_j + a_{j-2}=0$

2.9 Observation: By changing the value of k, if necessary, we may assume that $\boxed{a_0\neq 0}$. Assume this in the sequel. Then

i) $k^2-n^2=0$, i.e., $\boxed{k=\pm n}$ $\boxed{\text{The index equation}}$

ii) $a_1=0$

iii) $\underbrace{((k+j)^2-n^2)}_{(k+j+n)(k+j-n)}\,a_j=-a_{j-2}$ $\boxed{\text{Recursion equation}}$

We can solve iii) recursively: $\boxed{\text{Case 1: } k=n}$ $\Rightarrow (k+j)^2-n^2=j(j+2n)$

$j=2$: $a_2=\dfrac{-a_0}{2(2n+2)}=\dfrac{-a_0}{4(n+1)}$

$j=3$: $a_3=\dfrac{-a_1}{3(3+2n)}=0$,

$(n\geq 0)$ $0\neq 3(3+2n)$

$j=4$: $a_4=\dfrac{-a_2}{4(4+2n)}=\dfrac{a_0}{2^5(n+1)(n+2)}$,

etc.

General formula:

$a_{2l+1}=0$, if $\boxed{j=2l+1=\text{odd}}$

$a_{2l}=(-1)^l\dfrac{a_0\,n!}{2^{2l}\,l!\,(n+l)!}$, $\boxed{\text{if } j=2l=\text{even}}$

(Can be proved by induction).

Solution: (replace $l\to j$)

$$y(x)=a_0\,n!\sum_{j=0}^{\infty}\frac{(-1)^j\,x^{n+2j}}{2^{2j}\,j!\,(n+j)!}$$

$$=a_0\,2^n\,n!\sum_{j=0}^{\infty}\frac{(-1)^j}{j!\,(n+j)!}\left(\frac{x}{2}\right)^{n+2j}.$$

2.10 Defn. The function

$$J_n(x)=\sum_{j=0}^{\infty}\frac{(-1)^j}{j!\,(n+j)!}\left(\frac{x}{2}\right)^{n+2j}$$

is called Bessel's function of order n. (after its inventor).

Note: n is usually an integer ≥ 0, but it can also be a non-integer. It can even be complex: replace factorials by Γ-functions! (as long as $n\neq -1,-2,-3,\ldots$)

$y(x)=a_0\,2^n\,\Gamma(n+1)\,J_n(x)$, where

(5) $$J_n(x)=\sum_{j=0}^{\infty}\frac{(-1)^j}{j!\,\Gamma(n+j+1)}\left(\frac{x}{2}\right)^{n+2j}$$

(usually n is real).

Case 2 $\boxed{k=-n}$. The only difference is that the earlier expression

$k=n \Rightarrow (k+j-n)(k+j+n)=j(j+2n)$

is replaced by

$k=-n \Rightarrow (k+j-n)(k+j+n)=(j-2n)j$.

This causes no problems if $2n\neq$ integer. However, if $2n$ is an integer, then at some point we divide by zero (i.e. when $j=2n$). The recursion equation then tells us that

$0\cdot 2n\cdot a_{2n}=a_{2n-2}$, so

$a_{2n-2}=0$, and solving the recursion equation backwards we get

$$a_{2n-4}=0,\quad a_{2n-6}=0,\quad \dots .$$

There are two possibilities:

$\boxed{A:\ 2n = \text{odd integer, i.e., } n = \text{integer} + \tfrac{1}{2}}$

Then this forces the terms with odd indices to be zero. This does not matter: we still get a nonzero solution by taking all the odd terms to be zero, $a_0 \neq 0$, and the even terms to be nonzero.

In addition we seem to get one more solution by taking all even terms to be zero, odd terms to be zero for $j < 2n$, $a_{2n} \neq 0$, and odd terms with index $j > 2n$ to be nonzero. However, it turns out that this is the same solution that we get by taking $k=+n$ instead of $k=-n$, and taking the even terms in that series to be nonzero. (we normalized k so that the first coefficient is always $\neq 0$)

$\boxed{B:\ 2n = \text{even integer, i.e., } n = \text{integer}}$

In this case the same argument shows that all the odd coefficients vanish, and in addition, all the even coefficients also vanish for $j < 2n$. We do get a nonzero solution by taking $a_{2n} \neq 0$ and even terms with index $j > 2n$ to be nonzero, but this solution is again the same that we get by taking $k=+n$ instead of taking $k=-n$.

Thus: $k=-n$ gives us another solution of Bessels equation if and only if n is not an integer.

We can also argue as follows: In formula (5) we interpret $\frac{1}{\infty}=0$, and recall that $\Gamma(j+1-n)=\infty$ for $-n+j+1 \le 0$, i.e., $j \le n-1$. Then (5) gives (with $n \to -n$)

$$J_{(-n)}(x) = \sum_{j=n}^{\infty} \frac{(-1)^j}{j!\,\Gamma(j+1-n)} \left(\frac{x}{2}\right)^{2j-n}$$

$(j-n=\ell,\ j=n+\ell)$

$$= \sum_{\ell=0}^{\infty} \frac{(-1)^{n+\ell}}{(n+\ell)!\,\ell!} \left(\frac{x}{2}\right)^{2(n+\ell)-n}$$

$$= (-1)^n \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}}{\ell!\,(n+\ell)!} \left(\frac{x}{2}\right)^{2\ell-n}$$

$$= (-1)^n J_n(x).$$

Therefore, for $n=-1,-2,-3,\dots$ we define

2.11 Defn. For $n=1,2,3,4$ we define $J_{(-n)}$ by

$$\boxed{J_{-n}(x) = (-1)^n J_n(x)}.$$

(It can be shown that $\lim_{\mu\to -n} J_\mu = J_{-n}$ for all integers $-n<0$).

2.12 Note In Bessel's equation

$$(4) \quad x^2 y'' + x y' + (x^2 - n^2) y = 0$$

we can always take $n \geq 0$ (if n is real), since $n^2 = (-n)^2$. (Otherwise we may take $\operatorname{Re} n \geq 0$.)

2.13 Conclusion i) If $n \neq 0, 1, 2, 3, \ldots$, then we have found 2 linearly independent solutions of (4), namely J_n and J_{-n} (they are linearly independent because they have different blow-up rate at the point $x=0$; see formula (3) with $x_0 = 0$). The general solution of (4) is therefore given by

$$y(x) = c_1 J_n(x) + c_2 J_{-n}(x),$$

where c_1 and c_2 are constants. More about this in the next section.

ii) if $n = 0, 1, 2, \ldots$, then we have found only one solution $J_n(x)$ of (4). To get the general solution we must find another solution to (4). This other solution does not have a series expansion of the form (3). What does it look like?

2.14 The irregular case: The equations that we get when we try to equate the constants (see p. 39) are too complicated to be solved. They contain extra terms where $\lambda = -1, -2, -3, \ldots$, and we do not have a starting point for the process.

IV.3 Second order ODE's (review)

A general second order linear ODE is of the type

$$(1) \quad y'' + P y' + Q y = R$$

(we have divided the equation by the coefficient of y''). The equation is homogeneous if $R = 0$. The general solution of this equation is given by

$$(2) \quad y(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x),$$

where

i) y_p is an arbitrary ("particular") solution of (1), and

ii) y_1 and y_2 are two linearly independent solutions of the homogeneous equation

$$(3) \quad y'' + P y' + Q y = 0.$$

"Linear independence" means that there do not exist any two constants c_1 and c_2 (not both $=0$) such that

$$c_1 y_1(x) + c_2 y_2(x) \equiv 0.$$

If we know one solution $y(x)$ of (3), then we can find another solution by "varying the constants" c_1 and and c_2. We make the following Ansatz:

$$z(x) = v(x)\, y(x),$$

where v is a new unknown function. Differentiate, and substitute in (3):

$$\begin{array}{l|l} z = uy & Q \\ z' = u'y + uy' & P \\ z'' = u''y + 2u'y' + uy'' & 1 \\ \hline & \Sigma \end{array}$$

$$Quy + P(u'y + uy') + u''y + 2u'y' + uy'' = 0$$

The coefficient multiplying u is

$$u\,[\,Qy + Py' + y''\,] = 0$$

since y is a solution of (3). $\Rightarrow$

$$Pu'y + u''y + 2u'y' = 0, \quad \text{or}$$

$$y\,u'' + (Py + 2y')\,u' = 0.$$

Change unknown: $u' = v$, i.e., $u = \int v(x)\,dx$.

$\Rightarrow$

$$v' + \left(\frac{Py + 2y'}{y}\right) v = 0 \quad \Longleftrightarrow$$

$$v' = -\left(\frac{Py + 2y'}{y}\right) v$$

By the standard formula for a first order linear ODE,

$$v = c_1\, e^{-\int \left(P(x) + \frac{2y'}{y}\right) dx} \qquad (c_1 = \text{constant})$$

Note that $\int \frac{2y'}{y}\,dx = 2\int \left(\frac{d}{dx} \ln y\right) dx$

$$= 2 \ln y + c_2, \quad \text{so}$$

$$v = c_1\, e^{-\int P(x)dx}\, e^{-\ln y^2}\, e^{-c_2}$$

$$= \underbrace{c_1 e^{-c_2}}_{c_3}\, e^{-\int P(x)\,dx}\, \frac{1}{y^2(x)}, \quad \text{and}$$

$$u = \int v(x)\,dx = c_3 \int_a^x \frac{e^{-\int_a^s P(t)\,dt}}{y^2(s)}\,ds + c_4,$$

$$z = uy = c_4\, y + c_3\, y(x) \int_a^x \frac{e^{-\int_a^s P(t)\,dt}}{y^2(s)}\,ds.$$

We recovered the old solution (take $c_4 = 1$, $c_3 = 0$), and in addition we got a new one:

$$(4) \qquad z(x) = y(x) \int_a^x \frac{e^{-\int_a^s P(t)\,dt}}{y^2(s)}\,ds.$$

(Here you may choose the lower bound a in <u>any</u> convenient way).

We got

<u>3.1 Lemma.</u> If y is a solution of (3), then another solution of (3) is given by (4) above.

<u>Note</u>: Personally, I find it easier to <u>rederive</u> this expression whenever you need it (instead of memorizing it).

<u>3.2 Special case</u> $P(x) \equiv 0 \Rightarrow$

$$(5) \qquad z(x) = y(x) \int_a^x \frac{ds}{y^2(s)}$$

<u>3.3 Lemma:</u> The solution z above is always linearly independent of y.

Proof (outline): As $x \to a$, the integral $\int_a^x \frac{e^{-\int_a^s P(t)dt}}{y^2(s)}\,ds$ tends to zero, so $\frac{z(x)}{y(x)} \to 0$ as $x \to a$. If

$$c_1\, y(x) + c_2\, z(x) = 0, \quad \text{then}$$

necessarily $c_1 \neq 0$ and $c_2 \neq 0$ (because otherwise $y(x) \equiv 0$ or $z(x) \equiv 0$), and $\frac{z(x)}{y(x)} = -\frac{c_1}{c_2} \not\to 0$ as $x \to a$. Thus $c_1 = 0 \Rightarrow z(x) \equiv 0$ contradiction.

3.4 Lemma If we know 2 linearly independent solutions y_1 and y_2 of (3), then we can find a particular solution y_p to (1) by varying the constants: we look for a solution

$$y(x) = v_1(x)\, y_1(x) + v_2(x)\, y_2(x),$$

where v_1 and v_2 are two new unknown functions.

Proof later (when we need it).

Again: It is better to learn the method than to memorize the final (complicated-looking) result.

IV. 4 A second solution to Bessel's equation

Recall:

$$(4) \quad x^2 y'' + x y' + (x^2 - n^2) y = 0 .$$

If $n \neq 0, 1, 2, \ldots$, then we have two linearly independent solutions $J_n(x)$ and $J_{-n}(x)$. The general solution is

$$y(x) = c_1 J_n(x) + c_2 J_{-n}(x) .$$

If $n = 0, 1, 2, \ldots$, we have only one solution $J_n(x)$ (the solution $J_{-n} = -J_n$ is linearly dependent with this one).

Let us apply Lemma 3.1 ~~with~~ y $J_n(x)$. We get

$$z(x) = J_n(x) \left[c_1 \int_a^x \frac{e^{-\int_a^s P(t)\,dt}}{[J_n(s)]^2}\, ds + c_2 \right].$$

Here $P(x) = \frac{x}{x^2} = \frac{1}{x}$, and

$$-\int_a^x P(t)\,dt = -\int_a^x \frac{dt}{t} = -\ln(x) + \ln a$$

(must take $a \neq 0$).

$$z(x) = J_n(x) \left[c_2 + c_3 \int_a^x \frac{ds}{s\,[J_n(s)]^2} \right].$$

Looks BAD! But let us analyze the integral:

$$J_n(s) = s^n v_n(s) \quad \text{where } v_n(0) = 1 \neq 0$$

The function $\frac{1}{[v_n(s)]^2}$ is analytic at $s = 0$.

Expand this into a series

$$\frac{1}{[v_n(s)]^2} = \sum_{k=0}^{\infty} \alpha_k s^k , \quad \text{where } \alpha_0 = 1 \neq 0 .$$

Substitute into the integral: (still $a \neq 0$)

(Theorem: It is permitted to integrate a converging power series term by term. See course on analytic functions.)

$$\int_a^x \frac{ds}{s\,[J_n(s)]^2} = \int_a^x \sum_{k=0}^{\infty} \alpha_k s^{k-2n-1}\, ds$$

$$= \sum_{k=0}^{\infty} \int_a^x (\text{the integral function}) \quad \left(\begin{array}{l}\text{the substitution}\\ \text{at } s=a \text{ gives}\\ \text{a constant}\end{array}\right)$$

$$= \text{const} + \sum_{k=0}^{\infty} \alpha_k \begin{cases} \frac{1}{k-2n} x^{k-2n}, & k \neq 2n . \\ \ln x , & k = 2n . \end{cases}$$

Recall that $\alpha_0 = 1 \neq 0$.

Thus, we get a series of the type

(*)
$$\boxed{\begin{aligned} &\text{const} + \beta_{-2n} s^{-2n} + \beta_{-2n+1} s^{-2n+1} + \cdots \\ &\cdots + \beta_{-1} x^{-1} + \boxed{\beta_0 \ln x} + \beta_1 x + \cdots, \end{aligned}}$$

where $\beta_{-2n} \neq 0$.

<u>4.1 Conclusion</u> The <u>second</u> <u>solution</u> to Bessel's equation can be written in the form

$$z(x) = J_n(x)\,[\quad],$$

where [ ] is the box above. (= (*)).
This gives us the <u>***Neumann***</u> <u>***function***</u> $N_n(x)$. We return to this function later (Section V. 5).

<u>4.2 Properties.</u>

i) If $n \neq$ integer, then the logarithm is absent. We simply get a linear combination of J_{-n} and J_n.

ii) In all cases the coefficient $\beta_{-2n} \neq 0$, so near to $x = 0$, $N_n(x)$ blows up like x^{-n}. In particular,

$$|N_n(x)| \to \infty \quad \text{as} \quad x \downarrow 0.$$

(Often a boundary cond. at zero forces us to <u>exclude</u> this solution).

<u>Final conclusion:</u> Serial method for solution is OK, but it can sometimes lead to long and complicated computations!