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VIII Sturm–Liouville Theory

or alternatively,

Orthogonal solutions to differential equations

In this chapter we develop an orthogonality theory for the eigenfunctions of a differential operator $\mathcal{L}$ of the type

(1) $\mathcal{L}u = p_0(x)u''(x) + p_1(x)u'(x) + p_2(x)u.$

It would be possible to let the functions be complex, but to simplify the notations we assume in this chapter that

$\boxed{\text{all the functions are real.}}$ (with one small exception)

VIII.1 The Adjoint Differential Operator

1.1 Problem: Rewrite the expression $\langle v, \mathcal{L}u\rangle$ in such a way that derivatives on u have been replaced by derivatives on v. Here u and v are two times continuously differentiable functions, and we use the "standard" inner product

(2) $\langle v, u\rangle = \int_a^b v(x)u(x)\,dx$

(weight function $w(x) \equiv 1$, no complex conjugate since the functions are real).

Solution. "If you don't know what to do, then integrate by parts" (author unknown).

$$\langle v, \mathcal{L}u\rangle = \int_a^b v(x)\left[p_0(x)u'' + p_1 u' + p_2 u\right]dx$$

(over)

$$\int_a^b v(x)p_1(x)u'(x)\,dx = \Big/_a^b v(x)p_1(x)u(x) - \int_a^b [v(x)p_1(x)]'u(x)\,dx;$$

$$\int_a^b v(x)p_0(x)u''(x)\,dx$$
$$= \Big/_a^b v(x)p_0(x)u'(x) - \int_a^b [v(x)p_0(x)]'u'(x)\,dx$$
$$= \Big/_a^b \left\{v(x)p_0(x)u'(x) - [v(x)p_0(x)]'u(x)\right\}$$
$$+ \int_a^b [v(x)p_0(x)]''u(x)\,dx.$$

The sum is:

$$\langle v, \mathcal{L}u\rangle = \Big/_a^b \left\{v(x)p_1(x)u(x) + v(x)p_0(x)u'(x) - [v(x)p_0(x)]'u(x)\right\}$$

(3) $$+ \int_a^b \left\{[v(x)p_0(x)]'' - [v(x)p_1(x)]' + [v(x)p_2(x)]\right\}u(x)\,dx.$$

1.2 Defn. The differential operator

(4) $\mathcal{L}^*v = [p_0(x)v(x)]'' - [p_1(x)v(x)]' + p_2(x)v(x)$

is called the adjoint of $\mathcal{L}$.

(Note: In the complex case we must add complex conjugates on $p_0(x)$, $p_1(x)$ and $p_2(x)$ in the expression for $\mathcal{L}^*$.)

1.3 Thm. The following conditions are equivalent:

i) The substitution term in formula (3) vanishes

ii) $\langle v, \mathcal{L}u\rangle = \langle \mathcal{L}^*v, u\rangle$.

Proof: See computation above.

1.4 Problem Compute the adjoint of $\mathcal{L}^*$ (i.e., $(\mathcal{L}^*)^*$).

Solution. Carry out the differentiations in (4) to get

$$\mathcal{L}^* v = p_0 v'' + (2p_0' - p_1)v' + [p_0'' - p_1' + p_2]v$$
$$= q_0 v'' + q_1 v' + q_2 v, \quad \text{where}$$

$$q_0 = p_0, \quad q_1 = 2p_0' - p_1, \quad q_2 = p_0'' - p_1' + p_2.$$

Therefore

$$(\mathcal{L}^*)^* u = [q_0 u]'' - [q_1 u]' + q_2 u$$
$$= q_0 u'' + [2q_0' - q_1]u' + [q_0'' - q_1' + q_2]u$$

(substitute the values of q_0, q_1, q_2 above)

$$= p_0 u'' + [2p_0' - 2p_0' + p_1]u' + [p_0'' - (2p_0'' - p_1') + (p_0'' - p_1' + p_2)]u$$
$$= p_0 u'' + p_1 u' + p_2 u := \mathcal{L} u.$$

Thus, we get

1.5 Thm. $(\mathcal{L}^*)^* = \mathcal{L}$.

1.6. Defn. We call $\mathcal{L}$ self-adjoint ("självadjungerad") if $\boxed{\mathcal{L}^* = \mathcal{L}}$.

When is $\mathcal{L}$ self-adjoint?

1.7. Thm. The following conditions are equivalent:

i) $\mathcal{L}$ is selfadjoint

ii) $p_0' = p_1$

iii) $\mathcal{L} u = (p_0 u')' + p_2 u$.

Proof: With the notation on the preceding page, we have

$$\mathcal{L} = \mathcal{L}^* \iff p_0 = q_0, \; p_1 = q_1, \; p_2 = q_2$$
$$\iff 2p_0' - p_1 = p_1 \text{ and } p_0'' - p_1' + p_2 = p_2$$
$$\iff p_0' = p_1$$
$$\iff \mathcal{L} u = p_0 u'' + p_0' u' + p_2 u$$
$$\iff \mathcal{L} u = [p_0 u']' + p_2 u. \qquad \square$$

(Note: A similar but more complicated relationship is true in the complex case. Usually p_0, p_1 and p_2 are real.)

VIII.2 Second Order Selfadjoint Differential Equations

We can always rewrite the differential equation

(1) $\quad p_0(x) u''(x) + p_1(x) u'(x) + p_2(x) u(x) = 0$

or $\mathcal{L} u = 0$.

2.1. Defn. The equation (1) is self-adjoint iff $\mathcal{L}$ is self-adjoint.

Thus, from Thm. 1.7 above we get (over)

2.2 Thm. The following conditions are equivalent:

i) The equation (1) is self-adjoint

ii) $p_1 = p_0'$

iii) (1) can be written in the form

$$(p_0 u')' + p_2 u = 0,$$

2.3 Problem: What if $p_1 \neq p_0'$? Can we always rewrite (1) (without changing its set of solutions) so that it becomes self-adjoint?

Answer: Yes,

Solution: Multiply (1) by $r(x)$ to get

$$r p_0 u'' + r p_1 u' + r p_2 = 0, \quad \text{and}$$

require that $[r p_0]' = r p_1$. This means:

$$[r p_0]' = r p_1 \iff r' p_0 + r p_0' = r p_1$$

$$\iff r' p_0 = r(p_1 - p_0')$$

$$\iff \frac{r'}{r} = \frac{p_1 - p_0'}{p_0}$$

$$\iff \ln r = \int \frac{p_1}{p_0} dx - \ln p_0 \; (+\ln c)$$

$$\iff \ln(p_0 r) = \int \frac{p_1}{p_0} dx \implies$$

$$(2) \quad \boxed{r(x) = \frac{1}{p_0(x)} e^{\int \frac{p_1(x)}{p_0(x)} dx}}$$

(Thus, we begin by dividing (1) by $p_0(x)$!)

2.4 Thm. We can make (1) self-adjoint by multiplying by the function $r(x)$ in (2). The final result is of the type

$$(3) \quad [p(x) u'(x)]' + q(x) u(x) = 0,$$

where $p(x) = e^{\int \frac{p_1}{p_0} dx}$ and $q(x) = \frac{p_2}{p_0} e^{\int \frac{p_1}{p_0} dx}$.

(Thus, we first divide by p_0, and then multiply by $e^{\int \tilde{p}_1(x) dx}$, where $\tilde{p}_1(x) = \frac{p_1(x)}{p_0(x)}$ is the new p_1).

2.5 Corollary Every equation (1) can be rewritten in self-adjoint form (3).

2.6 Ex. Bessel's equation:

$$x^2 y'' + x y' + (x^2 - n^2) y = 0 \quad (\text{divide by } x^2)$$

$$y'' + \frac{1}{x} y' + \left(1 - \frac{n^2}{x^2}\right) y = 0.$$

This should be multiplied by $e^{\int \frac{dx}{x}} = e^{\ln x} = x \implies$

$$x y'' + y' + \left(x - \frac{n^2}{x}\right) y = 0 \iff$$

$$\boxed{(x y')' + \left(x - \frac{n^2}{x}\right) y = 0} = \text{self-adjoint Bessel's eq.}$$

2.7 Ex. Associated Legendres' equation:

$$(1-x^2) y'' - 2x y' + \left[n(n+1) - \frac{n^2}{1-x^2}\right] y = 0$$

$$\iff [(1-x^2) y']' + \left[n(n+1) - \frac{n^2}{1-x^2}\right] y = 0.$$

This equation is already self-adjoint.

2.8 Ex. Harmonic oscillator

$$y'' + \alpha^2 y = 0$$

is already self-adjoint ($p_0(x) \equiv 1$).

2.9 Ex. Generalized Bessel's equation:

$$x^2 y'' + (2\alpha+1)xy' + [\beta^2\gamma^2 x^{2\beta} - (\beta^2\nu^2 - \alpha^2)]y = 0.$$

First divide by x^2, then multiply by

$$e^{\int \frac{(2\alpha+1)x}{x^2}dx} = e^{(2\alpha+1)\ln x} = x^{2\alpha+1} \text{ to get}$$

$$[x^{2\alpha+1}y']' + [\beta^2\gamma^2 x^{2\beta+2\alpha-1} - (\beta^2\nu^2-\alpha^2)x^{2\alpha-1}]y = 0.$$

2.10 Ex. Euler's equation

$$x^2y'' + axy' + by = 0 \quad \text{leads to}$$

$$[x^a y']' + b x^{a-2} y = 0, \quad \text{i.e.,}$$

$$p(x) = x^a, \quad q(x) = bx^{a-2}.$$

VIII. 3 Self-adjoint Boundary Conditions

3.1 Problem. Rewrite $\langle v, \mathcal{L}u\rangle$ so that all derivatives on u has been replaced by derivatives on v. We suppose that

$$\mathcal{L}u = [pu']' + qu$$

is self-adjoint.

Note: This is the same as problem 1.1 on p. 123, but it is simpler since $\mathcal{L}$ is self-adjoint.

Solution Integrate by parts:

$$\langle v, \mathcal{L}u\rangle = \int_a^b v\,[(pu')' + qu]\,dx$$
$$= \Big|_a^b vpu' + \int_a^b(-v'pu' + vqu)\,dx$$
$$= \Big|_a^b[vpu' - v'pu] + \int_a^b\{[v'p]'u + vqu\}\,dx$$
$$= \Big|_a^b p(vu' - v'u) + \langle\mathcal{L}v, u\rangle.$$

3.2 Thm. If $\mathcal{L}$ is self-adjoint, then

$$\langle v, \mathcal{L}u\rangle - \langle\mathcal{L}v, u\rangle = \Big|_a^b p\,[vu' - v'u].$$

Thus, $\langle v, \mathcal{L}u\rangle = \langle\mathcal{L}v, u\rangle$ iff the substitution term vanishes.

Proof: See computation above. Compare this to Thm. 1.3 on p. 123.

3.3 Defn. We say that u and v (twice continuously differentiable) satisfy self-adjoint boundary conditions ("rand villkor") at the points a and b iff

$$(1) \quad \Big|_a^b p(x)\,[v(x)u'(x) - v'(x)u(x)] = 0.$$

3.3 Corollary. If $\mathcal{L}$ is self-adjoint, then $\langle v, \mathcal{L}u\rangle = \langle\mathcal{L}v, u\rangle$ if and only if u and v satisfy self-adjoint boundary conditions.

3.4 Ex. If $p(a) = p(b) = 0$, then we need no conditions on u and v, because the boundary term then always vanishes (warning: still, we cannot let u and v "blow up" at a and b, at least not "too badly").

3.5 Ex. If $p(a) = 0$ and $p(b) \neq 0$, then the condition becomes

$$v(b)u'(b) = v'(b)u(b).$$

3.6 Ex. Choose arbitrary constants $\alpha, \beta, \gamma, \delta$, so that at least one of α and β is $\neq 0$ and ——"—— γ and δ is $\neq 0$, and impose the "Robin" boundary conditions: or "Sturm–Liouville" or "mixed"

$$\underbrace{\begin{aligned}\alpha v(a)+\beta v'(a)&=0\\ \alpha u(a)+\beta u'(a)&=0\end{aligned}}_{\text{point } x=a} \qquad \underbrace{\begin{aligned}\gamma v(b)+\delta v'(b)&=0\\ \gamma u(b)+\delta u'(b)&=0\end{aligned}}_{\text{point } x=b.}$$

If, for example, $\beta \neq 0$, then

$$v'(a) = -\frac{\alpha}{\beta}v(a), \quad u'(a) = -\frac{\alpha}{\beta}u(a).$$

Insert this into (1) to get

$$p(a)\left[v(a)u'(a) - u(a)v'(a)\right] = 0.$$

A similar computation is valid at the point b. Thus, we get the following thm:

3.7 Thm. The boundary conditions

$$(2)\begin{cases}\alpha u(a)+\beta u'(a)=0 & (|\alpha|+|\beta|\neq 0)\\ \gamma u(b)+\delta u'(b)=0 & (|\gamma|+|\delta|\neq 0)\end{cases}$$

are self-adjoint (if we impose the same conditions on both u and v).

3.8 Defn. The boundary conditions (2) are called Sturm–Liouville (or Robin) boundary condition. (or (mixed))

3.9 Ex. Periodic boundary condition:

If u and v are periodic with period $b-a$, then

$$\begin{aligned}u(a)&=u(b),\\ v(a)&=v(b),\\ u'(a)&=u'(b),\\ v'(a)&=v'(b),\end{aligned}$$

and in this case, too, the substitution terms vanish.

The preceding examples cover all the examples which we have seen so far:

p. 17 (sine series) Ex. 3.7

p. 26, function Φ: Ex 3.9 (periodic)

p. 68 (hollow cylinder): Ex 3.7

pp. 70–71 (massive cylinder) Use Ex 3.5

p. 117, function Φ: Ex 3.9 (periodic)
Legendre's eq: Ex 3.4 (weight function zero).

Additional examples in the home works

VIII. 4 Eigenvalues and Eigenfunctions of Differential Operators

Repetition: In linear algebra we treat square matrices $A \in \mathbb{R}^{n\times n}$ or $A \in \mathbb{C}^{n\times n}$ and define:

λ is an eigenvalue of A with eigenvector v
$\Longleftrightarrow$ $Av = \lambda v$ and $v \neq 0$ (here $\lambda \in \mathbb{R}$ or $\lambda \in \mathbb{C}$).

More generally, we have two matrices A and B and look for all numbers λ and vectors $v \neq 0$ such that