IX Hermite Polynomials

The Hermite differential equation is

$$(1) \quad y'' - 2xy' + 2ny = 0,$$

n = a parameter (which for some values give us eigenvalues). Because of the underlying physical situation (described later), we are interested in the *infinite interval* $(-\infty, \infty)$, so this problem is a *singular* Sturm–Liouville problem (since $b-a=\infty$). We already found the self-adjoint form on p. 143:

$$(2) \quad (e^{-x^2} y')' + 2n e^{-x^2} y = 0,$$

i.e., $p(x) = e^{-x^2}$, $q(x) = 0$, $r(x) = 2e^{-x^2}$.

IX.1 Serial Solution

This problem is regular at all finite points. We can, e.g., expand round $x=0$. As we saw earlier, the index equation will be $k(k-1)=0$, so we can use the *simpler* series

$$y(x) = \sum_{j=0}^{\infty} a_j x^j$$

(instead of the more general $y(x) = \sum_{j=0}^{\infty} a_j x^{k+j}$). (see 2.3 at page 36). We get

$$\begin{array}{c|l} 2n & y(x) = \sum_{j=0}^{\infty} a_j x^j \\ -2x & y'(x) = \sum_{j=1}^{\infty} j a_j x^{j-1} \\ 1 & y''(x) = \sum_{j=2}^{\infty} j(j-1) a_j x^{j-2} \\ \hline \Sigma & 0 = \sum_{j=0}^{\infty} (j+2) a_{j+2} x^j + \sum_{j=0}^{\infty} (2n-2j) a_j x^j \end{array}$$

Thus, we may choose a_0 and a_1 freely, and we get the other coefficients from

$$(3) \quad a_{j+2} = -\frac{2(n-j)}{(j+1)(j+2)} a_j .$$

Taking $a_0 = 1$, $a_1 = 0$ we get an even sol.
Taking $a_0 = 0$, $a_1 = 1$ we get an odd solution.

Thus

$$a_2 = -\frac{2n}{2} a_0 = -n a_0$$

$$a_4 = -\frac{2(n-2)}{3 \cdot 4} a_2 = + \frac{2^2 n(n-2)}{4!} a_0$$

$$a_6 = -\frac{2^3 n(n-2)(n-4)}{6!} a_0, \text{ etc,}$$

$$a_3 = -\frac{2(n-1)}{2 \cdot 3} a_1$$

$$a_5 = -\frac{2(n-3)}{4 \cdot 5} a_3 = \frac{2^2 (n-1)(n-3)}{5!} a_1$$

$$a_7 = -2^3 \frac{(n-1)(n-3)(n-5)}{7!} a_1, \text{ etc.}$$

Even solution:
$$a_0 \left(1 - n x^2 + \frac{4n(n-2)}{4!} x^4 - \frac{8n(n-2)(n-4)}{6!} x^6 + \dots \right)$$

odd:
$$a_1 \left(x - \frac{2(n-1)}{3!} x^3 + \frac{4(n-1)(n-3)}{5!} x^5 - \frac{8(n-1)(n-3)(n-5)}{7!} x^7 \dots \right)$$

We get the *radius* of *convergence* by

$$\frac{1}{R} = \lim_{j \to \infty} \sqrt{\left| \frac{a_{j+2}}{a_j} \right|} = \left(\lim_{j \to \infty} \frac{2|(j-n)|}{(j+1)(j+2)} \right)^{1/2} = 0,$$

so the *radius* of *convergence* is $+\infty$ (the series converges for *all* $x \in \mathbb{C}$).

1.1 Thm. The series presented above converge for all $x \in \mathbb{C}$, and they are two (analytic and) linearly independent solutions of the Hermite differential eq.

As the Sturm-Liouville problem is singular, plain convergence is not enough, but we have to assume, in addition, that the solutions (and their derivatives) belong to $L^2(-\infty, \infty; e^{-x^2})$, i.e.,

$$\int_{-\infty}^{\infty} |y(x)|^2 e^{-x^2}\,dx < \infty.$$

A rather complicated computation shows that this is never true, unless the series is finite, i.e., from some index m_0 on, all $a_m = 0$. This is true for the even series iff n = even integer ≥ 0, and for the odd series iff n = odd integer ≥ 0. Thus, we must require that

$$\boxed{n = \text{integer} \geq 0}.$$

Usual normalization: Choose a_n or a_1 so that the coefficient of x^n is 2^n. We can then use the recursion formula (3) backwards to get

$$a_{n-2} = -\frac{n(n-1)}{4} a_n$$

$$a_{n-4} = +\frac{n(n-1)}{4} \cdot \frac{(n-2)(n-3)}{2\cdot 4} a_n$$

$$a_{n-6} = -\frac{n(n-1)(n-2)\cdots(n-5)}{2^3 \underbrace{2\cdot 4\cdot 6}_{2^3\cdot 3!}}$$

In general:

$$a_{n-2k} = (-1)^k \frac{n!}{(n-2k)!\,k!} (2x)^{n-2k}$$

1.2 Defn. The Hermite Polynomial H_n of degree n is

$$H_n = \sum_{k \leq n/2} (-1)^k \frac{n!}{k!\,(n-2k)!} (2x)^{n-2k}$$

(not binomial coefficients!)

Table: $(H_{-1} \equiv 0)$

$$H_0 = 1$$
$$H_1 = 2x$$
$$H_2 = 4x^2 - 2$$
$$H_3 = 8x^3 - 12x$$
$$H_4 = 16x^4 - 48x^2 + 12$$
$$H_5 = 32x^5 - 160x^3 + 120x \quad , \text{etc.}$$

IX. 2 Recursion Formulas and Generating Functions

According to Thm 6.5, p. 146, there is a recursion formula

$$\alpha_n H_{n+1} + (x+\beta_n) H_n + \gamma_n H_{n-1} = 0$$

for suitably chosen constants $\alpha_n, \beta_n, \gamma_n$. The simplest way to determine these is not the method used in the proof of that theorem, but to compare the coefficients of x^{n+1}, x^n, and x^{n-1} in this equation. This gives

2.1. Thm The Hermite polynomials H_n satisfy the recursion equation

$$\boxed{(1) \quad H_{n+1}(x) - 2x H_n(x) + 2n H_{n-1}(x) = 0}, \; n \geq 0.$$

As starting values we can use $H_{n-1} \equiv 0$ and $H_0(x) \equiv 1$ (Actually H_{-1} is not needed since $2n = 0$ when $n = 0$).

The Generating Function should be
$$\sum_{n=0}^{\infty} H_n(x) t^n.$$
Unfortunately, the normalization that we use (i.e., the highest term is $2^n x^n$) is not the right one for this. For example, taking $x=0$ we would get
$$\sum_{n=0}^{\infty} H_n(0) t^n, \quad \text{which for every } t \neq 0$$
a diverging series (radius of convergence $= 0$)
$$\sum_{n=0}^{\infty} H_n(0) t^n = \sum_{n=\text{even}} (-1)^{n/2} \frac{n!\, t^n}{(\frac{n}{2})!}.$$
The "correct" normalization is to divide H_n by $n!$.

2.2 Defn. The generating function $F(x,t)$ of the Hermite polynomials is
$$(2) \quad F(x,t) = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n.$$
When we computed the generating function of the Legendre polynomials we used only a recursion formula resembling the one in (1). We can "repeat" the same computation with the new recursion formula to get

2.3 Thm The generating function $F(x,t)$ for the Hermite polynomials H_n is given by
$$(3) \quad F(x,t) = e^{x^2-(x-t)^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n$$
$$= e^{2xt-t^2}.$$

2.4 Thm. $H_{n+1}'(x) = 2(n+1) H_n(x)$

Proof: Differentiate (3) w.r.t. x (note that $H_0' \equiv 0$)
$$\frac{\partial}{\partial x} F(x,t) = \sum_{n=1}^{\infty} \frac{H_n'(x)}{n!} t^n \iff$$
$$2t\, e^{2xt-t^2} = \sum_{n=1}^{\infty} \frac{H_n'(x)}{n!} t^n \iff$$
$$2 \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^{n+1} = \sum_{n=0}^{\infty} \frac{H_{n+1}'(x)}{(n+1)!} t^{n+1} \iff$$
$$\frac{H_{n+1}'}{(n+1)!} = 2\, \frac{H_n(x)}{n!} \iff H_{n+1}' = 2(n+1) H_n \quad \square.$$

IX.3 Representation Formula

By the general theory of analytic series: if
$$\varphi(t) = \sum_{n=0}^{\infty} a_n t^n,$$
then $a_n = \frac{\varphi^{(n)}(t)}{n!}\Big|_{t=0}$. Apply this to the generating function of Hermite's polynomials to get

3.1 Thm ('Rodrigues' formula)
$$(1) \quad H_n(x) = (-1)^n e^{x^2} \left(\frac{d^n}{dx^n}\right) e^{-x^2}$$

Proof:
$$\frac{H_n(x)}{n!} = \frac{1}{n!} \left(\frac{\partial}{\partial t}\right)^n e^{x^2-(x-t)^2} \Big|_{t=0}$$
$$= \frac{1}{n!} e^{x^2} \left(\frac{\partial}{\partial t}\right)^n e^{-(x-t)^2} \Big|_{t=0}.$$
(Change variable: $x-t=z$, $\frac{\partial}{\partial t} = -\frac{\partial}{\partial z}$, $t=0 \Rightarrow z=x$.)
$$= \frac{1}{n!} e^{x^2} \left(\frac{\partial}{\partial z}\right)^n e^{-z^2} \Big|_{z=x}$$
(over)

$$= \frac{1}{n!} e^{x^2} \left(\frac{d}{dx}\right)^n e^{-x^2}. \qquad \square$$

IX. 4 Orthogonality

The general Sturm-Liouville theory for (singular) problems gives

4.1 Thm The polynomials $H_n(x)$, $n \ge 0$, are orthogonal with respect to the (real) scalar product

$$\langle f, g \rangle = \int_{-\infty}^{\infty} e^{-x^2} f(x) g(x)\, dx,$$

and

$$\| H_n \|^2 = \int_{-\infty}^{\infty} e^{-x^2} (H_n(x))^2 dx = 2^n n! \sqrt{\pi}.$$

This sequence is complete in $L^2(-\infty, \infty; e^{-x^2})$

Proof: The orthogonality follows from the Sturm-Liouville theory; The substitution terms go to zero since $P_m(x) e^{-x^2} \to 0$ as $|x| \to \infty$ for all polynomials P_m. The theory gave us $r(x) = 2e^{-x^2}$ instead of e^{-x^2}, but we simply divide the integral by 2.

The formula for $\| H_n \|^2$ remains. To get this we use Lemma 6.6 on p. 148 and Rodrigue's formula:

$$\| H_n \|^2 = \int_{-\infty}^{\infty} e^{-x^2} \underbrace{2^n x^n}_{\text{Highest order term}} \underbrace{(-1)^n e^{x^2} \left(\frac{d}{dx}\right)^n e^{-x^2}}_{\text{Rodrigue's formula}} dx$$

$$= (-1)^n 2^n \int_{-\infty}^{\infty} x^n \left(\frac{d}{dx}\right)^n e^{-x^2} dx \quad \text{(integrate by parts } n \text{ times)}$$

$$= 2^n n! \int_{-\infty}^{\infty} e^{-x^2} dx = 2^n n! \sqrt{\pi}. \qquad \square$$

Completeness proof bypassed.

4.2 Defn. The Fourier-Hermite transformation of a function $f \in L^2(-\infty, \infty; e^{-x^2})$ is the sequence

$$F_n = \int_{-\infty}^{\infty} f(x) H_n(x) e^{-x^2} dx.$$

4.3 Thm. The function $f \in L^2(-\infty, \infty; e^{-x^2})$ can be reconstructed from its Fourier-Hermite series $\{F_n\}_{n=0}^{\infty}$ as an infinite sum

$$f(x) = \sum_{n=0}^{\infty} \frac{F_n}{2^n n! \sqrt{\pi}} H_n(x),$$

← Note: Not the "standard" normalization.

where the convergence takes place as explained on p. 149 (we always have norm-convergence, and in addition we have pointwise convergence if f is piece-wise differentiable).

Proof (outline). See Section XIII. 7. There only the regular case was discussed, but this singular case can be treated in a similar way (with a more complicated proof).

4.4 Thm (Non-normalized Parseval's identity) For every $f \in L^2(-\infty, \infty; e^{-x^2})$ we have

$$\| f \|^2 = \int_{-\infty}^{\infty} |f(x)|^2 e^{-x^2} dx = \sum_{n=0}^{\infty} \sqrt{\pi}\, 2^n n! \, |F_n|^2$$

Proof: See p. 149.

IX. 5 The Hermite Functions

As $H_n \perp H_m$, i.e.,

$$\int_{-\infty}^{\infty} H_n(x)\, H_m(x)\, e^{-x^2}\, dx = 0,$$

we also have

$$\int_{-\infty}^{\infty} \left[e^{-x^2/2} H_n(x)\right] \left[e^{-x^2/2} H_m(x)\right] dx = 0.$$

5.1 Defn. The function $h_n(x) = e^{-x^2/2} H_n(x)$ is called the Hermite function of order n.

Note: Not a polynomial.

5.2 Lemma $h_n(x) = (-1)^n e^{x^2/2} \left(\frac{d}{dx}\right)^n e^{-x^2}$

Proof: Thm 3.1, p. 158.

By "reinterpreting" Thm 4.3 on p. 160 we get:

5.3 Thm. The function sequence $\{h_n\}_{n=0}^{\infty}$ is a complete orthonormal sequence in $L^2(-\infty,\infty)$ with respect to the usual inner product

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\, dx,$$

and $\|h_n\|^2 = \int_{-\infty}^{\infty} |h_n(x)|^2 dx = 2^n n! \sqrt{\pi}$. Thus, an arbitrary function $f \in L^2(-\infty,\infty)$ can be written as a sum

$$f(x) = \sum_{n=0}^{\infty} \frac{c_n}{2^n n! \sqrt{\pi}}\, h_n(x),$$

(convergence in the norm of $L^2(-\infty,\infty)$) where

$$c_n = \int_{-\infty}^{\infty} f(x)\, h_n(x)\, dx.$$

IX. 6 The Fourier Transform on $(-\infty,\infty)$

The functions $h_n(x)$ are closely related to the standard Fourier transform on $(-\infty,\infty)$.

6.1 Defn. Let f be a piecewise continuous (or measurable) function on $(-\infty,\infty)$ which satisfies the condition

$$\int_{-\infty}^{\infty} |f(x)|\, dx < \infty$$

(i.e., $f \in L^1(-\infty,\infty)$). Then we define the Fourier transform of f by

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x} f(x)\, dx.$$

(Warning: Not everyone divides by $\sqrt{2\pi}$?)

6.2 Thm. The Fourier transform of the Hermite function $h_n(x)$ is

$$\hat{h}_n(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x} h_n(x)\, dx = (-i)^n h_n(\omega).$$

Interpretation: Observe that

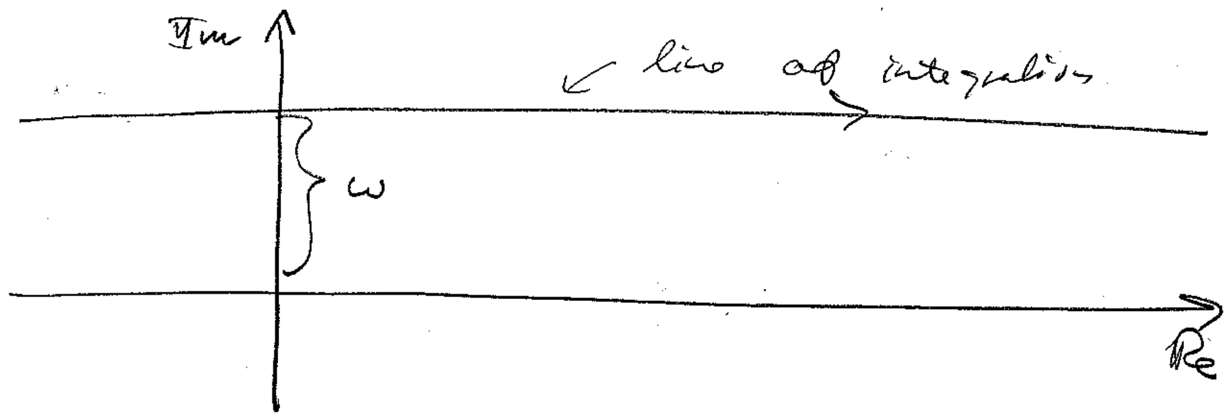
$$(-i)^n = \begin{cases} -i, & n = 1, 5, 9, 13, \dots \\ -1, & n = 2, 6, 10, 14, \dots \\ +i, & n = 3, 7, 11, 15, \dots \\ +1, & n = 0, 4, 8, 12, 16, \dots \end{cases}$$

We get $\widehat{h_n(\omega)}$ by simply multiplying $h_n(\omega)$ by this constant.

Proof: We start with the simplest case $n=0$.

$$\hat{h}_0(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x} e^{-x^2/2}\, dx$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x+i\omega)^2/2}\, e^{-\omega^2/2}\, dx$$
$$= \frac{1}{\sqrt{2\pi}} e^{-\omega^2/2} \int_{-\infty}^{\infty} e^{-(x+i\omega)^2/2}\, dx$$

This is a complex line integral along the line $\mathrm{Im}(z) = \omega$



By the general theory of analytic functions (see separate course), we can move this line down to the real axis instead, and get: $x + i\omega = v \quad dx = dv$

$$\hat{h}_0(\omega) = \frac{1}{\sqrt{2\pi}} e^{-\omega^2/2} \int_{-\infty}^{\infty} e^{-v^2/2}\, dv$$
$$= e^{-\omega^2/2} = h_0(\omega).$$

Thus, the theorem is true when $n=0$.

For the general case we use the generating function:

$$F(x,t) = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n = e^{2xt - t^2} \iff$$
$$\sum_{n=0}^{\infty} \frac{h_n(x)}{n!} t^n = e^{(-x^2/2 + 2xt - t^2)}. \implies$$

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \hat{h}_n(\omega)$$
$$= \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h_n(x) e^{-i\omega x}\, dx$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x} \left(\sum_{n=0}^{\infty} \frac{t^n}{n!} h_n(x) \right) dx$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x} e^{-(x^2/2 - 2xt + t^2)}\, dx$$
$$= \frac{e^{-t^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[x^2 - 2x(2t - i\omega) + (2t-i\omega)^2\right]}\, dx \quad \left(\times\, e^{\frac{1}{2}(2t-i\omega)^2}\right)$$
$$= \frac{1}{\sqrt{2\pi}} e^{(t^2 - 2i\omega t - \omega^2/2)} \underbrace{\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x - 2t - i\omega)^2}\, dx}_{=\sqrt{2\pi},\ \text{see the case } n=0}$$
$$= e^{t^2 - 2i\omega t} e^{-\omega^2/2}$$
$$= e^{-\omega^2/2} F(\omega, -it)$$
$$= e^{-\omega^2/2} \sum_{n=0}^{\infty} \frac{H_n(\omega)}{n!} (-it)^n$$
$$= e^{-\omega^2/2} \sum_{n=0}^{\infty} (-i)^n H_n(\omega) \frac{t^n}{n!}.$$

By comparing the coefficients of t^n we get

$$\hat{h}_n(\omega) = (-i)^n e^{-\omega^2/2} H_n(\omega) = (-i)^n h_n(\omega). \quad //$$

<u>6.3 Defn.</u> A complex number λ is an <u>eigenvalue</u> of the Fourier transformation with corresponding eigenfunction ϕ iff

$$\hat{\phi}(\omega) = \lambda \phi(\omega), \quad \omega \in \mathbb{R}.$$

(Compare this to the definition of an eigenfunction and eigenvector of a matrix.)

6.4 Corollary The Fourier transform has (at least) four eigenvalues, namely ± 1 and $\pm i$. All of these are multiple, with infinite multiplicity, i.e., to each eigenvalue there corresponds infinitely many linearly independent eigenfunctions.

Proof: All the functions h_n are eigenfunctions, which belong to one of $\pm 1, \pm i$.

Note: It can be shown that the Fourier transform has no other eigenvalues: If we repeat the Fourier transform 4 times we get back the original function

Therefore every eigenvalue λ must satisfy $\lambda^4 = 1$, i.e., $\lambda = \sqrt[4]{1} = 1, i, -1, -i$

6.5 Thm. The Fourier transformation $\hat{\phi}$ of an arbitrary function $\phi \in L^2(-\infty,\infty) \cap L^1(-\infty,\infty)$ can be computed in the following way: First we expand ϕ into a Fourier–Legendre series:

$$\phi = \sum_{n=0}^{\infty} \frac{c_n}{2^n n! \sqrt{\pi}} h_n(x), \quad \text{where}$$

$$h_n(x) = (-1)^n e^{x^2/2} \left(\frac{d}{dx}\right)^n e^{-x^2}, \quad \text{and}$$

$$c_n = \int_{-\infty}^{\infty} \phi(x)\, h_n(x)\, dx.$$

Then $\hat{\phi}(\omega)$ is given by

$$\hat{\phi}(\omega) = \sum_{n=0}^{\infty} \frac{(-i)^n c_n}{2^n n! \sqrt{\pi}} h_n(\omega).$$

The convergence is the "usual" convergence for generalized Fourier series.

Proof: Use Thm 5.3 on p. 161, and transform the series term by term. ⊲

[22]CHARLES HERMITE (1822—1901), French mathematician, is known by his work in algebra and number theory. The great HENRI POINCARÉ (1854—1912) was one of his students.

X Laguerre Polynomials

The Laguerre differential equation is

$$(1) \quad x y'' + (1-x) y' + n y = 0.$$

We found the self-adjoint form on p. 143:

$$(2) \quad (x e^{-x} y')' + n e^{-x} y = 0,$$

i.e., $p(x) = x e^{-x}$, $q(x) = 0$, $r(x) = e^{-x}$.

For physical reasons, this eq. should be valid on $(0,\infty)$. It is singular, since $(b-a) = \infty$, and also since $p(0) = 0$.

X.1 Serial solution

The point $x = 0$ is a regular singular point.

$$
\begin{array}{c|l}
n & y = \sum_{j=0}^{\infty} a_j x^{k+j} \\
1-x & y' = \sum_{j=0}^{\infty} (k+j)\, a_j\, x^{k+j-1} \\
x & y'' = \sum_{j=0}^{\infty} (k+j)(k+j-1)\, a_j\, x^{k+j-2} = \sum_{j=1}^{\infty} (k+j+1)(k+j)\, a_{j+1} x^{k+j-1} \\
\hline
\Sigma & 0 = \sum_{j=0}^{\infty} (n-k-j)\, a_j x^{k+j} + \sum_{j=0}^{\infty} (k+j)^2 a_j x^{k+j-1}.
\end{array}
$$

Index equation: Take $j=0$ in the second sum. This gives $k^2 a_0 = 0 \Rightarrow$ (if we take $a_0 \neq 0$) $k = 0$. $\Rightarrow$ First term of second sum drops out, and we can combine the two sums:

$$0 = \sum_{j=0}^{\infty} \left[(n-j)\, a_j + (j+1)^2 a_{j+1}\right] x^j, \quad \text{i.e.,}$$

$$(3) \quad a_{j+1} = -\frac{n-j}{(j+1)^2}\, a_j, \quad j \geq 0.$$

Note: we got only one solution, not two!