(1) $u_t = c^2 u_{xx} \qquad 0<x<L \,, \quad t>0$

(2) $\begin{cases} u(0,t)=0, & t>0 \\ u(L,t)=0, & t>0 \end{cases}$

(3) $u(x,0) = u_0(x), \quad 0<x<L$

c = a constant, $u_{xx} = \frac{\partial^2}{\partial x^2} u(x,t)$, $u_t = \frac{\partial}{\partial t} u(x,t)$.

(1) = partial differential equation
(2) = boundary condition ("randvillkor")
(3) = initial condition ("initialvillkor")

1.2 Separation of variables. We decide to try to find a solution of a "special type", where the variables have been "separated":

"Ansatz": $\qquad u(x,t) = F(x)\,G(t),$

where F depends only on x and G only on t. Substitute this into (1):

$$\frac{\partial}{\partial t} u(x,t) = \frac{\partial}{\partial t} F(x)G(t) = F(x)\,G'(t)$$

$$\frac{\partial^2}{\partial x^2} u(x,t) = \frac{\partial^2}{\partial x^2} F(x)G(t) = F''(x)\,G(t),$$

and (1) becomes

$$0 = u_t - c^2 u_{xx} = F(x)G'(t) - c^2 F''(x)G(t),$$

or

$$\boxed{F(x)\,G'(t) = c^2 F''(x)\,G(t), \quad 0<x<L,\ t>0}$$

We eliminate x on the left hand side, and eliminate t on the right hand side:
Divide by $c^2 F(x) G(t)$ to get

$$\frac{G'(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)}, \qquad 0<x<L,\ t>0.$$

Left hand side independent of t, Right hand side independent of x $\Big\} \Rightarrow$ they must both be constant:

$$\frac{G'(t)}{c^2 G(t)} = \text{constant} = \frac{F''(x)}{F(x)}.$$

Call this constant λ. This constant is real (since we assume F and G to be real). Three cases:

i) $\lambda>0 \qquad$ ii) $\lambda=0 \qquad$ iii) $\lambda<0$.

i) $\boxed{\lambda>0}$ Write $\boxed{\lambda = p^2}$ where $p>0$.

$$\frac{G'(t)}{c^2 G(t)} = p^2 = \frac{F''(x)}{F(x)} \quad \text{gives}$$

(a) $F''(x) = p^2 F(x) \qquad 0<x<L.$
(b) $G'(t) = p^2 c^2 G(t), \qquad t>0$

By standard theory for differential equations:

$$\begin{cases} F(x) = A e^{px} + B e^{-px} \\ (\text{the characteristic equation is } \lambda^2 = p^2, \\ \quad \text{with roots } \lambda = \pm p). \\ \\ G(t) = G(0)\, e^{p^2c^2t}, \qquad t>0, \end{cases}$$

and $u(x,t) = \left(Ae^{px} + Be^{-px}\right) G(0)\, e^{p^2c^2t}$.

Substitute into the boundary condition (2):

$$\begin{rcases} u(0,t) = (A+B)\,G(0)\,e^{p^2c^2t} = 0 \\ u(L,t) = \left(Ae^{pL} + Be^{-pL}\right) G(0)\, e^{p^2c^2t} = 0 \end{rcases} t>0$$

If $G(0)=0$, then $u(x,t)\equiv 0$, and we did not get anything interesting. A solution $u(x,t)\equiv 0$ is called a <u>trivial</u> solution.

If $G(0)\neq 0$, then we get

$$\begin{aligned} A+B&=0 \\ e^{pL}A+e^{-pL}B&=0 \end{aligned} \quad \left| \begin{array}{l} -e^{pL} \\ 1 \\ \hline \text{add} \end{array}\right.$$

$$\underbrace{(e^{pL}-e^{-pL})}_{\neq 0} B=0 \quad \Rightarrow B=0 \Rightarrow A=0 \Rightarrow$$

again we get a trivial solution.

Thus: The only possible solution in the case $\boxed{\lambda>0}$ is the <u>trivial</u> solution $u(x,t)\equiv 0$.

i) $\boxed{\lambda=0}$ Again we get only a trivial solution. This is a <u>homework</u> assignment.

ii) $\boxed{\lambda<0}$ Put $\lambda=-p^2$. As in case i), we get

$$\begin{cases} F''(x)= -p^2 F(x) & 0<x<L \\ G'(t)= -p^2c^2 G(t), & t>0 \end{cases}$$

$$\begin{cases} F(x)= A\cos px + B\sin px, & 0<x<L \\ G(t)= G(0)\, e^{-p^2c^2t}, & t>0 \end{cases}$$

$$u(x,t)= e^{-p^2c^2}G(0)\,(A\cos px + B\sin px).$$

If $G(0)=0$, then $u(x,t)\equiv 0$, <u>trivial</u>. Assume $G(0)\neq 0$.

The boundary condition (2) gives

$$u(0,t)= A\,G(0)\,e^{-p^2c^2t}=0, \quad t>0 \Rightarrow$$

$$\Rightarrow \boxed{A=0}$$

$$u(L,t)=0= e^{-p^2c^2t}\,G(0)\cdot B\sin(pL), \quad t\geq 0.$$

If $B=0$, then the solution is again trivial since then $F(x)\equiv 0$. Assume $B\neq 0$. Then

$$(*) \quad \boxed{\sin(pL)=0}.$$

Thus, $\boxed{\text{only certain values of } \lambda \text{ give a nontrivial solution}}$

Equation $(*)$ is equivalent to (recall that $p>0$)

$$p=p_n=\frac{n\pi}{L}, \quad n=1,2,3,\dots$$

Going back to the original variable $\lambda=-p^2$ we find that

$$\boxed{\lambda=\lambda_n= -\frac{n^2\pi^2}{L^2}, \quad n=1,2,3\dots}$$

This is called a <u>separation</u> <u>constraint</u>. The corresponding solution $u(x,t)$ is

$$(**) \quad \boxed{u_n(x,t)= e^{-\frac{n^2\pi^2c^2}{L^2}t}\, C_n \sin\frac{n\pi}{L}x,}$$

where C_n is an undetermined (=obertained) constant.

<u>Conclusion:</u> We have found infinitely many solutions of (1) with the boundary condition (2), namely those given by $(**)$ above.

The original problem also had an <u>initial</u> condition (3): $u(x,0)=u_0(x)$, $0<x<L$. How do we satisfy this condition?

Idea (= snilleblixt): The equations (1)-(2) are linear, i.e., if both u and v satisfy (1) and (2), then so do $u+v$ (check by substitution).

Thus: u_1+u_2 is a solution

$u_1+u_2+u_3$ — " —, etc., and

$$\text{(4)} \quad \sum_{n=1}^{N} u_n(x,t) = \sum_{n=1}^{N} c_n e^{-n^2\pi^2c^2t/L^2} \sin\frac{n\pi x}{L}$$

is also a solution, $N \in \mathbb{N}$ (= positive integer).

Case 1: If we are extremely lucky (= god tur), then u_0 is of the type

$$\text{(44)} \quad u_0(x) = \sum_{n=1}^{N} c_n \sin\frac{n\pi x}{L},$$

and we have found a solution to the full problem (1)-(3), namely (4) above.

Case 2: If we are not lucky, then we can still try to approximate the original $u_0(x)$ by means of the type (44) to get

$$u_0(x) = \lim_{N\to\infty} \sum_{n=1}^{N} c_n \sin\frac{n\pi x}{L},$$

and there is some hope that we may get an approximative solution

$$u(x,t) = \lim_{N\to\infty} u_N(x,t)$$

$$= \lim_{N\to\infty} \sum_{n=1}^{N} c_n e^{-n^2\pi^2c^2t/L^2} \sin\frac{n\pi x}{L}.$$

The study of the validity (= giltighet) of this formula requires some deeper studies, and we run into the notion of a Fourier Series.

1.3 Conclusion The preceding geometry was unusually simple, and its solution turned out to be a "simple" Fourier series expansion of the initial function, multiplied by exponentials with different "decaying rates" (= dämpnings-hastighet). In more complicated situations (a ball, or a cylinder, or a shell) and other equations (Laplace, wave eq.) then the computation proceeds in the same way, but we get more complicated expansions in terms of special functions, and we need a theory for "generalized Fourier series".

III.2 The Helmholz Equation

2.1 Problem Solve the heat equation

$$(1) \quad u_t(x,t) = c^2 \nabla^2 u(x,t)$$

in 3 space dimensions.

One way is to separate the time variable from the space variables as we did before. Another way is as follows: We try to get rid of the time variable by using Laplace transforms.

Warning: The purpose of the following computation is to give us some indication or hint to what we should expect. We will not use the Laplace transform later in actual proofs.

Thus, in the equation

(1) $u_t(x,t) = c^2 \nabla^2 u(x,t)$

we think of x as a constant, and take a Laplace transform w.r.t. (= with respect to) time. We know (2) that

$$\left(\mathcal{L}\left(\frac{\partial}{\partial t} u\right)\right)(s) = s\,\mathcal{L}(u) - u(0)$$

Apply this rule to (1). Denote $\mathcal{L}(u)$ by ψ. Then we get

$$s\,\psi(x,s) - u(0) = c^2 \nabla^2 \psi(x,s).$$

If $u(0) = 0$ and $s > 0$ then this is an example of <u>Helmholz equation</u>

(2) $\nabla^2 \psi \pm k^2 \psi = 0$,

where $\psi = \psi(x)$ and $k > 0$. (Heat eq: minus sign / Wave eq: plus sign)

The preceding argument gives us a reason (a motivation) to study Helmholz equation. (Another way to get to the same equation is to "separate variables".)

III.3 The Laplacian in Cylindrical Coordinates. (Arfken, Sect. 8,3)

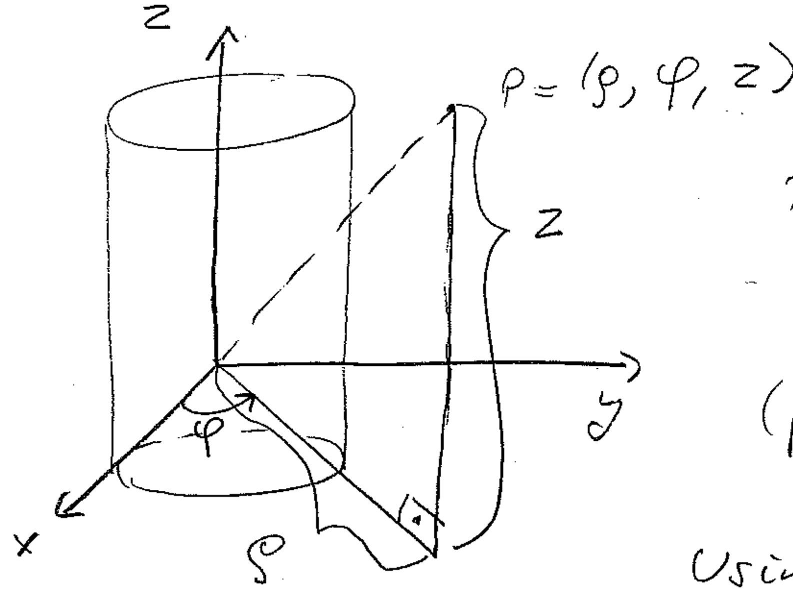
What is ∇^2 in equations (1) and (2). It is called the Laplacian (Laplace-operatorn).

In cartesian (= euclidean = euklidisk = kartesisk) coordinates we define

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2},$$

where $u = u(x,y,z)$ is a scalar (= reell-värd, skalär) function of 3 variables x, y, z (= "space variables" = rums variabler)

We convert this to <u>cylindrical coordinates</u>



Thus, $\begin{cases} x = \rho \cos\varphi \\ y = \rho \sin\varphi \\ z = z \end{cases}$ (polar coordinates in the xy-plane).

Using the chain rule of differentiation we find:
(A formal derivation of this formula is found in Arfken, and it is <u>not</u> required in this course. It is something you learn in multidimensional analysis.)

$$\nabla^2 \psi = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \psi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \psi}{\partial \varphi^2} + \frac{\partial^2 \psi}{\partial z^2}$$

Thus, Helmholz equation in cylindrical coordinates is given by

(3) $\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \psi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \psi}{\partial \varphi^2} + \frac{\partial^2 \psi}{\partial z^2} \pm k^2 \psi = 0.$

To this equation we add boundary <u>conditions</u> which depend on the particular situation.

<u>Separation of variables.</u> Try to find a solution of the form

$$\psi(\rho, \varphi, z) = P(\rho)\,\Phi(\varphi)\,Z(z).$$

Then (over)

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) = \frac{\Phi Z}{\rho}\frac{d}{d\rho}\left(\rho P'(\rho)\right)$$

$$\frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\varphi^2} = \frac{PZ}{\rho^2}\Phi''(\varphi)$$

$$\frac{\partial^2\psi}{\partial z^2} = P\Phi Z''(z)$$

$$\pm k^2\psi = \pm k^2 P\Phi Z$$

Add these $\Rightarrow$

$$\frac{\Phi Z}{\rho}\frac{d}{d\rho}\left(\rho P'(\rho)\right) + \frac{PZ}{\rho^2}\Phi''(\varphi) + P\Phi Z''(z) \pm k^2 P\Phi Z = 0.$$

One possible solution is $\psi = P\Phi Z \equiv 0$. This is the trivial solution. If $\psi \not\equiv 0$, then, at least for some values of ρ, φ, z we will have $P(\rho)\Phi(\varphi)Z(z) \neq 0$, and can divide by this (keep your fingers crossed; this may lead to problems later!)

Dividing by $\psi = P\Phi Z$ gives us

$$\underbrace{\underbrace{\frac{1}{\rho P(\rho)}\frac{d}{d\rho}\left(\rho P'(\rho)\right)}_{\text{depend only on } \rho} + \underbrace{\frac{\Phi''(\varphi)}{\rho^2\Phi(\varphi)}}_{\text{depends on } \rho \text{ and } \varphi}}_{\text{do not depend on } z} + \underbrace{\frac{Z''(z)}{Z(z)}}_{\text{depends only on } z} \pm \underbrace{k^2}_{\text{constant}} = 0$$

If we move the two last terms to the other side, then the left hand side is independent of z, and the right hand side depends only on z. Therefore, each part is a <u>constant</u>. In particular, $\frac{Z''(z)}{Z(z)}$ is a constant.

Depending on the boundary conditions, we may discover later that the constant is e.g. >0 or <0. To simplify later computations we denote it by l^2 (l real means that $l^2>0$, l pure imaginary means that $l^2<0$). Thus

(4) $\quad Z''(z) = l^2 Z(z)$, which has the solution

$$Z(z) = Ae^{lz} + Be^{-lz} \qquad (l \text{ real or imaginary}).$$

Substitute this into the equation above to get

(5) $$\underbrace{\frac{\rho}{P(\rho)}\frac{d}{d\rho}\left(\rho P'(\rho)\right)}_{\text{does not depend on } \varphi} + \underbrace{\frac{\Phi''(\varphi)}{\Phi(\varphi)}}_{\text{depends only on } \varphi} + \underbrace{(\pm k^2 + l^2)\rho^2}_{\text{does not depend on } \varphi} = 0$$

Thus, $$\frac{\Phi''(\varphi)}{\Phi(\varphi)} =: \text{constant}.$$

We have some <u>natural</u> <u>boundary</u> <u>conditions</u> for Φ. Since (ρ, φ, z) and $(\rho, \varphi+2\pi, z)$ represent the <u>same</u> point (we have gone once around the z-axis), Φ must be <u>periodic</u> with period 2π. If the constant is >0, then we get solutions of the type

$$\Phi(\varphi) = Ae^{\alpha\varphi} + Be^{-\alpha\varphi}$$

which are <u>not</u> periodic. Therefore the constant must be ≤ 0, and Φ is of the type

$$\Phi(\varphi) = A\cos(\alpha\varphi) + B\sin(\alpha\varphi)$$

(if the constant is $-\alpha^2$ with $\alpha \geq 0$). This function is 2π-periodic if and only if

$$\boxed{\alpha = \alpha_m = 0, 1, 2, 3 \ldots}$$

Thus, the equation for Φ becomes

(6) $\boxed{\Phi''(\varphi) = -m^2 \Phi(\varphi),}$

with the <u>separation</u> condition

(7) $\boxed{m = 0, 1, 2, \dots}$

and the solution

$$\boxed{(8) \quad \Phi(\varphi) = \Phi_m(\varphi) = C_m \cos(m\varphi) + D_m \sin m\varphi}$$

(so we appear to get a <u>Fourier</u> <u>series</u> <u>in</u> φ)

Substitute (6) back into (5) to get

$$(9) \quad \rho \frac{d}{d\rho}\left(\rho \frac{dP}{d\rho}\right) + \left[(\pm k^2 + l^2)\rho^2 - m^2\right] P = 0.$$

The solutions of this equation are various <u>Bessel</u> <u>functions</u> of order m (exactly which type of Bessel function depends on the value of $\pm k^2 + l^2$).

<u>Summary</u> i) The equation (4) for Z has the solution

$$Z(z) = Ae^{lz} + Be^{-lz},$$

where l is real or imaginary depending on the particular boundary conditions for Z.

ii) The equations (5) and (6) lead to trigonometric solutions

$$\Phi_m(\varphi) = C_m \cos(m\varphi) + D_m \sin(m\varphi), \quad m = 0, 1, 2, \dots$$

for Φ.

iii) Equation (9) for P leads to Bessel function solutions for P.

<u>Note 1.</u> Physically, the variable ρ in equation (9) stands for

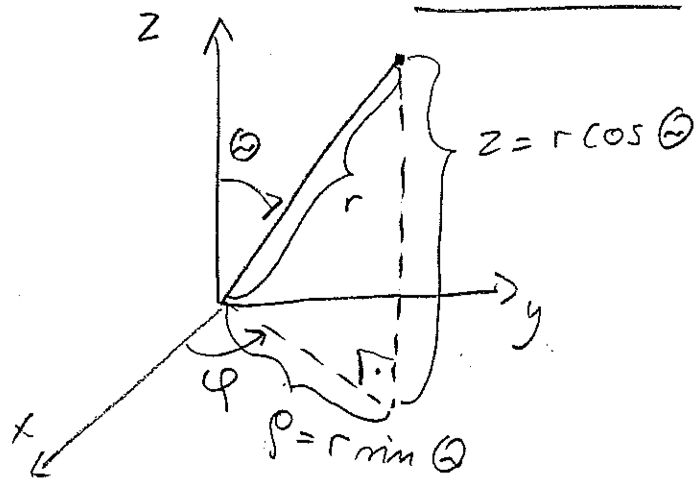
$$\boxed{\rho = \text{distance to the } z\text{-axis}},$$

so the natural domain for ρ is $0 < \rho < \infty$. The cylindrical coordinate system has a singularity at $\rho = 0$ (φ is not defined when $\rho = 0$) and we shall see later that equation (9) has a <u>singularity</u> (= singularitet) at $\rho = 0$.

<u>Note 2.</u> Most of the work still remains to be done!

- What boundary conditions are fearable? (= vettiga, gängbara)
- What are the Bessel functions?
- Can we expand the solution into some sort of Bessel function series?

III.4 The Laplacian in sperical coordinates (Arfken, sect. 8.3)



$$\begin{cases} x = r \sin\Theta \cos\varphi \\ y = r \sin\Theta \sin\varphi \\ z = r \cos\Theta \end{cases}$$

$$\begin{cases} x^2 + y^2 + z^2 = r^2 \\ x^2 + y^2 = r^2 \sin^2\Theta \end{cases}$$

On the earth: $\Theta = 0$ on the equator
is the latitude ($= \left\{ \begin{matrix} \text{nordlig} \\ \text{sydlig} \end{matrix} \right\}$ bredd)

$\varphi =$ longitude ($= \left\{ \begin{matrix} \text{ostlig} \\ \text{västlig} \end{matrix} \right\}$ längd)

$r =$ the radius of the earth.

On the sky. In principle same coordinates (no r)

Note: $r=0$ is a singular point (Θ and φ are not determined when $r=0$), so we can expect singularities in our equations when $r=0$ (of the type "divide by zero").

4.1 Problem. Solve the wave equation in a spherical symmetry!

By using the Fourier transform in the time-variable we can reduce this to a simpler problem:

4.2 Problem. Solve the Helmholz equation in spherical coordinates.

$$(1) \quad \nabla^2 \psi + k^2 \psi = 0$$

(note: wave equation + Fourier trans form gives $+k^2$, heat equation + Laplace transf. gives $-k^2$)

To solve this we need to know (see Artikon) that

$$\nabla^2 \psi = \frac{1}{r^2 \sin\Theta} \left[\sin\Theta \frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r}\right) + \frac{\partial}{\partial \Theta}\left(\sin\Theta \frac{\partial \psi}{\partial \Theta}\right) + \frac{1}{\sin\Theta} \frac{\partial^2 \psi}{\partial \varphi^2} \right]$$

We argue as in the preceding section: we first make an "Ansatz":

$$\psi = R(r)\, T(\Theta)\, \Phi(\varphi),$$

substitute into (1), and divide by ψ. This gives

$$(2) \quad \underbrace{\frac{1}{Rr^2} \frac{d}{dr}\left(r^2 \frac{dR}{dr}\right)}_{\mathrm{I}} + \underbrace{\frac{1}{Tr^2 \sin\Theta} \frac{d}{d\Theta}\left(\sin\Theta \frac{dT}{d\Theta}\right)}_{\mathrm{II}} + \underbrace{\frac{1}{\Phi r^2 \sin^2\Theta} \frac{d^2\Phi}{d\varphi^2}}_{\mathrm{III}} = -k^2.$$

(we would have had $+k^2$, if we had started with the heat équation).

Abbreviation of (2) $= \mathrm{I} + \mathrm{II} + \mathrm{III} = -k^2$

We observe: $\begin{cases} \mathrm{I} & \text{depends only on } r \\ \mathrm{II} & \text{depends on } r \text{ and } \Theta \\ \mathrm{III} & \text{depends on } r, \Theta, \varphi. \end{cases}$

$\Rightarrow$ There is no separation.

Brilliant idea: Multiply by $\boxed{r^2 \sin^2\Theta}$ to get rid of r and Θ in the last term. After this:

III depends only on φ
the others do not depend on φ

and we have achieved separation! Thus,

$$\boxed{\frac{\Phi''(\varphi)}{\Phi(\varphi)} = \text{constant}}$$

The <u>same</u> argument as we used in section III. 3 gives

(3) $\Phi''(\varphi) = -m^2 \Phi(\varphi)$

(4) $m = 0, 1, 2, \ldots$

(5) $\Phi_m(\varphi) = A_m \cos(m\varphi) + B_m \sin(m\varphi)$.

Substitute into (2) to get

(6) $I + II - \frac{m^2}{r^2 \sin^2\theta} = -k^2$.

This simplifies if we multiply by r^2.

i) $(I + k^2) r^2$ depends only on r

ii) the other terms do not depend on r.

Therefore $r^2 (I + k^2) = \text{constant} = Q \overset{>}{=} 0$

$$\frac{1}{R} \frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) + r^2 k^2 = Q, \quad \text{or}$$

(7) $r^2 R'' + 2rR' + (r^2k^2 - Q)R = 0$. $\quad Q = n^2$

This equation can be **reduced** to a Bessel's equation $\boxed{\text{for } R.}$ Substitute (7) into (6) to get

$$\frac{1}{\sin\theta} \frac{d}{d\theta}\left(\sin\theta \frac{dT}{d\theta}\right) - \frac{m^2}{\sin^2\theta} T + QT = 0,$$

$\Longleftrightarrow$

$$T'' + \frac{\cos(\theta)}{\sin(\theta)} T' - \frac{m^2}{\sin^2\theta} T + QT = 0.$$

Change variable: $s = \cos\theta$, $ds = -\sin\theta \, d\theta$ $\quad \boxed{\begin{array}{l}\theta = 0 \Rightarrow s = 1 \\ \theta = \pi \Rightarrow s = -1\end{array}}$

Denote $T(\theta)$ by $v(s)$, so that $\boxed{T(\theta) = v(\cos(\theta))}$, and use the chain rule:

$$\begin{array}{c|l}
\left(Q - \frac{m^2}{\sin^2\theta}\right) & T(\theta) = v(s) \\
\frac{\cos\theta}{\sin\theta} & T'(\theta) = -\sin\theta \, v'(s) \\
1 & T''(\theta) = -\cos\theta \, v'(s) + \sin^2\theta \, v''(s) \\
 & \quad = (1 - \cos^2\theta) v''(s) - \cos\theta \, v'(s) \\
 & \quad = (1 - s^2) v''(s) - s v'(s)
\end{array}$$

<u>Add</u>

$$\begin{array}{ll}
(1-s^2) v'' - s v'(s) & (= T''(\theta)) \\
- s v'(s) & \left(= \frac{\cos\theta}{\sin\theta} T'(\theta)\right) \\
+ \left(Q - \frac{m^2}{1-s^2}\right) v(s) & \left(= \left(Q - \frac{m^2}{\sin^2\theta}\right) T(\theta)\right) \\
= 0 &
\end{array}$$

$\Rightarrow$ (8) $\boxed{(1-s^2) v'' - 2sv' + \left[Q - \frac{m^2}{1-s^2}\right] v = 0.}$

For $m = 0$ we get <u>Legendre's equation</u>.

$$(1-s^2) v'' - 2sv' + Qv = 0, \quad -1 \le s \le 1$$

For $m \neq 0$ we get the <u>associated</u> <u>Legendre's eq.</u>

<u>Physical interpretation</u>. $s = \cos(\theta) = \frac{z}{r}$, and $z = sr$. If we require that the solution behaves well as $s = \pm 1$ (= the north and the south poles) then we get an extra <u>separation</u> condition for θ, to which we return later.

<u>Summary</u> Different <u>geometries</u> give different <u>ordinary</u> <u>equations</u>. The geometry is more important than the equation: Many equations (heat equation, wave eq., Helmholz eq., Schrödinger eq.; which all contain ∇^2) lead to the <u>same</u> <u>ordinary diff. eq.</u> (with different values of the parameters.)

<u>Next</u> $\boxed{\text{How do we solve these ODE's?}}$