"Matrix pencil"

$$Av = \lambda Bv.$$

Interpretation: A and B turn the vector v in the same direction.

Note: If B is invertible then we can reduce this problem to a standard eigenvalue problem: Let $w = B^{-1}x$. Then

$$Av = AB^{-1}w = \lambda BB^{-1}w = \lambda w,$$

so λ is an eigenvalue of AB^{-1} with eigenvector w.

We apply the same idea to the differential operator L. We suppose that L is self-adjoint:

(1) $Lu = [pu']' + qu$,

and let B represent the operator

(2) $Bu = r(x)u(x)$,

where $r(x)$ is another given (real) function. We get:

4.1 Problem: Find all twice differentiable functions $u \not\equiv 0$ and the corresponding constants λ for which $Lu + \lambda Bu = 0$, i.e.,

(3) $[pu']' + (q + \lambda r)u = 0$.

Note: For historical reasons, there is a different sign convention in the Sturm-Liouville theory than in linear algebra: We use $+\lambda$ instead of $-\lambda$.

Note: The function r appears in a natural way: If we want to solve a problem of the type

$$p_0 u'' + p_1 u' + p_2 u + \lambda u = 0,$$

then we first make it self-adjoint by multiplying with $r(x)$, and get

$$[pu']' + qu + \lambda ru = 0.$$

(see section VIII.2). There are also other cases where $r \not\equiv 1$ appears naturally when we separate variables.

4.2 Defn. We call Problem 4.1 a Sturm-Liouville problem. It is regular iff $\boxed{p(x) > 0}$ and $\boxed{r(x) > 0}$ for all $x \in [a,b]$ (including the end points). It is irregular if either

i) the interval (a,b) is of infinite length, or

ii) $p(x) > 0$ and $q(x) > 0$ for all $x \in (a,b)$, but $p(x)$ or $q(x)$ is zero when $x = a$ or $x = b$.

Note: The case where, e.g., $p(c) = 0$ for some $c \in (a,b)$ can be reduced to two problems: one on (a,c) and another on (c,b). They are both irregular since $p(c) = 0$.

4.3 Defn. The constant λ in Problem 4.1 is an eigenvalue of L, and u is the corresponding eigenfunction.

Note: The solution to (3) depends not only on L, but also on the boundary conditions. We assume in the sequel that

$\boxed{\text{the boundary conditions are self-adjoint}}$.

(see Defn. 3.3).

4.4 Thm. The eigenvalues of a Sturm–Liouville problem are *real*, and any two functions which belong to two *different* eigenvalues are *orthogonal* to each other with respect to the inner product

$$(4) \qquad \langle f, g\rangle = \int_a^b f(x)\, g(x)\, \underbrace{r(x)\,dx}_{\text{(weight)}}$$

Proof. (Compare this to the proof of the orthogonality of the different Bessel functions given in pp. 69–71.)

Take two eigenfunctions u and v, and the corresponding eigenvalues λ and μ. In this computation we allow u, v, λ and μ to be *complex*, but still require p, q, r to be real. We also use *complex* inner product. (Recall that the eigenvalues of a *real* matrix may be *complex*!)

We have
$$\left.\begin{aligned} [pu']' + [q+\lambda r]u &= 0 \\ [pv']' + [q+\mu r]v &= 0 \end{aligned}\right\} \iff$$

$$\left\{\begin{aligned} \mathcal{L}u + \lambda r u &= 0 \\ \mathcal{L}v + \mu r v &= 0 \end{aligned}\right. \quad \begin{array}{|l} \langle v, \cdot\rangle \\ \langle \cdot, u\rangle \end{array} \quad \Longrightarrow$$

$$(*) \qquad \begin{aligned} \langle v, \mathcal{L}u\rangle + \bar{\lambda}\langle v, ru\rangle &= 0 \\ \langle \mathcal{L}v, u\rangle + \mu\langle rv, u\rangle &= 0 \end{aligned} \quad \begin{array}{|l} 1 \\ -1 \end{array}$$

Clearly,
$$\langle rv, u\rangle = \int_a^b r(x)\, v(x)\, \overline{u(x)}\, dx = \int_a^b v(x)\, \overline{[r(x)u(x)]}\, dx = \langle v, ru\rangle$$
(since r is real), and by repeating the proof of Thm 1.7 with complex u and v we get $\langle v, \mathcal{L}u\rangle = \langle \mathcal{L}v, u\rangle$. Thus, from the equation $(*)$:
$$(\bar{\lambda} - \mu)\langle v, ru\rangle = 0.$$

First take $\lambda = \mu$, $u = v$. Then
$$(\bar{\lambda} - \lambda)\langle u, ru\rangle = 0,$$
and $\langle u, ru\rangle = \int_a^b |u(x)|^2 r(x)\, dx > 0$ since $u \not\equiv 0$.

Therefore $\lambda = \bar{\lambda}$, i.e., λ is real!

Next take $\lambda \neq \mu$ (and $\lambda = \bar{\lambda}$). Then we get
$$(\lambda - \mu)\langle v, ru\rangle = 0, \quad \text{so}$$
$$\langle v, ru\rangle = 0, \quad \text{i.e.,}$$
$v \perp u$ w.r.t. the inner product (4), $\square$

4.5 Thm. It is always possible to choose the eigenfunctions of a Sturm–Liouville problem to be real.

Proof. Write $u(x) = v(x) + i w(x)$. Since p, q, r are real, both $v(x) = \operatorname{Re} u(x)$ and $w(x) = \operatorname{Im} u(x)$ satisfy the equation
$$\mathcal{L}u + \lambda r u = 0$$
whenever u satisfies this eq. Thus, if u is an eigenfunction, then so are v and w (if they are $\not\equiv 0$). Replace u by either v or w.

4.6 Defn. An eigenvalue λ is *simple* ("enkelt") if all the eigenfunctions belonging to this eigenvalue are *multiples* of *one fixed eigenfunction*. Otherwise the eigenvalue is *multiple* ("mångdubbelt"). It has *multiplicity* n ("multiplicitet") if all the eigenfunctions can be written as linear combinations of n linearly independent eigenfunctions.

4.7 Ex. The harmonic oscillator

$$y'' + \lambda y = 0, \quad \text{with}$$

periodic boundary conditions: $y(0) = y(1)$
$y'(0) = y'(1)$

Case 1: $\boxed{\lambda = 0}$ gives $y(x) = Ax + B$. The condition $y'(0) = y'(1)$ is always true. $y(0) = y(1)$ gives $A = 0$. Thus, $\lambda = 0$ is a simple eigenvalue with eigenfunction $u(x) \equiv 1$ (or $u(x) \equiv$ constant $\neq 1$)

Case 2: $\lambda = -\alpha^2 < 0$ gives $y = Ae^{\alpha x} + Be^{-\alpha x}$.

$$\left.\begin{aligned} y(0) = y(1) &\Rightarrow A + B = Ae^{\alpha} + Be^{-\alpha} \\ y'(0) = y'(0) &\Rightarrow \alpha(A - B) = \alpha(Ae^{\alpha} - Be^{-\alpha}) \end{aligned}\right\} \Rightarrow$$

$$\begin{aligned} (e^{\alpha} - 1)A + (e^{-\alpha} - 1)B &= 0 \quad \Big| +1 \quad \Big| +1 \\ (e^{\alpha} - 1)A - (e^{-\alpha} - 1)B &= 0 \quad \Big| -1 \quad \Big| +1 \end{aligned}$$

$\Rightarrow A = B = 0 \Rightarrow$ no $\lambda < 0$ is an eigenvalue.

Case 3. $\lambda = \alpha^2 > 0$ gives $y = A\cos(\alpha x) + B\sin(\alpha x)$
The periodicity condition forces $\alpha = 2\pi n$, $n =$ integer, and we find that all

$$\lambda_n = (2\pi n)^2 \qquad (n = 1, 2, 3, \ldots)$$

are double eigenvalues, with eigenfunctions $\cos(2\pi n x)$ and $\sin(2\pi n x)$ (and linear combinations of these two).

Note: In most cases the eigenvalues are simple. This follows from the next theorem. First we prove two lemmas:

4.8 Lemma. Let u be a solution of

$$(5) \quad p_0(x)u'' + p_1(x)u' + p_2(x)u = 0$$

in the interval (a, b), with $p_0(x) > 0$ for $x \in (a, b)$. If $u(c) = u'(c) = 0$ for some $c \in (a, b)$, then $u(x) \equiv 0$, $x \in (a, b)$.

Proof: By the course in differential equations, for any two fixed numbers α and β, equation (5) has a unique solution u in $[c, b)$ satisfying the initial condition $u(c) = \alpha$, $u'(c) = \beta$. Take $\alpha = 0 = \beta$. One obvious solution satisfying $u(c) = 0$ and $u'(c) = 0$ is $u(x) \equiv 0$. Another is the one mentioned in the lemma. By uniqueness, they are the same, i.e., $u(x) = 0$ for all $x \in [c, b)$.
A similar argument (replace x by $-x$) shows that $u(x) = 0$ also for $x \in (a, c]$.

4.9 Lemma. Two solutions $u(x)$ and $v(x)$ of the same eq. (5) are linearly dependent if and only if their Wronskian ("Wronski's determinant")

$$W(x) = \begin{vmatrix} u(x) & u'(x) \\ v(x) & v'(x) \end{vmatrix} = u(x)v'(x) - u'(x)v(x)$$

is identically zero. (Note: same $p_2 \Rightarrow$ same eigenvalue)

Note: linearly dependent $\Leftrightarrow$ one is a multiple of the other.

Proof. If e.g. $u(x) \not\equiv 0$, then $u(x_0) \neq 0$ in some point, and there is some $\varepsilon > 0$ so that $u(x) \neq 0$ in $[x_0 - \varepsilon, x_0 + \varepsilon]$. Define $z(x) = \dfrac{v(x)}{u(x)}$, $x \in [x_0 - \varepsilon, x_0 + \varepsilon]$. If $W(x) \equiv 0$, then $z' = \dfrac{v'u - u'v}{u^2} \equiv 0$ in $[x_0 - \varepsilon, x_0 + \varepsilon] \Rightarrow$ z is a constant, and $v(x) = cu(x)$ for some constant c. Define $y(x) := v(x) - cu(x)$.

Then y is a solution of (5), and $y \equiv 0$ in $[x_0-\varepsilon, x_0+\varepsilon]$. In particular, $y(x_0) = 0 = y'(x_0)$. By lemma 4.8, $y(x) \equiv 0$ in (a,b), i.e., $v(x) = cu(x)$ for $x \in (a,b)$.

Conversely, if they are linearly dependent then obviously $W(x) \equiv 0$. $\square$

4.10 Thm. If the boundary condition at least in one of the two end points a and b is of the type

$$\alpha u(a) + \beta u'(a) = 0 \qquad (|\alpha| + |\beta| \neq 0),$$

then all eigenvalues are simple.

Proof. Suppose that u and v are two real eigenfunctions with the same eigenvalue λ. Then

$$(pu')' + (q+\lambda r)u = 0 = (pv')' + (q+\lambda r)v.$$

Define the "Wronskian" ("Wronski's determinant") (multiplied by $p(x)$):

$$W(x) = p(x)\begin{vmatrix} u(x) & u'(x) \\ v(x) & v'(x) \end{vmatrix} = p(x)[uv' - u'v]$$

$$\Rightarrow \quad W'(x) = \cancel{u'pv'} + u[pv']' - \cancel{pu'v'} - [pu']'v$$

$$= u[-q-\lambda r]v - [-q-\lambda r]uv$$

$$= 0.$$

Thus, $W(x) = \text{constant} = W(a)$. By the computation on page 130 we get $W(a) = 0$, so $W(x) \equiv 0$. Since $p(x) \neq 0$ for $x \in (a,b)$, we get

$$u(x)v'(x) - u'(x)v(x) \equiv 0.$$

By lemma 4.9, u and v are linearly dependent. $\square$

4.10a Thm If $p(a) = 0$ or $p(b) = 0$ (this is the singular case with finite a, b), then all the eigenvalues are simple (when we require the eigenfunctions and their derivatives to be bounded at the critical point).

Proof: The proof is the same as on the previous page. (Recall that we always require $p(x) > 0$ for $a < x < b$.)

4.10b Ex. Take the equation

$$y'' + \lambda y = 0$$

with <u>periodic</u> boundary conditions:

$$y(0) = y(1), \qquad y'(0) = y'(1).$$

For each eigenvalue $\boxed{\lambda = (n\pi)^2}$ $n = 1, 2, 3, \dots$, we get <u>two</u> linearly independent eigenfunctions, namely $\sin(n\pi x)$ and $\cos(n\pi x)$. However for the eigenvalue $\boxed{n=0}$ we get <u>only one</u> eigenfunction $y(x) \equiv 1$.

4.11 Ex. $y'' + \lambda y = 0$, $y(0) = y(1) = 0$.

Eigenvalues $\lambda_n = (n\pi)^2$, $n = 1, 2, 3, \dots$,
eigenfunctions $\sin(n\pi x)$.
The eigenvalues are simple.

4.12 Ex. Legendre's diff. eq.

$$(1-x^2)y'' - 2xy' + \lambda y = 0$$

is self-adjoint:

$$[(1-x^2)y']' + \lambda y = 0, \quad \text{with}$$

$$p(x) = 1-x^2, \quad -1 \le x \le 1, \quad r(x) \equiv 1.$$

This is a singular problem since $p(-1) = p(1) = 0$. The only bounded solutions that we found were $P_n(x)$, with $\lambda = n(n+1)$, $n = 0, 1, 2, \dots$. The eigenvalues are: $\lambda_n = n(n+1)$, $n = 0, 1, 2, \dots$, and the eigenfunctions are $P_n(x)$. These are orthogonal since $p(-1) = p(1) = 0$ (Ex. 3.4, p. 129). The eigenvalues are still simple (if we require solutions to be bounded).

4.13 Ex. Bessel's equation $(xy')' + \lambda^2 xy = 0$ (parameter $n = 0$). Here $p(x) = x$ with $p(0) = 0$, so this problem is also singular (irregular = not regular). If we require the solution to be bounded (= begränsad) at the origin and to satisfy $y(1) = 0$ then we get the eigenvalues α_k, where α_k are the zeros of $J_0(x)$ in $(0, \infty)$, and the eigenfunctions are $J_0(\alpha_k x)$. Because of Thm 4.10a the eigenvalues are simple.

4.14 Thm The highest possible multiplicity is two.

Proof. All eigenfunctions v corresponding to the eigenvalue λ_n satisfy the same diff. eq.

$$\circledast \quad (pv')' + (q + \lambda_n r)v = 0.$$

The general solution of a second order differential equation such as $\circledast$ is given by

$$v = c_1 v_1 + c_2 v_2,$$

where v_1 and v_2 are solutions of $\circledast$, and c_1 and c_2 constants. $\Rightarrow$ At most two linearly independent solutions remain after we impose bdry. cond.

VIII. 5 An Infinite Interval and Other Irregular Problems

Sometimes we need to study a Sturm–Liouville problem

$$(1) \quad (pv')' + qv + \lambda r v = 0$$

on an infinite interval, e.g. $[0, \infty)$ or $(-\infty, \infty)$. These problems are always regarded to be irregular (singular) because of the interval. In this case we can run into difficulties in the computation of the inner product

$$\langle f, g \rangle = \int_a^b f(x)\overline{g(x)}\, r(x)\, dx.$$

5.1 Defn. We say that f belongs to the space $L^2(a,b;r)$ ("L^2 over the interval (a,b) with weight function r") iff (f is Lebesgue measurable and) f's norm is finite:

(over)

$$\|\phi\|^2 = \int_a^b |\phi(x)|^2 r(x)\,dx < \infty.$$

5.2 Lemma If both $\phi \in L^2(a,b;r)$ and $g \in L^2(a,b;r)$, then the inner product

$$\langle \phi, g \rangle = \int_a^b \phi(x)\overline{g(x)}\, r(x)\,dx$$

is well-defined, i.e., the integral converges (even absolutely).

Proof: Take $a < A < B < b$, and use Schwartz inequality:

$$\left(\int_A^B |\phi(x)\overline{g(x)}|\, r(x)\,dx\right)^2 \le \int_A^B |\phi(x)|^2 r(x)\,dx \int_A^B |g(x)|^2 r(x)\,dx$$

$$\le \|f\|^2 \|g\|^2.$$

Let $A \to a$ and $B \to b$ to get

$$\int_a^b |\phi(x)\overline{g(x)}|\, r(x)\,dx < \infty.$$

5.2 Comment $\boxed{\text{MOST OF THE THEORY REMAINS VALID}}$.

However, we must re-interpret the boundary conditions: We start by integrating over a finite interval $[A,B]$, where $a < A < B < b$, and get a substitution term (see p. 129)

$$\Big/_A^B\, p(vu' - v'u),$$

which is required to go to zero as $A \to a$ and $B \to b$.

5.3 Defn. The functions u and v satisfy self-adjoint boundary conditions iff

$$(2) \quad \lim_{\substack{A \to a \\ B \to b}} \Big/_A^B\, p(vu' - v'u) = 0.$$

Note: If $p(x) \to 0$ as $x \to 0$ or as $x \to b$, then we can even allow unbounded (= icke begränsade) functions u and v, as long as the condition (2) is satisfied, and all the functions which we need belong to $L^2(a,b;r)$.
("All the functions" $= u, u', \cancel{u''}, v, v', \cancel{v''}$).

5.4 Note The same approach is used for a singular Sturm-Liouville problem over a finite interval. For example Bessel's equation and Legendre's equation should be treated this way. The "true" reason to why the alternative solutions N_n and Q_n must be discarded is not really that they are unbounded, but the fact that their derivatives do not belong to $L^2(a,b;r)$, and this leads to problems with the inner products

$$\langle v, \mathcal{L}u \rangle$$

which everything is based on.
(There will be a homework related to this).

5.5. Ex. Laguerre's diff. eq.

(3) $\quad xy'' + (1-x)y' + \lambda y = 0,$

on the interval $(0, \infty)$. $p_0(x) = x$, $p_1(x) = 1-x$, so

$$\int \frac{p_1}{p_0}dx = \int \frac{1-x}{x}dx = \ln x - x, \quad \text{and}$$

$$\frac{1}{p_0} e^{\int p_1/p_0 dx} = \frac{1}{x} e^{\ln x - x} = \frac{1}{x}\cdot x\cdot e^{-x} = e^{-x},$$

so we should multiply (3) by e^{-x} to get the self-adjoint version

(4) $\quad (xe^{-x}y')' + \lambda e^{-x}y = 0.$

Thus $p(x) = xe^{-x}$, $r(x) = e^{-x}$. We shall return to this equation later. The appropriate solutions will be the Laguerre polynomials, and the boundary conditions are automatically self-adjoint since $p(0) = 0$ and $y(x)\,p(x) \to 0$ as $x \to \infty$ for all (polynomial) solutions y. The natural inner product is

$$\langle \phi, g\rangle = \int_0^\infty \phi(x)\overline{g(x)}\, e^{-x}\, dx.$$

5.6. Ex. Hermite's equation on $(-\infty, \infty)$:

(5) $\quad y'' - 2xy' + 2\lambda y = 0.$

$p_0 \equiv 1$, $p_1 = -2x$, $\int \frac{p_1}{p_0}dx = -x^2$, so we multiply (5) by e^{-x^2}:

(6) $\quad (e^{-x^2}y')' + 2\lambda e^{-x^2}y = 0,$ i.e.,

$p(x) = e^{-x^2}$, $r(x) = 2e^{-x^2}$.

Polynomial solutions satisfy the boundary condition (2) automatically, and the inner product is

$$\langle \phi, g\rangle = \int_{-\infty}^{\infty} \phi(x)\overline{g(x)}\, e^{-x^2}\, dx.$$

VIII. 6 Sequences of Orthogonal Polynomials

6.1 Repetition. Legendre's polynomial $P_n(x)$ is of degree n, and these polynomials are orthogonal with weight function $r(x) \equiv 1$ on the interval $(-1, 1)$, i.e.,

$$\int_{-1}^{1} P_n(x)\, P_m(x)\, dx = 0, \quad n \neq m.$$

Note: The weight function $r(x) \equiv 1$ on any other finite interval can be reduced to this by rescaling.

6.2 More general setting. Take some "arbitrary" sequence p_n of polynomials of (exact) degree n, i.e.,

$$p_n(x) = a_n^n x^n + a_{n-1}^n x^{n-1} + \dots + a_0^n, \quad a_n^n \neq 0,$$

(here n = a superscript in a^n, not a power). We also suppose that the sequence is orthogonal w.r.t. some weight function r on some interval (a, b) (finite or infinite).

6.2 Thm. Every polynomial q of degree n can be written in the form

$$q = c_n p_n + c_{n-1}p_{n-1} + \dots + c_0 p_0.$$

Proof. Write $q(x) = b_n x^n + b_{n-1}x^{n-1} + \dots + b_0$. The theorem is equivalent to the following: The system of equations below always have a solution (compare the coefficients of the different powers of x):

$$c_n a_n^n = b_n \qquad (\Rightarrow c_n = b_n / a_n^n)$$

$$c_n a_{n-1}^n + c_{n-1} a_{n-1}^{n-1} = b_{n-1} \qquad (\Rightarrow c_{n-1} = \cdots)$$

etc. This system indeed always has a solution, since we assume that $a_m^m \neq 0$ for all $0 \le m \le n$: First equation gives c_n, second gives c_{n-1}, third gives c_{n-2}, etc.

6.3 Thm. If q is an arbitrary polynomial of degree $m < n$, then $q \perp p_n$, i.e.,

$$\int_a^b q(x) \overline{p_n(x)}\, r(x) dx = 0 .$$

Proof. Write $q(x) = c_m p_m(x) + c_1 p_{m-1}(x) + \cdots + c_0 p_0$. As

$$\int_a^b p_k(x) \overline{p_n(x)}\, dx = 0 \quad \text{for all } k < n,$$

we must also have $\int_a^b q(x) \overline{p_n(x)}\, dx = 0$.

Conversely: If $p_n(x) \perp q(x)$ for all polynomials q of degree $< n$, then the sequence $\{p_n\}$ is orthogonal.

(Proof trivial).

6.4 Thm. Suppose that all the polynomials p_n are real. Then each polynomial p_n has exactly n zeros in the open interval (a,b) (and no other zeros).

Compare this to homework 37).

Proof. Case $n=0$ is trivial (p_0 is a constant). Case $n \ge 1$. Suppose that p_n changes sign k times in the interval (a,b). Denote the points where p_n changes sign by

$$a < x_1 < x_2 < \cdots < x_k < b.$$

There must be at least one such point since (recall that we always assume $r(x) > 0$ for $a < x < b$): $p_n \perp p_0$, so

$$0 = \int_a^b p_0\, p_n(x)\, r(x) dx = \text{const} \cdot \int p_n(x) \underbrace{r(x)}_{>0} dx.$$

Make a new polynomial:

$$q(x) = (x - x_1)(x - x_2) \cdots (x - x_k).$$

Then we must have

$$\int_a^b \underbrace{q(x) p_n(x)}_{\ge 0} \underbrace{r(x)}_{>0} dx \neq 0$$

(zero when $x = x_k$)

By Theorem 6.3, this is impossible if $k < n$. Thus $k = n$. ∎

6.5 Thm. The polynomials (orthogonal sequence p_k of) satisfy the following recursion equation:

$$(1) \qquad \alpha_n p_{n+1} - (x + \beta_n) p_n + \gamma_n p_{n-1} = 0,$$

where $\alpha_n = \dfrac{a_n^n}{a_{n+1}^{n+1}}$, $\beta_n = \left(\dfrac{a_n^{n+1}}{a_{n+1}^{n+1}} - \dfrac{a_{n-1}^n}{a_n^n} \right)$

$$\gamma_n = \frac{\overline{a_{n-1}^{n-1}}}{a_n^n} \frac{\|p_n\|^2}{\|p_{n-1}\|^2} \quad \left(= \alpha_{n-1} \frac{\|p_n\|^2}{\|p_{n-1}\|^2} \text{ if the coeff. are real} \right)$$

Note: $\alpha_n = 1$ if $a_k^k = 1$ for all k

$$\|p_n\|^2 = \int_a^b |p_n(x)|^2 r(x) dx.$$

Proof: First we adjust α_n so that the polynomial $\alpha_n p_{n+1} - x p_n$ is of degree $\le n$, i.e., $\alpha_n = a_n^n / a_{n+1}^{n+1}$. This gives

(2) $\alpha_n p_{n+1} - x p_n = h(x)$, with degree $\le n$.

Write $h(x) = \sum_{l=0}^{n} c_l p_l = c_n p_n + c_{n-1} p_{n-1} + \dots + c_0 p_0$.

Multiply (2) by $\overline{p_k}(x) r(x)$, $k \le n$ and integrate:

$$\langle h, p_k \rangle = c_k \|p_k\|^2 = c_k \int_a^b |p_k(x)|^2 r(x)\,dx$$
$$= \int_a^b (\alpha p_{n+1} - x p_n)\, \overline{p_k}\, r \, dx \Rightarrow$$

(*) $$c_k = \frac{1}{\|p_k\|^2} \int_a^b \left[\alpha_n p_{n+1}(x) - x p_n(x)\right] \overline{p_k(x)}\, r(x)\,dx.$$

Thm 6.3 gives $c_k = 0$ for $k \le n-2$, since $p_k(x)$ and $x p_k(x)$ are polynomials of degree $\le n-1$. Thus,

$$\alpha_n p_{n+1} - x p_n = c_n p_n + c_{n-1} p_{n-1}.$$

We can find the coefficient $c_n = \beta_n$ by comparing the powers of x^n:

$$\alpha_n a_n^{n+1} - a_{n-1}^{n} = c_n a_n^n, \quad \text{and thus}$$

$$\beta_n = c_n = (\alpha_n a_n^{n+1} - a_{n-1}^n)/a_n^n = \frac{a_n^{n+1}}{a_{n+1}^{n+1}} - \frac{a_{n-1}^n}{a_n^n}.$$

The coefficient $c_{n-1} = -\gamma_n$ we get, e.g., from (*) with $k = n-1$

$$-\gamma_n = c_{n-1} = \frac{-1}{\|p_{n-1}\|^2} \int_a^b p_n(x)\, x\, \overline{p_{n-1}(x)}\, r(x)\,dx$$
$$= \frac{-1}{\|p_{n-1}\|^2} \int_a^b p_n(x) \left[\overline{a}_{n-1}^{n-1} x^n + \text{lower order terms}\right] r(x)\,dx$$
$$= -\frac{\overline{a}_{n-1}^{n-1}}{\|p_{n-1}\|^2} \int_a^b x^n p_n(x)\, r(x)\,dx.$$

Then we apply Lemma 6.6 below. ∎

6.6. Lemma $$\int_a^b x^n \overline{p_n(x)}\, r(x)\,dx = \frac{1}{a_n^n} \|p_n\|^2.$$

Proof: $$\|p_n\|^2 = \int_a^b \left[a_n^n x^n + \text{lower order terms}\right] \overline{p_n(x)}\, r(x)\,dx$$
$$= \int_a^b a_n^n x^n \overline{p_n(x)}\, r(x)\,dx$$
$$= a_n^n \int_a^b x^n \overline{p_n(x)}\, r(x)\,dx. \quad ∎$$

6.7. Generalized Rodriguez formula: All the polynomials in this course satisfy, in addition, a formula of the type

$$p_n(x) = c_n \frac{1}{r(x)} \frac{d^n}{dx^n} \frac{[p(x)]^n}{r(x)^{n-1}},$$

where c_n is a constant. For example: Legendre has $r(x) \equiv 1$ and $p(x) = (1-x^2)$. This formula requires p and r to have suitable properties: the formula must produce a polynomial of degree n! (although r and p themselves need not be polynomials, see p. 143).

VIII.7 Generalized Fourier Series for Regular Sturm–Liouville Problems

We now suppose that the problem is regular: the interval is finite, and $p(x) > 0$ and $r(x) > 0$ for all $x \in [a,b]$. We also use Sturm–Liouville boundary conditions:

$$\begin{cases} \alpha y(a) + \beta y'(a) = 0 & |\alpha| + |\beta| \ne 0, \\ \gamma y(b) + \delta y'(b) = 0 & |\gamma| + |\delta| \ne 0. \end{cases}$$

7.1 Thm. In this case (see above) the eigenvalues are simple and real, and we can take the eigenfunctions to be real.

Proof: Eigenvalues simple: Thm 4.10, p.138
— " — real: Thm 4.4, p.134
Eigenfunctions real: Thm 4.5, p.135.

7.2 Thm: There always exists infinitely many eigenvalues λ_n with corresponding eigenfunctions v_n, and $\lambda_n \to \infty$ as $n \to \infty$. Moreover, the sequence $\{v_n\}$ is complete.

Proof: See course on "Hilbert rum". (rather long).

By the general theory in Chapter VI, every ϕ ($\in L^2(a,b;r)$) can be written as a generalized Fourier series

$$\phi(x) = \sum_{n=0}^{\infty} \frac{\langle \phi, v_n \rangle}{\|v_n\|^2} v_n ,$$

where $\langle \phi, g \rangle = \int_a^b \phi(x) g(x) r(x) dx$.

The series converges in the norm $\|\cdot\|$ induced by the inner product $\langle , \rangle$, i.e., $\|\phi - \phi_N\| \to 0$ as $N \to \infty$, where

$$\phi_N = \sum_{n=0}^{N} \frac{\langle \phi, v_n \rangle}{\|v_n\|^2} v_n$$

is a truncated ("kapad") Fourier-series. We also have

Parseval's identity:

$$\|f\|^2 = \int_a^b |\phi(x)|^2 r(x) dx = \sum_{n=0}^{\infty} \left| \frac{\langle \phi, v_n \rangle}{\|v_n\|} \right|^2$$

Thus, if we denote the (non-normalized) generalized Fourier series by

$$c_n = \frac{\langle \phi, v_n \rangle}{\|v_n\|^2} , \quad \text{then}$$

$$\phi(x) = \sum_{n=0}^{\infty} c_n v_n , \quad \text{and}$$

$$\boxed{\|\phi\|^2 = \sum_{n=0}^{\infty} \|v_n\|^2 |c_n|^2} \quad \boxed{\text{Non-normalized Parseval}}$$

7.3 Thm. The series above converges at every point $x \in [a,b]$ at least in the following case:

- $\phi(x)$ is continuous in $[a,b]$.
- $\phi'(x)$ is piece-wise continuous in $[a,b]$.

(that is, we allow ϕ' to have jump discontinuities).

Proof. Course in Fourier transforms (maybe?) This is not the least possible result).

7.4 Integral Transform Interpretation: The "Generalized Fourier transform" of ϕ is the sequence

$$F(n) = \frac{\langle \phi, v_n \rangle}{\|v_n\|^2} ,$$

and we recover ϕ from its transform by

$$\phi = \sum_{n=0}^{\infty} F(n) v_n .$$

Note: Other normalizations do exist. For example,

$$F(n) = \langle \phi, v_n \rangle, \quad \phi = \sum_{n=0}^{\infty} \frac{F(n)}{\|v_n\|^2} v_n, \quad \text{etc.}$$

7.5 Ex. Sines-series and cosine-series are of this type: The Sturm-Liouville problem is

$$y''+\lambda y=0,\quad y(0)=y(L)=0 \quad (\tfrac{1}{2}\text{ wave sines})$$

$$y''+\lambda y=0,\quad y'(0)=y'(L)=0 \quad (\tfrac{1}{2}\text{ wave cosines})$$

$$y''+\lambda y=0,\quad y'(0)=y(L)=0 \quad (\tfrac{1}{4}\text{ wave cosines})$$

The standard (mixed sines and cosines) series (the full wave series) comes from the same equation with periodic boundary conditions

$$y''+\lambda y=0,\qquad \begin{matrix} y(0)=y(L)\\ y'(0)=y'(L).\end{matrix}$$

7.6. Ex. We studied the heat equation in a hollow pipe on pages 67–68. We got Bessel's equation with index $n=0$:

$$R''+\tfrac{1}{\rho}R'+\lambda^2R=0,$$

and solutions $R_k(\rho)=a_kJ_0(\alpha_k\rho)+b_kN_0(\alpha_k\rho)$. The boundary conditions $R(r_1)=0$, $R'(r_2)=0$ were of Sturm-Liouville type. Also otherwise the problem is regular. The boundary conditions gave us:

$$\begin{cases} J_0(\alpha_kr_1)a_k+N_0(\alpha_kr_1)b_k=0\\ J_0'(\alpha_kr_2)a_k+N_0'(\alpha_kr_2)b_k=0\end{cases}$$

which has a solution $(a_k,b_k)\neq(0,0)$ iff

$$\begin{vmatrix} J_0(\alpha_kr_1) & N_0(\alpha_kr_1)\\ J_0'(\alpha_kr_2) & N_0'(\alpha_kr_2)\end{vmatrix}=0.$$

$$\Longleftrightarrow\quad J_0(\alpha_kr_1)N_0'(\alpha_kr_2)=J_0'(\alpha_kr_2)N_0(\alpha_kr_1)$$

By the general theory, this equation has infinitely many solutions α_k, and $\alpha_k\to\infty$ as $k\to\infty$. We can, for example take $a_k=1$ and

$$b_k=-\frac{J_0(\alpha_kr_1)}{N_0(\alpha_kr_1)}=-\frac{J_0'(\alpha_kr_2)}{N_0'(\alpha_kr_2)}\;)$$

or we can choose $a_k=\dfrac{1}{J_0(\alpha_kr_1)}$ and $b_k=-\dfrac{1}{N_0(\alpha_kr_1)}$

$$R_k(\rho)=\frac{J_0(\alpha_k\rho)}{J_0(\alpha_kr_1)}-\frac{N_0(\alpha_k\rho)}{N_0(\alpha_kr_1)}=a_kJ_0(\alpha_k\rho)+b_kN_0(\alpha_k\rho)$$

(this is a more symmetric choice). By the general theory, this gives us an orthogonal and complete sequence with respect to the inner product

$$\int f(\rho)\,\overline{g(\rho)}\,\rho\,d\rho.$$

[16]JACQUES CHARLES FRANÇOIS STURM (1803—1855), was born and studied in Switzerland and then moved to Paris, where he later became the successor of Poisson in the chair of mechanics at the Sorbonne.

JOSEPH LIOUVILLE (1809—1882), French mathematician and professor in Paris, contributed to various fields in mathematics and is particularly known by his important work in complex analysis (Liouville's theorem; Sec. 13.6), special functions, differential geometry, and number theory.