VII.2 Recursion Formulas for Legendre Polynomials

The basic recursion formulas for Legendre polynomials can be derived in many different ways. A simple method which involves a lot of work is the following:

Idea: $P_{n+1}(x)$ contains the powers $x^{n+1}, x^{n-1}, x^{n-3} \dots$

$x P_n(x)$ contains the same powers.

By adjusting the coefficient α in the expression $P_{n+1}(x) - \alpha x P_n(x)$ we can kill off the highest order term x^{n+1}!

Coefficient of x^{n+1} in P_{n+1} is $\dfrac{(2n+2)!}{2^{n+1}((n+1)!)^2}$

Coefficient of x^{n+1} in $xP_n(x)$ is $\dfrac{(2n)!}{2^n(n!)^2}$.

Choose $\alpha = \dfrac{(2n+2)(2n+1)}{2(n+1)^2} = \dfrac{2n+1}{n+1}$.

$\Rightarrow P_{n+1}(x) - \dfrac{2n+1}{n+1} x P_n(x)$ is a polynomial of degree $\leq n-1$.

By subtracting a multiple of P_{n-1} we can kill off the x^{n-1}-term. The result is rather surprising: "By accident" all the other terms are eliminated, too.

2.1 Thm. $\boxed{(n+1) P_{n+1}(x) - (2n+1) x P_n(x) + n P_{n-1}(x) = 0}$

Interpretation: $xP_n(x)$ is a "weighted average" of P_{n+1} and P_{n-1}:

$$x P_n(x) = \frac{n}{2n+1} P_{n-1} + \frac{n+1}{2n+1} P_{n+1} \quad \left(\begin{array}{l}\text{sum of} \\ \text{coefficients} \\ \text{is } =1\end{array}\right)$$

We can also use the Generating Function.

$$F(x,t) = \sum_{n=0}^{\infty} P_n(x) t^n$$

(the Z-transformation of the sequence $P_n(x)$)

Problem: Compute $F(x,t)$

We proceed as in the case of Bessel functions:

$$\frac{\partial}{\partial t} F(x,t) = \sum_{n=1}^{\infty} n P_n(x) t^{n-1} \underset{n-1 \to n}{=} \sum_{n=0}^{\infty} (n+1) P_{n+1} t^n$$

(use recursion formula) $[(n-1 \to n)]$

$$= \sum_{n=0}^{\infty} \left[-n P_{n-1}(x) + (2n+1) x P_n(x)\right] t^n$$

$$= \sum_{n=0}^{\infty} -(n+1) P_n(x) t^{n+1} + x \sum_{n=0}^{\infty} (2n+1) P_n t^n$$

$$= -t^2 \sum_{n=0}^{\infty} n P_n t^{n-1} - t \sum_{n=0}^{\infty} P_n t^n$$

$$+ 2xt \sum_{n=0}^{\infty} n P_n t^{n-1} + x \sum_{n=0}^{\infty} P_n t^n$$

$$= \underbrace{(2xt - t^2) \frac{\partial}{\partial t} F(x,t)}_{\text{move over}} + (x-t) F(x,t) \Rightarrow$$

$$\frac{\partial}{\partial t} F(x,t) = \frac{x-t}{1-2xt+t^2} F(x,t)$$

$$= -\frac{1}{2} \frac{-2x+2t}{1-2xt+t^2} F(x,t)$$

$$= -\frac{1}{2} \left[\frac{d}{dt} \ln(1-2xt-t^2)\right] F(x,t).$$

Separate variables (divide by $F(x,t)$), integrate $\Rightarrow$

$$F(x,t) = \frac{C(x)}{(1-2xt+t^2)^{1/2}}$$

Determine $C(x)$ by setting $t=0 \Rightarrow$

$$F(x,0) = C(x) = P_0(x) \equiv 1.$$

2.2 Thm. The generating function

$$F(x,t) = \sum_{n=0}^{\infty} P_n(x)\, t^n \qquad \boxed{-1 \le x \le 1,\ |t| < 1}$$

of the Legendre polynomials $(n = 0, 1, 2, \dots)$ is given by

$$F(x,t) = \frac{1}{\sqrt{1-2xt+t^2}}.$$

From this formula we can deduce many new recursion formulas:

2.3 Thm. The following recursion formulas are valid for Legendre polynomials:

i) $\displaystyle\sum_{n=0}^{\infty} P_n(x)\, t^n = \frac{1}{\sqrt{1-2xt+t^2}}$, $-1 \le x \le 1$, $|t|$ little.

ii) $P_n(1) = 1$ for all n,

iii) $P_n(-1) = (-1)^n$ — " — .

iv) $P_n = P_{n+1}' - 2x P_n' + P_{n-1}'$

v) $(n+1)P_{n+1} - (2n+1)xP_n + nP_{n-1} = 0$

vi) $(n+1)P_{n+1}' - (2n+1)(xP_n)' + nP_{n-1}' = 0$

vii) $P_{n+1}' = (n+1)P_n + xP_n'$

viii) $(2n+1)P_n = P_{n+1}' - P_{n-1}'$

ix) $P_{n-1}' = xP_n' - nP_n$

take $P_{-1} \equiv 0$, $P_0 \equiv 1$, $P_1(x) = x$

Proofs: i) = Thm 2.2

v) = Thm 2.1

vi) = derivative of v)

vii) = eliminate P_{n-1}' from iv) and vi)

viii) = eliminate xP_n' from vi) and vii)

ii) Choose $x = 1$ in i):

$$\sum_{n=0}^{\infty} P_n(1)\, t^n = \frac{1}{(1-2t+t^2)^{1/2}} = \frac{1}{1-t}$$

$$= 1 + t + t^2 + \dots \implies P_n(1) = 1 \text{ for all } n.$$

Radius of convergence

$$F(x,t) = \frac{1}{\sqrt{1-2xt+t^2}}, \qquad -1 \le x \le 1.$$

Singular where $t^2 - 2xt + 1 = 0$, and the radius of convergence is the absolute value of the singularity closest to the origin.

$$t^2 - 2xt + 1 = 0 \iff$$

$$t = x \pm \sqrt{x^2-1} \qquad (|x| \le 1)$$

$$= x \pm i\sqrt{1-x^2}, \text{ so}$$

$$|t|^2 = x^2 + (1-x^2) = 1, \text{ i.e.}$$

The radius of convergence is $\boxed{1}$ for all x, $\boxed{-1 \le x \le 1}$

$(t = \pm 1$ for $x = 0$, $t = \pm i$ for $x = 0)$

iii) A similar computation (take $t=-1$ in ii))

ix) Eliminate P_{n+1}' from iv) and vii).

The only "hard" one is iv) =

iv): Differentiate $F(x,t)$ w.r.t. x:

$$\sum_{n=0}^{\infty} P_n'(x)\,t^n = \frac{\partial}{\partial x}\,\frac{1}{(1-2xt+t^2)}$$

$$= \cdots = \frac{t}{1-2xt+t^2}\,F(x,t), \text{ so}$$

$$F(x,t) = \sum_{n=0}^{\infty} P_n(x)t^n = \frac{1-2xt+t^2}{t}\sum_{n=0}^{\infty} P_n'(x)t^n$$

$$= \cdots = \sum_{n=0}^{\infty}\left(P_{n+1}' - 2xP_n' + P_{n-1}'\right)t^n \Rightarrow$$

$$P_n(x) = P_{n+1}' - 2xP_n' + P_{n-1}'$$

(this computation was abbreviated by $\frac{1}{3}$ page)

VII.3 Representation Formulas for Legendre Polynomials

3.1 Thm (Rodrigue's formula)

$$P_n(x) = \frac{1}{2^n n!}\left(\frac{d}{dx}\right)^n (x^2-1)^n.$$

Proof: "Easy" but a lot of work. Write

$$(x^2-1)^n = \sum_{k=0}^{n}\binom{n}{k}(-1)^k(x^2)^{n-k}$$

$$= \sum_{k=0}^{n}\frac{n!\,(-1)^k}{k!\,(n-k)!}\,x^{2n-2k},$$

and differentiate n times. This leads to the formula for P_n given on p. 102.

3.2 Thm $$P_n(x) = \frac{1}{2^{n+1}\pi i}\oint_\gamma \frac{(\xi^2-1)^n}{(\xi-x)^{n+1}}\,d\xi,$$

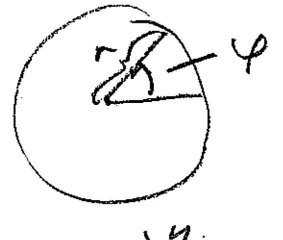
where the curve γ encircles (= går runt) the point x counterclockwise (= moturs)

This follows from a general result for analytic functions which says that for all analytic functions,

$$f^{(n)}(x) = \frac{n!}{2\pi i}\oint_\gamma \frac{f(\xi)}{(\xi-x)^{n+1}}\,d\xi.$$

If we here choose $f(x)=(x^2-1)$ then this becomes Rodrigue's formula.

From here we can derive a trigonometric integral formula for $P_n(x)$ as follows: We choose γ to be a circle with center x and radius r: $\xi = x + re^{i\varphi}$. Then $d\xi = re^{i\varphi}\cdot i\,d\varphi$, and



$$P_n(x) = \frac{1}{2^{n+1}\pi i}\int_0^{2\pi}\frac{\left((x+re^{i\varphi})^2-1\right)^n}{(re^{i\varphi})^{n+1}}\,ire^{i\varphi}\,d\varphi$$

$$= \frac{1}{2^{n+1}\pi}\int_0^{2\pi}\frac{\left(x^2-1+2rxe^{i\varphi}+r^2e^{2i\varphi}\right)^n}{(re^{i\varphi})^n}\,d\varphi$$

Let w be a complex square root to x^2-1: write x^2-1 in polar form

$$x^2-1 = |x^2-1|\,e^{i\eta},$$

where we choose the polar angle η so that $-\pi < \eta \le \pi$. Then we take

$$w = \sqrt{|x^2-1|}\cdot e^{i\psi}, \quad \text{where } \psi = \eta/2.$$

Observe that $-\pi/2 < \psi \le \pi/2$, so $\operatorname{Re} w \ge 0$.

Denote $\rho = \sqrt{|x^2-1|}$. The case $\rho = 0$, i.e., $x = \pm 1$ is special, and we return to it later. Right now, let $\rho \neq 0$, and choose $r = \rho$. Then $w = \rho e^{i\psi} = re^{i\psi}$
$\Rightarrow$ $\boxed{r = \rho = w e^{-i\psi}}$, and

$$P_n(x) = \frac{1}{2^{n+1}\pi} \int_0^{2\pi} \left[\frac{w^2 + 2x\rho e^{i\varphi} + \rho^2 e^{2i\varphi}}{\rho e^{i\varphi}}\right]^n d\varphi$$

$$= \frac{1}{2^{n+1}\pi} \int_0^{2\pi} \left[\frac{w^2 + 2xw\, e^{i(\varphi-\psi)} + w^2 e^{2i(\varphi-\psi)}}{w e^{i(\varphi-\psi)}}\right]^n d\varphi$$

$$= \frac{1}{2^{n+1}\pi} \int_0^{2\pi} \left[2x + w\left(e^{-i(\varphi-\psi)} + e^{i(\varphi-\psi)}\right)\right]^n d\varphi$$

$$= \frac{1}{2^{n+1}\pi} \int_0^{2\pi} 2^n \left(x + w\cos(\varphi-\psi)\right)^n d\varphi$$

(the function inside the integral is 2π-periodic)

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(x + \sqrt{x^2-1}\,\cos\varphi\right)^n d\varphi$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(x + i\sqrt{1-x^2}\cos\varphi\right)^n d\varphi$$

"complex square root" with Real part ≥ 0

This leads to:

<u>3.3 Thm.</u> $$P_n(x) = \frac{1}{\pi} \int_0^{\pi} \left(x + \sqrt{x^2-1}\cos\varphi\right)^n d\varphi$$

<u>Proof</u>: The computation above is valid if $x \neq \pm 1$. If $x = \pm 1$, then the result is also true by direct inspection!

<u>VII. 4 Orthogonality of the Legendre polynomials</u>

<u>4.1 Thm orthogonal product</u> The Legendre polynomials P_n are with respect to the inner

$$\langle f, g\rangle = \int_{-1}^{1} f(x)\,\overline{g(x)}\,dx,$$ and

$$\|P_n\|^2 = \int_{-1}^{1} (P_n(x))^2 dx = \frac{1}{n+1/2}.$$

<u>Proof</u>. To prove orthogonality of P_n and P_m where $m<n$ it suffices to show that $P_n \perp x^k$ for all $k \leq m$ (since P_m is of order m). We do this, integrating by parts, and using Rodrigue's formula: For all $k \leq n$:

$$\int_{-1}^{1} x^k P_n(x)\,dx = \frac{1}{2^n n!}\int_{-1}^{1} x^k \left(\frac{d}{dx}\right)^n (x^2-1)^n\,dx$$

We integrate by parts, differentiating x^k and integrating the rest a total of k times. All the substitution terms vanish if $k<n$, because of the factor $(x^2-1)^{n-r}$. Thus: After k integrations by part:

r = number of integrations by part

$$= \begin{cases} (-1)^k \dfrac{k!}{2^n n!} \displaystyle\int_{-1}^{1} \frac{d^{n-k-1}}{dx^{n-k-1}} (x^2-1)^n = 0 & \text{if } k<n \\ \dfrac{(-1)^n}{2^n} \displaystyle\int_{-1}^{1} (x^2-1)^n dx & (\text{if } k=n) \end{cases}$$

$$= \frac{1}{2^{n-1}}\int_0^1 (x^2-1)^n\,dx \qquad \begin{pmatrix} x^2 = t \\ x = \sqrt{t} \\ dx = \frac{1}{2}\frac{dt}{\sqrt{t}} \end{pmatrix}$$

$$= \frac{1}{2^n}\int_0^1 (1-t)^n t^{-1/2}\,dt$$

$$= \frac{1}{2^n} B(n+1, \tfrac{1}{2}) = \frac{1}{2^n}\frac{\Gamma(n+1)\Gamma(\frac{1}{2})}{\Gamma(n+1+1/2)}$$

$$= \frac{1}{2^n}\frac{n!\sqrt{\pi}}{(n+1/2)(n-1/2)(n-3/2)\cdots\frac{1}{2}\Gamma(1/2)} = \cdots = \frac{2^{n+1}(n!)^2}{(2n+1)!}$$

Thus, we get $\int_{-1}^{1} P_m(x)\, P_n(x)\, dx = 0$ if $m < n$, and if $m = n$ then only the highest order term in P_n produces a nonzero result. Write

$$P_n(x) = \sum_{k=0}^{n} a_k x^k, \quad \text{then}$$

$$\int_{-1}^{1} P_n(x) P_n(x)\, dx = \int_{-1}^{1} a_n x^n P_n(x)\, dx$$

$$= (\text{as above}) = a_n \frac{2^{n+1}(n!)^2}{(2n+1)!}.$$

Substituting the value of a_n (from the formula in p. 102) we get

$$\int_{-1}^{1} [P_n(x)]^2 dx = \frac{2}{2n+1} = \frac{1}{n+1/2}. \quad \square$$

VII.5 Fourier-Legendre Series

5.1 Defn. Let $f \in C(-1,1)$ satisfy $\int_{-1}^{1} [f(x)]^2 dx < \infty$. Then we call the sequence $\{c_n\}_{n=0}^{\infty}$, where

$$c_n = (n+\tfrac{1}{2}) \int_{-1}^{1} f(x) P_n(x)\, dx = \frac{\langle f, P_n\rangle}{\|P_n\|^2}$$

the Fourier-Legendre transformation, and the series

$$\sum_{n=0}^{\infty} c_n P_n(x)$$

$\boxed{\text{Not normalized}}$

is the Fourier-Legendre series of f.

Note normalization: $n+\frac{1}{2} = \|P_n\|^{-2}$.

For these series similar results are true as for Fourier-series, sinus-series, cosinus-series, Fourier-Bessel series, etc. For example:

5.2 Thm i) The Fourier-Legendre series converges to f in the norm induced by the inner product $\langle f, g\rangle = \int_{-1}^{1} f(x)\overline{g(x)}\, dx$, i.e.,

$$\lim_{N\to\infty} \int_{-1}^{1} |f(x) - f_N(x)|^2 dx = 0, \quad \text{where}$$

$$f_N(x) = \sum_{n=0}^{N} c_n P_n(x). \quad \text{That is, the sequence}$$

$P_n(x)$ is complete.

ii) Parseval's identity gives

$$\int_{-1}^{1} [f(x)]^2 dx = \sum_{n=0}^{\infty} \frac{c_n^2}{n+1/2}$$

Note: If f is a polynomial of order m, then $c_n = 0$ for $n > m$, i.e.,

$$f(x) = \sum_{n=0}^{m} c_n P_n(x)$$

in this case.

5.3 Ex. Expand x^2 in a Fourier-Legendre series.

Method 1: Use the formulas above

Method 2: Do it directly: we know that $P_0(x) \equiv 1$, $P_1(x) = x$, $P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$ $\Rightarrow$

$$x^2 = c_0 + c_1 x + c_2\left(\tfrac{3}{2}x^2 - \tfrac{1}{2}\right)$$

$$= \left(c_0 - \frac{c_2}{2}\right) + c_1 x + \frac{3c_2}{2} x^2 \Rightarrow$$

$\frac{3c_2}{2} = 1$, $c_1 = 0$, $c_0 = c_2/2$, so

$$x^2 = \tfrac{1}{3} P_0(x) + \tfrac{2}{3} P_2(x).$$

5.4 Thm Let f be an arbitrary function. Then the polynomial

$$f_N(x) = \sum_{k=0}^{N} c_k P_k(x)$$

(a truncated (= "kapad") Fourier-Legendre series) has the following property: This polynomial is the best possible polynomial approximation of order N of f, if we by "best possible" mean that the "error integral"

$$\int_{-1}^{1} |f(x) - f_N(x)|^2 dx$$

is as small as possible.

Proof: See next section.

VII.6 Best Norm Approximation

Thm. 5.4 above is a special case of:

6.1 Thm. Let f be an arbitrary function, and let $\{\varphi_n\}_{n=0}^{\infty}$ be an orthogonal sequence. Define $f_N = \sum_{n=0}^{N} \frac{\langle f, \varphi_n \rangle}{\|\varphi_n\|^2} \varphi_n$ (a "truncated" Fourier series). Then f_N is the best approximation to f_N among all linear combinations $g = \sum_{n=0}^{N} c_n \varphi_n$ of $\{\varphi_0, \dots, \varphi_N\}$ in the following sense: The error integral

$$\|f - g\|^2 = \int_a^b |f(x) - g(x)|^2 w(x) dx,$$

where $g(x) = \sum_{n=0}^{N} c_n \varphi_n$, is minimized if we choose $c_n = \frac{\langle f, \varphi_n \rangle}{\|\varphi_n\|^2}$.

Proof. Simplify by defining $\chi_n = \varphi_n / \|\varphi_n\|$. Then $\chi_n = \varphi_n / \|\varphi_n\|$. Put $f_N = \sum_{n=0}^{N} \langle f, \chi_n \rangle \chi_n$.

Then $\langle f - f_N, \chi_n \rangle = \langle f, \chi_n \rangle - \langle f_N, \chi_n \rangle$

$$= \langle f, \chi_n \rangle - \langle f, \chi_n \rangle = 0$$

for $n = 0, 1, \dots, N$, so

$$\boxed{f - f_N \perp \chi_n \quad \text{for } n = 0, 1, 2, \dots N}$$

Therefore if we take $g = \sum_{n=0}^{N} c_n \chi_n$, then

$$g - f_N = \sum_{n=0}^{N} (c_n - \langle f, \chi_n \rangle) \chi_n \perp f - f_N,$$

and $\|f - g\|^2$

$$= \langle f - g, f - g \rangle = \langle f - f_N + f_N - g, f - f_N + f_N - g \rangle$$

$$= \langle f - f_N, f - f_N \rangle + \underbrace{\langle f - f_N, f_N - g \rangle}_{=0}$$

$$+ \underbrace{\langle f_N - g, f - f_N \rangle}_{=0} + \langle f_N - g, f_N - g \rangle$$

$$= \|f - f_N\|^2 + \|f_N - g\|^2.$$

Thus, $\|f - g\|^2 = \|f - f_N\|^2 + \|f_N - g\|^2 \geq \|f - f_N\|^2$. We get equality by taking $g = f_N$. ☐

VII.7 A Second Solution to Legendre's Differential Equation

In Section VII.1 we found one polynomial solution P_n and another solution which was unbounded close to ± 1. A more frequently used solution is (Arfken, p. 761)

$$Q_n(x) = \frac{1}{2} P_n(x) \ln \frac{1+x}{1-x} - \sum_{k \leq \frac{n+1}{2}} \frac{2n-4k-1}{(2k+1)(n-k)} P_{n-2k-1}(x).$$

This is the "analogue of the Neumann function", and $|Q_n(x)| \to \infty$ as $x \to \pm 1$.

VII. 8 Associated Legendre Functions

8.1 Defn. Let P_n = Legendre's polynomial, and Q_n = the function in the preceding section. Then we call

$$P_n^m(x) = (1-x^2)^{m/2} \frac{d^m}{dx^m} P_n(x) \quad \text{and}$$

$$Q_n^m(x) = (1-x^2)^{m/2} \frac{d^m}{dx^m} Q_n(x)$$

associated Legendre functions.

P_n^m = "Type 1" $\quad Q_n^m$ = "Type 2", $0 \leq m \leq n$.

8.2 Note: By combining this with Rodrigue's formula we get

$$P_n^m(x) = \frac{1}{2^n n!} (1-x^2)^{m/2} \underbrace{\frac{d^{m+n}}{dx^{m+n}} (x^2-1)^n}_{\text{polynomial of order } n-m.}$$

8.3 Note: When we derived Legendre's differential equation we used a change of variables $x = \cos\theta$ (= polar angle). For the physically relevant case $-1 \leq x \leq 1$ we have $\sqrt{1-x^2} = \sqrt{1-\cos^2\theta} = \sqrt{\sin^2\theta} = \sin\theta$, and

$$P_n^m(\cos\theta) = \sin^m\theta \left. \frac{d^m}{dx^m} P_n(x) \right|_{x=\cos\theta}$$

$$Q_n^m(\cos\theta) = \sin^m\theta \left. \frac{d^m}{dx^m} Q_n(x) \right|_{x=\cos\theta}$$

Here is a list of the first functions P_n^m:

Table: $\quad P_n^m(\cos\theta) = P_n(\cos\theta)$ (all n)

$$P_0^0(\cos\theta) = 1$$

$$P_1^0(\cos\theta) = \cos\theta; \quad P_1^1(\cos\theta) = \sin\theta$$

$$P_2^0(\cos\theta) = \tfrac{1}{4} + \tfrac{3}{4}\cos(2\theta)$$

$$P_2^1(\cos\theta) = \tfrac{3}{2}\sin(2\theta)$$

$$P_2^2(\cos\theta) = \tfrac{3}{2} - \tfrac{3}{2}\cos(2\theta) \quad \text{etc.}$$

$P_0 \equiv 1$, $P_1(x) = x$; $P_2(x) = \frac{1}{2}(3x^2-1)$;
$P_3(x) = \frac{1}{2}(5x^3-3x)$; $P_4(x) = \frac{1}{8}(35x^4-30x^2+3)$, ...

8.4 Thm The functions P_n^m and Q_n^m are solutions to Legendre's associated differential equation

$$(1) \quad (1-x^2)y''(x) - 2xy'(x) + \left[n(n+1) - \frac{m^2}{1-x^2}\right] y(x) = 0$$

Note: We encountered this equation in Section III. 4 (page 32), when we separated variables for the spherical Laplacian.

Proof: About $1\frac{1}{2}$ page long. Start from Legendre's differential equation, differentiate m times, and simplify.

VII. 9 Sperical Harmonics
("Sfärernas Harmonier")

9.1 Task: Solve the Laplace equation

$$(1) \quad \nabla^2 V(r, \theta, \varphi) = 0.$$

in the unit ball $\boxed{r \leq 1}$. This equation is static (no time dependence). We give a boundary condition on the surface $r = 1$:

$$V(1,\theta,\varphi) = v(\theta,\varphi) \quad \text{(given)}$$

($r=1$ is the equation of the surface of a ball of radius 1).

Note: This equation also appears in a time-dependent problem after we have separated the time-variable.

Solution: Separate variables: See p. 30

$$V(r,\theta,\varphi) = R(r)\,F(\theta,\varphi).$$

As in Section III. 4 (p. 31) (with $b=0$) we get

(2) $(r^2R')' - \lambda R = 0$ and

(3) $\sin\theta \frac{\partial}{\partial\theta}\left[\sin\theta \frac{\partial F}{\partial\theta}\right] + \frac{\partial^2 F}{\partial\varphi^2} + \lambda \sin^2\theta\, F = 0$

We start by solving (2). This is Euler's equation (= special case of generalized Bessel's equation). (In a time-dependent problem we get more complicated equations.) This equation is easy to solve:

Ansatz: $R = r^\alpha$:

$$\begin{array}{l|l} R = r^\alpha & -\lambda \\ R' = \alpha r^{\alpha-1} & 2r \\ R'' = \alpha(\alpha-1) r^{\alpha-2} & r^2 \\ \hline & \Sigma \end{array}$$

$(\lambda + 2\alpha + \alpha^2 - \alpha)\, r^\alpha = 0$

$\Leftrightarrow \quad \alpha^2 + \alpha - \lambda = 0$

$$\alpha = -\tfrac{1}{2} \pm \sqrt{\tfrac{1}{4}+\lambda} = -\tfrac{1}{2}\left(1 \pm \sqrt{1+4\lambda}\right).$$

It can be shown that the solution of Laplace's equation must be differentiable infinitely many times in the ball $r<1$, in particular, also at $r=0$.

$\Rightarrow$ α must be an integer ≥ 0. Put $\boxed{\alpha = n}$

we must discard the minus-sign, and put

$$\sqrt{\tfrac{1}{4}+\lambda} = \left(n+\tfrac{1}{2}\right) \quad (n = 0,1,2,\dots), \text{ i.e.,}$$

$$\lambda = \left(n+\tfrac{1}{2}\right)^2 - \tfrac{1}{4} = n^2 + n = n(n+1).$$

So we get the same separation condition as in Legendre's equation, namely

$$\boxed{\lambda = n(n+1), \quad n = 0,1,2,3,\dots}$$

We continue to separate θ and φ:

$$F(\theta,\varphi) = \Theta(\theta)\,\Phi(\varphi), \quad \text{and get}$$

(cf. pages 31–34):

(4) $\sin^2\theta\, \Theta'' + \sin\theta\cos\theta\, \Theta' + [n(n+1)\sin^2\theta - \nu]\,\Theta = 0$

(5) $\Phi'' + \nu\Phi = 0.$

Since φ is an azimutal angle Φ must be 2π-periodic (see p. 31), and

$$\nu = m^2, \quad m = 0,1,2,\dots$$

Substitute into (4), make a change of variables $s = \cos\theta$ (see pages 31–32):

$$(1-s^2)\,u''(s) - 2su' + \left[n(n+1) - \frac{m^2}{1-s^2}\right]u = 0,$$

i.e., the associated Legendre equation.

This equation has two linearly independent solutions

$$P_n^m(\cos\theta) \quad \text{and} \quad Q_n^m(\cos\theta).$$

As $Q_n^m(\cos(\theta))$ is unbounded at $\theta = 0$ and $\theta = \pi$ (north and south poles) we must discard $Q_n^m(\cos\theta)$, and get solutions of the type

same m (p. 114)

$$\begin{aligned} u_n^m(\theta,\varphi) &= P_n^m(\cos\theta)\cos(m\varphi) \\ &= \sin^m\theta \cos(m\varphi) \frac{d^m}{ds^m} P_m(s)\Big|_{s=\cos\theta}, \\ v_n^m(\theta,\varphi) &= P_n^m(\cos\theta)\sin(m\varphi) \\ &= \sin^m\theta \sin(m\varphi) \frac{d^m}{ds^m} P_m(s)\Big|_{s=\cos\theta}. \end{aligned}$$

These functions have a name:

9.2 Defn. The functions u_n^m and v_n^m are the so called <u>sperical harmonics</u> ("sfärharmoniska")
Their linear combinations

$$\sum_{m=0}^{n} P_n^m(\cos\theta)\left[A_n^m \cos(m\varphi) + B_n^m \sin(m\varphi)\right]$$

are <u>spherical functions</u> ("klotytfunktioner") of order n. If we multiply these by r^n we get a solution of Laplace's equation (next page)

9.3 Comment The solutions of the Laplace equation are called <u>harmonic functions</u> ("harmoniska") (har med "övertoner" att göra) (överton = harmonic)

We still need to adjust the coefficients A_n^m and B_n^m to satisfy the boundary condition

$$v(1,\theta,\varphi) = u(\theta,\varphi) \quad \text{(given)}$$

The complete solution is of the type

$$v(r,\theta,\varphi) = \sum_{n=0}^{\infty} r^n \left(\sum_{m=0}^{n} P_n^m(\cos\theta)\left[A_n^m \cos(m\varphi) + B_n^m \sin(m\varphi)\right]\right)$$

Note:
- A power series in r ("potensserie")
- A Fourier series in φ
- An associated Legendre series in θ.

Taking $r = 1$ we get the boundary condition:

$$u(\theta,\varphi) = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\left[A_n^m u_n^m(\theta,\varphi) + B_n^m v_n^m(\theta,\varphi)\right],$$

$$v(r,\theta,\varphi) = \sum\sum r^n [\quad] \quad (+\text{constant} = \text{value at } \infty)$$

a so called <u>Laplace series</u> in (θ,φ).
More about this series in the next section.

9.4 Task. Solve the Laplace equation in the <u>outside</u> of a ball of radius 1:

$$\nabla^2 V(r,\theta,\varphi) = 0, \quad \boxed{r \geq 1}. \quad V(\infty,\theta,\varphi) = C$$

(to exclude constant solution)

Same boundary condition as before. The solution stays the same with one <u>exception</u>: when we separate the r-variable require the solution to be <u>analytic</u> and vanish at ∞ i.e., $V(\frac{1}{r},\theta,\varphi)$ should be analytic and vanish at zero.

The index cond. on p. 117 gives $\alpha = -1, -2, -3, \ldots$; we discard the plus-sign in the square root, and get (as before)

$$\lambda = n(n+1), \quad n = 0,1,2,3,\ldots; \quad \text{and}$$
$$R(r) = r^{-(n+1)}.$$

We get the <u>same</u> solution as before, except that we must replace $\boxed{r^n \to r^{-(n+1)}}$

9.5 Ex. Suppose we have the boundary function

$$u(\theta,\varphi) = 3 + 2\cos\theta - \sin\theta\sin\varphi.$$

The first functions P_n^m are given in the table on p. 115. A comparison gives

$$u(\theta,\varphi) = 3P_0^0(\cos\theta)\cos(0\varphi) + 2P_1^0(\cos\theta)\cos(0\varphi) - P_1^1(\cos\theta)\sin(1\cdot\varphi),$$ so

the solution of the Laplace equation outside the unit ball is

$$v(r,\theta,\varphi) = \underbrace{\frac{3}{r}}_{\text{constant charge}} + \underbrace{\frac{2}{r^2}\cos\theta}_{\text{dipole}} - \underbrace{\frac{1}{r^2}\sin\theta\sin\varphi}_{\text{dipole}}.$$

VII. 10 Laplace Series

A lengthy but "easy" computation gives

10.1 Thm. The function sequence $\{u_n^m(\theta,\varphi), v_n^m(\theta,\varphi)\}$, where $0 \le m \le n < \infty$, is an orthogonal sequence (doubly indexed) with respect to the inner product

$$\langle f,g\rangle = \int_{\varphi=-\pi}^{\pi}\int_{\theta=0}^{\pi} f(\theta,\varphi)\,\overline{g(\theta,\varphi)}\,\sin(\theta)\,d\varphi\,d\theta,$$

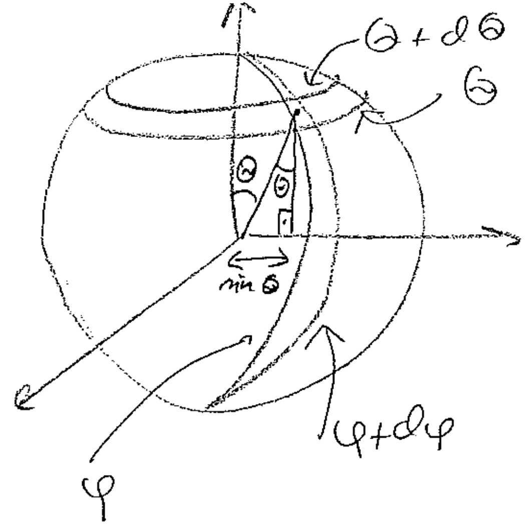
i.e., $\langle u_n^m, v_l^k\rangle = 0$ for all $0 \le m \le n$, $0 \le k \le l$.

$$\langle u_n^m, u_l^k\rangle = 0 = \langle v_n^m, v_l^k\rangle \text{ if } k \ne m \text{ or } l \ne n.$$

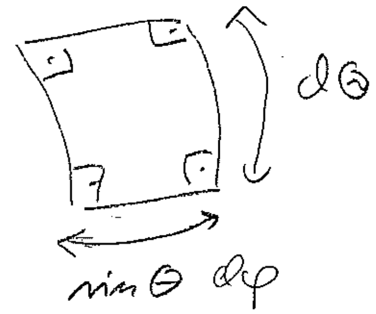
Moreover: $\|u_n^m\|^2 = \|v_n^m\|^2 = \dfrac{\pi}{n+\frac12}\dfrac{(n+m)!}{(n-m)!}$ $\quad 1 \le m \le n$

$v_n^0 = 0 \qquad \|u_n^0\|^2 = \dfrac{2\pi}{n+1/2}$ $\quad$ (note: $m=0$ is special)

10.2 Note: $\sin\theta\, d\theta\, d\varphi$ is the standard surface element when we integrate in spherical coordinates.



Blow up:



Area $\sin\theta\,d\varphi\,d\theta$.

Note: Often people write $\sin\theta\,d\varphi\,d\theta = d\sigma$, where $d\sigma$ = a two-dimensional surface element.

10.3 Thm The sequence in Thm 10.1 is complete.

Proof bypassed.

(i.e. normalizable)

10.4 Coroll. Parseval's identity is valid.

Formulas (special case of general formula):

$$A_n^0 = \frac{n+\frac12}{2\pi}\int_{\varphi=-\pi}^{\pi}\int_{\theta=0}^{\pi} u(\theta,\varphi)\,u_n^0(\theta,\varphi)\,d\sigma$$

B_n^0 = arbitrary (since $v_n^0 = 0$), e.g. $B_n^0 = 0$.

For $m \ge 1$:

$$A_n^m = \frac{n+\frac12}{\pi}\frac{(n-m)!}{(n+m)!}\int_{\varphi=-\pi}^{\pi}\int_{\theta=0}^{\pi} u(\theta,\varphi)\,u_n^m(\theta,\varphi)\,d\sigma$$

$$B_n^m = \frac{n+\frac12}{\pi}\frac{(n-m)!}{(n+m)!}\int_{\varphi=-\pi}^{\pi}\int_{\theta=0}^{\pi} u(\theta,\varphi)\,v_n^m(\theta,\varphi)\,d\sigma$$

Here $d\sigma = \sin\theta\,d\varphi\,d\theta$, and this is a "standard" integral over the surface of the sphere with radius 1.