As in the chapter about the Hermite polynomials we put

<u>1.1 Thm.</u> The series above <u>converges</u> for <u>all</u> $x \in \mathbb{C}$, and it defines an analytic function $y(x)$ which satisfies (1) and (2).

However, we must also require that the solution belong to $L^2(0,\infty; xe^{-x})$ (as in Sections <u>VIII.5</u> and <u>IX.1</u>). Again, this will be true <u>only</u> when the series is finite, i.e., $y(x)$ is a polynomial, and this is true iff

$$\boxed{n = \text{integer} \geq 0},$$

The most common normalization is $a_0 = 1$.

<u>1.2 Defn</u> The polynomials

$$L_n(x) = \sum_{k=0}^{n} (-1)^k \frac{n!}{(n-k)!\,(k!)^2} x^k$$

$$= \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{x^k}{k!}$$

Binomial coeff.

are the <u>Laguerre polynomials</u>.

<u>Table</u>:

$$\begin{aligned} (L_{-1} &\equiv 0) \\ L_0 &\equiv 1 \\ L_1 &= 1 - x \\ L_2 &= 1 - 2x + \tfrac{1}{2}x^2 \\ L_3 &= 1 - 3x + \tfrac{3}{2}x^2 - \tfrac{1}{6}x^3 \\ L_4 &= 1 - 4x + 3x^2 - \tfrac{2}{3}x^3 + \tfrac{1}{24}x^4 \quad \text{etc.} \end{aligned}$$

[21] EDMOND LAGUERRE (1834—1886), French mathematician, who did research work in geometry and the theory of infinite series.

X.2 Properties

<u>2.1 Rodrigue's formula</u>

$$L_n(x) = \frac{1}{n!} e^x \left(\frac{d}{dx}\right)^n (x^n e^{-x}), \quad n \geq 0.$$

<u>2.2 Generating Function</u>

$$F(x,t) = \frac{1}{1-t} e^{-\frac{xt}{1-t}} = \sum_{n=0}^{\infty} L_n(x) t^n$$

<u>2.3 Recursion Formula</u>

$$(n+1) L_{n+1}(x) + (x - 2n - 1) L_n(x) + n L_{n-1}(x) = 0$$

<u>2.4 Derivative Formulas</u>

$$L_n' = L_{n-1}' - L_{n-1} \qquad (n \geq 0) \qquad (L_{-1} \equiv 0)$$

$$x L_n' = n(L_n - L_{n-1}) \qquad \text{— " —}$$

<u>2.5 Orthogonality</u> $L_n \perp L_m$, i.e., $\langle L_n, L_m \rangle = 0$ where

$$\langle f, g \rangle = \int_0^\infty f(x) \overline{g(x)} e^{-x} dx.$$

In addition, $\|L_n\| = 1$, i.e.,

$$\int_0^\infty (L_n(x))^2 e^{-x} dx = 1.$$

<u>2.6 Fourier-Laguerre series</u>. The sequence $\{L_n\}_{n=1}^{\infty}$ is complete and ortho<u>normal</u> in $L^2(0,\infty; e^{-x})$. The generalized Fourier Laguerre transform $F(n)$ of $f \in L^2(0,\infty; e^{-x})$ is

$$F_n = \langle f, L_n \rangle = \int_0^\infty f(x) L_n(x) e^{-x} dx, \quad \text{and}$$

$$f(x) = \sum_{n=0}^{\infty} F_n L_n(x).$$

<u>Parseval</u>:

$$\int_0^\infty |f(x)|^2 e^{-x} dx = \sum_{n=0}^{\infty} |F_n|^2$$

X.3 The Laguerre Functions

We can divide e^{-x} in the formula

$$\langle \phi, g \rangle = \int_0^\infty \phi(x) \overline{g(x)}\, e^{-x}\, dx = \int_0^\infty [e^{-x/2}\phi(x)][e^{-x/2}\overline{g}(x)]\, dx$$

(as in Section IX, 5), and get

3.1 Defn. The functions

$$\ell_n(x) = e^{-x/2} L_n(x) = \frac{1}{n!} e^{x/2} \left(\frac{d}{dx}\right)^n (x^n e^{-x})$$

are the Laguerre functions of order n (no longer polynomials).

As in section IX.5 we get

3.2 Thm. The sequence $\{\ell_n\}_{n=0}^\infty$ is complete and orthonormal in $L^2(0,\infty)$ with the usual inner product

$$\langle \phi, g \rangle = \int_0^\infty \phi(x) \overline{g(x)}\, dx.$$

X.4 The Laplace Transform

4.1 Defn. The Laplace transform of the function $\phi \in L^2(0,\infty)$ is given by

$$\hat{\phi}(z) = \int_0^\infty e^{-zx} \phi(x)\, dx$$

for all $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$ (put $z = \alpha + i\beta$, with $\alpha > 0$; then $|e^{-zx}| = e^{-\alpha x}$ and the integral converges absolutely. see p. 3.)

4.2 Problem. What is the Laplace transform of the Laguerre functions?

Solution:
$$\hat{\ell}_n(z) = \int_0^\infty e^{-zx} \ell_n(x)\, dx = \int_0^\infty e^{-zx} e^{-x/2} L_n(x)\, dx = \int_0^\infty e^{-(z+\frac{1}{2})x} L_n(x)\, dx = \hat{L}_n(z+\tfrac{1}{2}).$$

Thus, it is enough to compute the Laplace transform of the Laguerre polynomials. We get these from the generating function, or alternatively, from Rodrigues' formula:

$$\hat{L}_n(z) = \int_0^\infty e^{-zx} \frac{1}{n!} e^{x} \left(\frac{d}{dx}\right)^n (x^n e^{-x})\, dx = \frac{1}{n!} \int_0^\infty e^{-(z-1)x} \left(\frac{d}{dx}\right)^n (x^n e^{-x})\, dx$$

(take $\operatorname{Re}(z-1) > 0$ and integrate by parts n times)

$$= \frac{1}{n!}(-1)^n (-(z-1))^n \int_0^\infty e^{-(z-1)x} x^n e^{-x}\, dx$$

$$= \frac{(z-1)^n}{n!} \int_0^\infty x^n e^{-zx}\, dx \quad \left(\text{integrate by parts } n \text{ times}\right)$$

$$= (z-1)^n (-1)^n \frac{1}{(-z)^n} \int_0^\infty e^{-zx}\, dx$$

$$= \left(1 - \frac{1}{z}\right)^n \Big|_0^\infty \frac{e^{-zx}}{-z} = \frac{1}{z}\left(1 - \frac{1}{z}\right)^n.$$

Thus we get:

4.3 Thm.
$$\hat{L}_n(z) = \frac{(z-1)^n}{z^{n+1}} \quad \text{and}$$

$$\hat{\ell}_n(z) = \frac{(z-1/2)^n}{(z+1/2)^{n+1}}.$$

This gives us the following method to compute Laplace transforms: we expand $\phi \in L^2(0,\infty)$ into a Laguerre series:

$$\phi(x) = \sum_{n=0}^{\infty} c_n \ell_n(x),$$

where $c_n = \langle \phi, \ell_n \rangle = \int_0^\infty \phi(x)\, \ell_n(x)\, dx$

$$\boxed{c_n = \int_0^\infty \phi(x)\, L_n(x)\, e^{-x/2}\, dx,}$$

and transform the series term by term to get

$$\boxed{\hat{\phi}(z) = \sum_{n=0}^{\infty} c_n \hat{\ell}_n(z) = \sum_{n=0}^{\infty} c_n \frac{(z-1/2)^n}{(z+1/2)^{n+1}}}$$

X.4 Associated Laguerre Polynomials

If we replace the Laguerre equation

$$xy'' + (1-x)y' + ny = 0$$

by the associated Laguerre equation

$$(1) \qquad xy'' + (k+1-x)y' + ny = 0,$$

we get another family of polynomial solutions L_n^k, the associated Laguerre polynomials

$$L_n^k(x) = (-1)^k \left(\frac{d}{dx}\right)^k L_{n+k}(x) \qquad \text{(by Rodrigue's formula)}$$

$$= \frac{e^x x^{-k}}{n!} \left(\frac{d}{dx}\right)^n \left(x^{n+k} e^{-x}\right).$$

The crucial inner product is

$$\langle f, g \rangle = \int_0^\infty f(x)\, \overline{g(x)}\, x^k e^{-x}\, dx.$$

See Arfken pp. 779–781.

XI Chebyshev's Polynomials

Note: Many different spellings: Tschebyscheff, Chebyshef, etc (transliterated from Cyrillic).

Chebyshev's differential equation is

$$(1) \qquad (1-x^2)y'' - xy' + n^2 y = 0.$$

Compare this to the associated Legendre's eq:

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0.$$

The factor $1-x^2$ has the same explanation: this equation is obtained through a change of variable $x = \cos\theta$, θ = polar angle.

Self-adjoint form. We should multiply by

$$\frac{1}{p_0} e^{\int \frac{p_1}{p_0} dx} = \frac{1}{1-x^2} e^{\int \frac{-x}{1-x^2} dx}, \text{ where}$$

$$\int \frac{-x}{1-x^2} dx = \frac{1}{2} \int \frac{-2x}{1-x^2} dx = \frac{1}{2} \ln(1-x^2),$$

so

$$\frac{1}{1-x^2} e^{\int \frac{p_1}{p_0}} = \frac{1}{1-x^2} e^{\frac{1}{2}\ln(1-x^2)} = \frac{\sqrt{1-x^2}}{1-x^2} = \frac{1}{\sqrt{1-x^2}}$$

The self-adjoint form is thus

$$(2) \qquad \left(\sqrt{1-x^2}\, y'\right)' + \frac{n^2}{\sqrt{1-x^2}}\, y = 0,$$

so $p(x) = \sqrt{1-x^2}$ and $r(x) = \frac{1}{\sqrt{1-x^2}}$ $(q(x) = 0)$

The natural interval is $(-1,1)$. Note that (as before) $\sqrt{1-x^2} = \sqrt{1-\cos^2\theta} = \sin\theta$!

The problem is singular since $p(-1) = p(1) = 0$. No boundary conditions needed, but the solution should belong to

$$L^2\left(-1,1; \frac{1}{\sqrt{1-x^2}}\right).$$

Note: we anticipated later computations and wrote the eigenvalue parameter λ as $\lambda = n^2$. You can get $\lambda < 0$ by takin $n =$ pure imaginary.

XI. 1 Serial Solution

The point $x=0$ is a regular point. We proceed as usual (with $k=0$, since regular)

$$\begin{array}{r|l} n^2 & y = \sum_{j=0}^{\infty} a_j x^j \\ -x & y' = \sum_{j=1}^{\infty} j a_j x^{j-1} \\ 1-x^2 & y'' = \sum_{j=2}^{\infty} j(j-1) a_j x^{j-2} \end{array}$$

$$\Sigma \qquad 0 = \sum_{j=0}^{\infty} \left[(n^2 - j - j(j-1))\, a_j + (j+2)(j+1)\, a_{j+2} \right] x^j$$

We can choose a_0 and a_1 arbitrarily, and the others should then satisfy

$$a_{j+2} = \frac{j^2 - n^2}{(j+1)(j+2)}\, a_j \qquad \boxed{\lim_{n\to\infty} \frac{a_{j+2}}{a_j} = 1}$$

Taking $a_0 \neq 0$, $a_1 = 0$, we get an even solution
$a_0 = 0$, $a_1 \neq 0$, — " — odd — " —.

1.1 Thm. The preceding two series converge for all $|x| < 1$, and they define two analytic and linearly independent solutions of (1) and (2). Radius of conv. is 1
Singularities at ± 1.

Unfortunately, these solutions do not belong to $L^2(-1,1; \frac{1}{\sqrt{1-x^2}})$ unless the series is finite, so we once more get a separation condition

$$\boxed{n = \text{integer} \geq 0}$$

$n =$ even $\Rightarrow$ we get an even polynomial
$n =$ odd $\Rightarrow$ — " — — " — odd polynomial.

1.2 Defn. The polynomial

$$T_n(x) = \frac{n}{2} \sum_{k \leq n/2} (-1)^k \frac{(n-k-1)!}{k!\,(n-2k)!} (2x)^{n-2k}$$

are the Chebyshev polynomials (of the first kind).

1.3 Thm. These polynomials are eigenfunctions of the S-L-problem (1) with eigenvalues n^2.

XI. 2 Chebyshev Polynomials of the Second Kind

We get these by replacing Chebyshev's differential equation by

(1) $(1-x^2)y'' - 3xy' + n(n+2)y = 0,$

where we have written the separation parameter λ in the form $\lambda = n(n+2)$. This equation is a special case of the following ultraspherical equation (and so are Legendre's and Chebyshev's equations).

(2) $(1-x^2)y'' - (2\alpha+1)xy' + n(n+2\alpha)y = 0.$

The self-adjoint form of (1) is (homework)

(3) $((1-x^2)^{3/2}y')' + n(n+2)\sqrt{1-x^2}\, y = 0.$

A computation similar to those in Sections IX.1, X.1, XI.1 gives: we get two solutions, but only polynomial solutions belong to $L^2(-1,1; \sqrt{1-x^2})$. These solutions appear only when

$$\boxed{\lambda = n(n+2), \quad n = \text{integer} \geq 0}$$

(radius of conv $= 1$, singularities at ± 1

2.1 Defn. The polynomials

$$U_n(x) = \sum_{k\le n/2} (-1)^k \frac{(n-k)!}{k!\,(n-2k)!} (2x)^{n-2k}$$

are the Chebyshev polynomials of the second kind.

X.3 Orthogonality

According to the general Sturm–Liouville Theory, the polynomials T_n are a complete orthogonal series with respect to the inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)\,\overline{g(x)}\,\frac{dx}{\sqrt{1-x^2}},$$

and the polynomials U_n are a complete orthonormal series w.r.t. the inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)\,\overline{g(x)}\,\sqrt{1-x^2}\,dx.$$

In the next section we compute their norms, and get:

$$\|T_n\|^2 = \int_{-1}^{1} (T_n(x))^2 \frac{dx}{\sqrt{1-x^2}} = \begin{cases} \frac{\pi}{2}, & n\ne 0, \\ \pi, & n=0. \end{cases}$$

$$\|U_n\|^2 = \int_{-1}^{1} (U_n(x))^2 \sqrt{1-x^2}\,dx = \pi/2.$$

As usual, we can expand functions in $L^2(-1,1;\frac{1}{\sqrt{1-x^2}})$ and $L^2(-1,1;\sqrt{1-x^2})$ in orthogonal series w.r.t. these "basis" functions.

[20] PAFNUTI CHEBYSHEV (1821—1894), Russian mathematician, is known by his work in approximation theory and the theory of numbers. Another transliteration of the name is TCHEBICHEF.

XI.4 Trigonometric Formulas

Let's make a change of variable in Chebyshev's equation: $x = \cos(\theta)$

$$(1-x^2)y'' - xy' + n^2 y = 0$$

Same as on p. 31 (Legendre's eq)

$$v(\theta) = y(\cos\theta) = y(x)$$
$$v'(\theta) = -\sin\theta\, y'(\cos\theta)$$
$$v''(\theta) = \underbrace{\sin^2\theta\, y''(\cos(\theta))}_{(1-x^2)y''} \underbrace{- \cos\theta\, y'(\cos\theta)}_{-xy'}$$

$\Rightarrow$ $v'' + n^2 v = 0$, and

$$v(\theta) = A\cos(n\theta) + B\sin(n\theta), \text{ and}$$
$$y(x) = A\cos(n \arccos x) + B\sin(n \arccos x).$$

Now:

$$\cos(n\theta) + i\sin(n\theta) = e^{in\theta} = (e^{i\theta})^n$$

$$= \sum_{k=0}^{n} \binom{n}{k} i^k \cos^{n-k}\theta \sin^k\theta \quad \left(\begin{matrix} k=2l \text{ or} \\ k=2l+1 \end{matrix}\right)$$

$$= \sum_{l\le n/2} (-1)^l \binom{n}{2l} \cos^{n-2l}\theta\,(1-\cos^2\theta)^l \quad (\text{even})$$

$$i\sum_{l\le n/2} (-1)^l \binom{n}{2l+1} \cos^{n-2l-1}\theta \sin^{2l+1}\theta \quad (\text{odd})$$

$$= \sum_{l\le n/2} (-1)^l \binom{n}{2l} x^{n-2l} (1-x^2)^l$$

$$+ i\sum_{l\le n/2} (-1)^l \binom{n}{2l+1} x^{n-2l-1} (1-x^2)^l \sqrt{1-x^2}.$$

Thus,

$$\left.\begin{cases} \cos(n\arccos x) = \sum_{l\le n/2} (-1)^l \binom{n}{2l} x^{n-2l} (1-x^2)^l \\ \dfrac{\sin(n\arccos x)}{\sqrt{1-x^2}} = \sum_{l\le n/2} (-1)^l \binom{n}{2l+1} x^{n-2l-1} (1-x^2)^l \end{cases}\right\} \text{ are polynomials}$$

On the other hand, we know that the only polynomial solution of Chebyshev's equation is a constant times T_n. Thus,

$$T_n(x) = A\cos(n\arccos x).$$

Substituting $x = 0$ gives $T_n(0) = A$, so $A = 1$. This proves the first half of the following theorem

<u>4.1 Thm.</u> $T_n(x) = \cos(n\arccos x)$ and

$$U_n(x) = \frac{1}{\sqrt{1-x^2}}\sin(n\arccos x).$$

<u>Proof</u>: We already did the formula for T_n. The proof of the formula for U_n is similar: One shows that

$$\frac{1}{\sin\theta}\sin((n+1)\theta)$$

is a solution of the equation that we get by substituting $x = \cos\theta$ in the equation in Section XI.2. □

A similar computation proves:

<u>4.2 Thm.</u> The general solution of Chebyshev's differential equation is

$$y = AT_n(x) + B\sqrt{1-x^2}\,U_{n-1}(x),$$

and the general solution of equation (1) in Section XI.2 is (second kind)

$$y = A\frac{1}{\sqrt{1-x^2}}T_{n+1}(x) + BU_n(x).$$

<u>4.3 Defn.</u>

$$V_n(x) = \sqrt{1-x^2}\,U_{n-1}(x),$$

$$W_n(x) = \frac{1}{\sqrt{1-x^2}}T_{n+1}(x).$$

Then

$$\begin{cases} T_n(\cos\theta) = \cos n\theta \\ V_n(\cos\theta) = \sin n\theta \\ U_n(\cos\theta) = \sin(n+1)\theta / \sin\theta \\ W_n(\cos\theta) = \sin\theta\cos(n-1)\theta \end{cases}$$

and these functions have the following orthogonality properties:

$$\int_{-1}^{1} T_m(x)T_n(x)\frac{dx}{\sqrt{1-x^2}} = \begin{cases} 0, & m \neq n \\ \frac{\pi}{2}, & m = n \neq 0 \\ \pi, & m = n = 0 \end{cases}$$

$$\int_{-1}^{1} W_m(x)W_n(x)\sqrt{1-x^2}\,dx = \begin{cases} 0, & m \neq n \\ \frac{\pi}{2}, & m = n \neq 0 \\ 0, & m = n = 0 \end{cases}$$

$$\int_{-1}^{1} U_m(x)U_n(x)\sqrt{1-x^2}\,dx = \begin{cases} 0, & m \neq n \\ \frac{\pi}{2}, & m = n \end{cases}$$

$$\int_{-1}^{1} V_m(x)V(x)\frac{dx}{\sqrt{1-x^2}} = \begin{cases} 0, & m \neq n \\ \frac{\pi}{2}, & m = n \end{cases}$$

All these series are complete. T_n and U_n are polynomials; the others are not.

XI. 5 Recursion Formulas Etc

By using the trigonometrical formulas for T_n and U_n we can derive lots of additional formulas, such as

5.1 Recursion Formulas

$$\begin{cases} T_{n+1} - 2x T_n + T_{n-1} = 0 \\ U_{n+1} - 2x U_n + U_{n-1} = 0 \end{cases}$$

5.2 Derivative Formulas

$$\begin{cases} (1-x^2) T_n' + n x T_n = n T_{n-1} \\ (1-x^2) U_n' + n x U_n = (n+1) U_{n-1} \end{cases}$$

5.3 Generating Functions

$$\frac{1-t^2}{1-2xt+t^2} = \sum_{n=0}^{\infty} T_n(x) t^n$$

$$\frac{1}{1-2xt+t^2} = \sum_{n=0}^{\infty} U_n(x) t^n$$

(Recall: $\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n$, P_n = Legendre).

5.4 Values at ±1

$T_n(1) = 1$, $T_n(-1) = (-1)^n$.

5.5 Zeros of T_n

If $T_n(x) = 0 \Rightarrow \cos(n\Theta) = 0$ where $x = \cos\Theta$ and $0 \le \Theta \le \pi \Rightarrow$

$$n\Theta = (\tfrac{1}{2}+k)\pi, \text{ so}$$

$$\Theta = \frac{(2k+1)\pi}{2n},$$ and the zeros are

$$\boxed{x_k = \cos\frac{(2k+1)\pi}{2n}}$$

5.6 Zeros of U_n

$U_n(x) = 0 \Leftrightarrow \sin(n\Theta) = 0$ where $x = \cos\Theta$, i.e., $n\Theta = k\pi$ and $\Theta = \frac{k\pi}{n}$, so

$$\boxed{x_k = \cos\frac{k\pi}{n}}.$$

5.7 Size estimate:

$$|T_n(x)| \le 1, \quad -1 \le x \le 1$$

$$|U_n(x)| \le \frac{1}{\sqrt{1-x^2}}, \quad -1 < x < 1,$$

since $|\sin(n\Theta)| \le 1$ and $|\cos(n\Theta)| \le 1$.

In addition

$$|T_n(x)| = 1$$

in exactly $n+1$ points in $[-1,1]$ (out of which $n-1$ lie in the open interval $(-1,1)$), and

$$|U_n(x)| = \frac{1}{\sqrt{1-x^2}} \text{ in } (n-1) \text{ points in } (-1,1).$$

5.8 Use

The Chebyshev polynomials are used e.g. in approximations where the maximal error should be as small as possible.

The End