Special Functions

Credits: 10

Responsible person: Olof Staffans

Period: Periods 3 and 4

Format: 6 hours of teaching per week. Weekly exercises

Level: Advanced undergraduate

Form of assessment: Homework and written examination

Prerequisites: Analysis

Study Forms: Lectures and home work.

Aim(s): To familiarize the participants with the most common special mathematical functions and their applications to different physical and technical problems.

Contents: The basic theory of the Gamma and Beta functions. A general study of how special functions appear when one solves partial differential equations by separation of variables. Serial solutions of boundary value problems. The theory of orthogonal function series. Presentation of the Bessel, Legendre, and Laguerra functions, and the general hypergeometric functions.

Literature: Chapters 8 - 13 from G. B. Arfken and H. J. Weber, Mathematical Methods for Physicists, plus lecture notes.

Time: The two spring semesters 2011, Tue. 13-15, Wed.10-12, and Thu. 10-12 in Hilbertrummet (ASA B329). The first lecture will be given on Tuesday January 11, 2011.

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273012 Special Functions 2011

<u>Lectures</u>

Tue 13-15
We 10-12
Th 10-12

11 weeks, starting 11.1.11

<u>Homework</u> About 4 home work assignments per week, which are handed in and judged on a course scale. These contribute to 50% of the final grade in the final exam at the end of the course. They are not counted (or counted only partially) in later exams. To get maximal points for the home assignments it is enough to solve 75% of all the given home work. Some of these need access to a computer. They are typically to be handed in on wednesday.

<u>Final Exam:</u> at the end of the course contributes to one half of the final grade

<u>Useful Computer Programs:</u> Mathematica (Aton), Matlab (Aton, PC-class), Mathcad, Maple.

<u>Course material:</u> Lecture notes will be handed out. We mainly follow the book by Arfken: Mathematical Methods for Physicists

<u>Language:</u> English (or Swedish).

<u>Next time:</u> This course will <u>not</u> be offered during at least the next two academic year.

I Introduction

QUEST. 1. What is a SF (= special function)?

Ans. A. Not found on small calculator
Ans B. Found in books of tables
Ans C. Found in mathematical universal programs (Mathematica, Maple, Matlab, Mathcad)

(Pictures!)

Comment. Strictly speaking, $\sin x$, $\cos x$, e^x, $\ln x$ are also "special functions", but they are too common to be considered "special".

Conclusion: Functions like sin, cos, $\ln x$, e^x, but more "special" (more complicated).

QUEST 2. Where are SF:s used?

Ans A. Solutions of PDE:s (partial differential equations)

Ans B. Produce orthogonal basis functions, and lead to generalized "Fourier series": "Arbitrary" functions can be written as infinite sums of special functions.

Ans C. Give nice examples on functions with special properties.

QUEST 3. Where do SF:s come from?

Ans A. Separation of variables in PDE:s
Ans B. Solutions of boundary value problems for ODE:s (ordinary diff. eqs.)
Ans C. Certain power series give rise to SF:s
Ans D. Certain infinite products — " —
Ans E. Certain integrals — " —
Ans F. Certain recursion formulas — " —

QUEST 4. Properties of SF:s ?

Ans A. Are analytic (complex arguments are needed)
Ans B. Are often eigenfunctions for differential operators
Ans C. Are often orthogonal in some sense
Ans D. — " — symmetrical — " —
Ans E. Have often an characteristic asymptotic behavior as $t \to \infty$ or $t \to 0$.

II. The Γ-function and related functions (Arfken, Chapter 10)

Γ = "Gamma".

Idea. The "factorial" (= fakultet)

$$n! = 1\cdot 2\cdot 3 \cdot \dots \cdot n \quad (n \text{ is a positive integer})$$

is well-defined for $n \in \mathbb{N}$ (= integers ≥ 1).
Can we define $x!$ for $x \notin \mathbb{N}$?

II 1. Definition of $\Gamma(z)$

There are many possible (equivalent) definitions. One such definition is:

1.1 Defn (as a limit): We define

$$(1) \quad \Gamma(z) = \lim_{n\to\infty} \frac{1\cdot 2\cdot 3 \cdots n}{z(z+1)(z+2)\cdots(z+n)}\, n^z$$

for those $z \in \mathbb{R}$ (real) or $z \in \mathbb{C}$ (complex) for which the limit exists.

1.2 Problem. What does t^z mean for $t>0$ and $z \in \mathbb{C}$?

Solution. Formally, $t^z = e^{\ln(t^z)} = e^{z\ln t}$, so we can use this formula as a definition

1.3 Defn. For all $t>0$ and all $z = \alpha + i\beta \in \mathbb{C}$ we define

$$t^{\alpha+i\beta} = e^{(\alpha+i\beta)\ln t}$$

Recall: $e^{\alpha+i\beta} = e^{\alpha}(\cos\beta + i \sin\beta)$

1.4 Lemma Let $t>0$, $\alpha + i\beta \in \mathbb{C}$. Then

i) $t^{\alpha+i\beta} = t^{\alpha}[\cos(\beta \ln t) + i \sin(\beta \ln t)]$

ii) $|t^{\alpha+i\beta}| = t^{\alpha}$.

Proof. According to Defn. 1.3,

i)
$$\begin{aligned} t^{\alpha+i\beta} &= e^{(\alpha+i\beta)\ln t} \\ &= e^{\alpha \ln t} e^{i\beta \ln t} \\ &= e^{\ln(t^{\alpha})} e^{i\beta \ln t} \\ &= t^{\alpha}[\cos(\beta \ln t) + i \sin(\beta \ln t)] \end{aligned}$$

ii) $|t^{\alpha+i\beta}| = t^{\alpha}\, |e^{i\beta \ln t}| = t^{\alpha}$. □

1.5 Prop.

i) $\Gamma(1) = 1$

ii) $\Gamma(z+1) = z\Gamma(z)$ if the limit in (1) in Defn. 1.1 exists

iii) $\Gamma(n+1) = n!$, $n = 0, 1, 2, 3, \dots$

Proof. i) $\Gamma(1) = \lim_{n\to\infty} \frac{1\cdot 2 \cdots n}{1\cdot 2 \cdot \dots \cdot n(n+1)}\, n^1$

$$= \lim_{n\to\infty} \frac{n}{n+1} = 1.$$

ii)
$$\Gamma(z+1) = \lim_{n\to\infty} \frac{1\cdot 2\cdot 3 \cdots n}{(z+1)(z+2)\cdots(z+n+1)}\, n^{z+1}$$

$$= \underbrace{\lim_{n\to\infty} \frac{1\cdot 2\cdot 3 \cdots n}{z(z+1)\cdots(z+n)}\cdot n^z}_{\to \Gamma(z)} \cdot \underbrace{\frac{z}{z+n+1}\cdot n}_{\to z}$$

$$= z\,\Gamma(z).$$

iii) Follows from i) and ii). □

1.6 Cor.

i) If $\Gamma(z)$ exists, then so does $\Gamma(z+1)$

ii) If $\Gamma(z+1)$ exists and $z \neq 0$, then $\Gamma(z)$ exists.

Proof: See the preceding proof.

Alternative definition of $\Gamma(z)$ (integral)

1.7 Defn. We define

$$(2) \quad \tilde{\Gamma}(z) = \int_0^\infty e^{-t} t^{z-1} \, dt$$

for those ($z \in \mathbb{R}$ or) $z \in \mathbb{C}$ for which the integral converges.

Note: "converges" means:

$$\lim_{\substack{M \to \infty \\ \varepsilon \to 0+}} \int_\varepsilon^M e^{-t} t^{z-1} \, dt \quad \text{exists (and is finite)!}$$

1.8 Lemma. The integral (2) converges absolutely if and only if $\mathrm{Re}\, z > 0$.

Note: "absolute convergence" means that

$$\lim_{\substack{M \to \infty \\ \varepsilon \to 0+}} \int_\varepsilon^M |e^{-t} t^{z-1}| \, dt \quad \text{exists (and is finite).}$$

Proof. Denote $\mathrm{Re}\, z = \alpha$. Then

$$|e^{-t} t^{z-1}| = e^{-t} t^{\mathrm{Re}(z-1)} = e^{-t} t^{\alpha-1}, \text{ so}$$

$$\int_0^\infty |e^{-t} t^{z-1}| \, dt = \int_0^\infty e^{-t} t^{\alpha-1} \, dt$$

$$= \left(\int_0^1 + \int_1^\infty \right) e^{-t} t^{\alpha-1} \, dt.$$

A closer inspection shows that

$$\int_1^\infty e^{-t} t^{\alpha-1} \, dt < \infty$$

for all $\alpha \in \mathbb{R}$. Thus, only $\int_0^1$ can cause problems. We have

$$e^{-1} \int_0^1 t^{\alpha-1} \, dt < \int_0^1 e^{-t} t^{\alpha-1} \, dt < \int_0^1 t^{\alpha-1} dt,$$

so this integral converges if and only if $\int_0^1 t^{\alpha-1} \, dt < \infty$, i.e., $\alpha > 0$.

Note: Every "absolutely converging" integral also "converges". The proof is similar to the proof that "every absolutely converging sum converges". This is (or should be) treated in "Analysis".

1.9 Thm. $\Gamma(z) = \tilde{\Gamma}(z)$ for all $z \in \mathbb{C}$ with $\mathrm{Re}\, z > 0$. In particular, the limit (1) in Defn. 1.1 exists when $\mathrm{Re}\, z > 0$.

Idea of Proof. We know that $e^{-t} = \lim_{n \to \infty} \left(1 - \frac{t}{n}\right)^n$.

It seems plausible (but needs to be proved) that

$$(*) \quad \int_0^\infty e^{-t} t^{z-1} \, dt = \lim_{n \to \infty} \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1} dt.$$

Integration by parts gives

$$(**) \quad \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1} \, dt = \frac{1 \cdot 2 \cdots n}{z(z+1) \cdots (z+n)} n^z.$$

Thus, there is some hope that:

$$\tilde{\Gamma}(z) = \int_0^\infty e^{-t} t^{z-1} \, dt = \lim_{n \to \infty} \frac{1 \cdot 2 \cdots n}{z(z+1) \cdots (z+n)} n^z = \Gamma(z)$$

$\square$

(over)

<u>Note</u>. The proof of (*) is rather long. The proof of (**) is a mechanical computation (see Artken)

<u>Note</u>: Formula (**) is the <u>motivation</u> for why we introduced Definition 1.1 in the first place.

<u>1.10 Prop.</u> $\Gamma(z)$ exists for all $z \in \mathbb{C}$, except for $z = 0, -1, -2, -3, \dots$

<u>Proof</u>. By Lemma 1.8 and Thm. 1.9, $\Gamma(z)$ exists for all $\operatorname{Re} z > 0$. This combined with Cor. 1.6 gives existence for all $z \in \mathbb{C}$ except for $z = 0, -1, -2, \dots$. $\square$

<u>Note</u>: The limit (1) in Defn. 1.1 is <u>not defined</u> for $z = 0, -1, -2, \dots$ (zero is the denominator).

Third definition of $\Gamma(z)$ (infinite product):

<u>1.11 Defn.</u>

$$(3) \quad \tilde{\Gamma}(z) = \frac{1}{z e^{\gamma z}} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n},$$

where $\gamma = \lim_{n \to \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}\right) - \ln n$

(γ = "the Euler constant").

<u>Note</u>: (3) means: $\tilde{\Gamma}(z) = \lim_{N \to \infty} \frac{1}{z e^{\gamma z}} \prod_{n=1}^{N} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n}$.

<u>1.12 Thm.</u> $\tilde{\Gamma}(z) = \Gamma(z)$.

<u>Proof</u>: See Artken. $\square$

II.2. The Beta-function

The Beta-function $B(z,w)$ is a function of <u>two</u> (complex) variables, which is closely related to the Γ-function.

<u>2.1 Defn.</u>

$$(1) \quad B(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1}\, dt$$

for those $z \in \mathbb{C}$, $w \in \mathbb{C}$ for which the integral converges.

<u>2.2 Lemma</u> This integral converges if and only if both $\operatorname{Re} z > 0$ and $\operatorname{Re} w > 0$.

<u>Proof</u>. Homework.

<u>Note</u>: It is possible to <u>extend</u> the <u>domain</u> of definition of $B(z,w)$ by using a different definition (or "analytic continuation").

<u>2.3 Thm.</u> For all $\operatorname{Re} z > 0$ and $\operatorname{Re} w > 0$ we have

$$(2) \quad B(z,w) = \frac{\Gamma(z)\,\Gamma(w)}{\Gamma(z+w)}.$$

<u>Proof</u>. This proof is based on various manipulations of integrals:
- change of integration variable
- change of order of integration in a double integral.

These manipulations are always allowed: if either
i) the integrand is continuous and the interval of integration is finite, or
ii) the integrand is continuous in the <u>interior</u> of the interval, and the integrals converge <u>absolutely</u> (proof rather long).

We have for $\mathrm{Re}\, z>0$, $\mathrm{Re}\, w>0$,

$$\Gamma(z)\Gamma(w) = \int_0^\infty e^{-t} t^{z-1} dt \int_0^\infty e^{-s} s^{w-1} ds$$

$$\begin{pmatrix}\text{double}\\ \text{integral}\end{pmatrix} = \int_0^\infty\int_0^\infty t^{z-1} s^{w-1} e^{-(s+t)} ds\, dt$$

$$(r=s+t) = \int_{t=0}^\infty \int_{r=t}^\infty t^{z-1}(r-t)^{w-1} e^{-r} dr\, dt$$

$$\begin{pmatrix}\text{change}\\ \text{order};\ r\geq t\end{pmatrix} = \int_{r=0}^\infty \int_{t=0}^r t^{z-1}(r-t)^{w-1} dt\, e^{-r} dr$$

$$\begin{pmatrix} t=ru\\ dt=r\,du\end{pmatrix} = \int_{r=0}^\infty \int_{u=0}^1 (ru)^{z-1}[r(1-u)]^{w-1} e^{-r} r\, du\, dr$$

$$= \underbrace{\int_{r=0}^\infty r^{z+w-1} e^{-r} dr}_{=\tilde{\Gamma}(z+w)} \underbrace{\int_{u=0}^1 u^{z-1}(1-u)^{w-1} du}_{B(z,w)}$$

$$= \Gamma(z+w)\, B(z,w).$$

Thus, $B(z,w) = \dfrac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}$

if we know that $\Gamma(z+w) \neq 0$. This we will prove later. □

2.4 Note. By using (2) as a definition of $B(z,w)$ we can extend the domain of defn. of $B(z,w)$ to all those $z, w \in \mathbb{C}$ for which

$$z \neq 0, -1, -2, \ldots$$
$$w \neq 0, -1, -2, \ldots$$
$$(z+w) \neq 0, -1, -2, \ldots$$

(The last requirement can be removed if we interpret $1/\Gamma(z+w) = 0$ for $z+w = 0, -1, -2, \ldots$)

Thm. 2.5. For all $\mathrm{Re}\, z>0$ and $\mathrm{Re}\, w>0$,

$$(3)\quad B(z,w) = \int_0^\infty \frac{u^{z-1}}{(1+u)^{z+w}} du = \int_0^\infty \frac{u^{w-1}}{(1+u)^{z+w}} du.$$

Proof: Use a change of variable $t = \frac{u}{1+u}$ in formula (1).

II. 3 Properties of the Γ-function

We know already:

$$(1)\quad \Gamma(z) = \lim_{n\to\infty} \frac{1\cdot 2\cdots n}{z(z+1)\cdots(z+n)} n^z, \qquad z\in\mathbb{C},\ z\neq 0,-1,-2,\ldots$$

$$(2)\quad \Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \qquad \mathrm{Re}\, z>0,$$

$$(3)\quad \Gamma(z) = \frac{1}{z e^{\gamma z}} \prod_{n=1}^\infty \frac{e^{z/n}}{(1+\frac{z}{n})}, \quad z\in\mathbb{C},\ z\neq 0,-1,-2,\ldots$$

$$(4)\quad \Gamma(z+1) = z\Gamma(z) \quad \text{(the difference formula)}$$

$$(5)\quad \Gamma(n+1) = n! \quad (n=0,1,2,\ldots)$$

Note: Because of (5), some people denote $\Gamma(z)$ by $(z-1)!$, allowing a complex argument in the factorial, but it is still the same function which we have defined above. (For example, Arfken uses this notation.)

(= analytisk)

3.1 Thm $\Gamma(z)$ is analytic (it has a complex derivative) except at the points $z = 0, -1, -2, -3, \dots$, where it has simple poles. It has no zeros (= nollställen). It is real on the real axis. It satisfies, in addition,

(6) $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin(\pi z)}$, $z \neq$ integer (= heltal)

(7) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ $\qquad \Gamma'(z) = \int_0^\infty e^{-t}\frac{d}{dz}t^{z}\,dt = \dots$

(8) $\Gamma(2z) = \dfrac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma(z+\frac{1}{2})$

(9) $\Gamma(z) > 0$ for $z > 0$ (z real)

Note: (6) = "The reflection formula"

(8) = "The doubling formula".

Idea of proof. – We bypass the proof of analyticity (needs knowledge about analytic functions)

– We bypass the proof of the reflection formula (again needs extra knowledge) (residue calculus)

– Real on the real axis: See formula (1).

– Zeros and poles. $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin(\pi z)}$.

Right hand side: No zeros. Simple poles (i.e., the denominator has simple zeros) at $z = 0, \pm 1, \pm 2, \dots$.

Compare this to the left hand side $\Rightarrow$ $\Gamma(z)$ has no zeros. To determine the poles (= the places where $|\Gamma(z)| \to \infty$) we write this as

(*) $\Gamma(z) = \dfrac{\pi}{\Gamma(1-z)\sin(\pi z)}$.

We know from (1) that the only possible poles of $\Gamma(z)$ are located at $z = 0, -1, -2, \dots$. If z is real and ≤ 0, then $1-z \ge 1$, and $\Gamma(1-z) > 0$, hence it follows from (*) that Γ has simple poles at $0, -1, -2, \dots$ (i.e., $1/\Gamma(z)$ has simple zeros at $z = 0, -1, -2, \dots$).

– Doubling formula. $B(z,z) = \dfrac{\Gamma(z)\Gamma(z)}{\Gamma(2z)} \Rightarrow$

$$\Gamma(z)^2 = \Gamma(2z)B(z,z)$$
$$= \Gamma(2z)\int_0^1 [u(1-u)]^{z-1}\,du \qquad \left(u = \tfrac{1+s}{2},\ du = ds/2\right)$$
$$= \Gamma(2z)\,2^{1-2z}\int_{-1}^1 (1-s^2)^{z-1}\,ds \qquad \text{(even function)}$$
$$= 2^{2-2z}\Gamma(2z)\int_0^1 (1-s^2)^{z-1}\,ds \qquad \left(s = \sqrt{v},\ ds = \tfrac{1}{2}\tfrac{dv}{\sqrt{v}}\right)$$
$$= 2^{1-2z}\Gamma(2z)\int_0^1 v^{-1/2}(1-v)^{z-1}\,dv$$
$$= 2^{1-2z}\Gamma(2z)B(\tfrac{1}{2}, z)$$
$$= 2^{1-2z}\frac{\Gamma(\frac{1}{2})\Gamma(z)\Gamma(2z)}{\Gamma(z+1/2)} \Rightarrow$$

$$\Gamma(2z) = \frac{2^{2z-1}\,\Gamma(z+1/2)}{\Gamma(1/2)} = \frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma(z+\tfrac{1}{2}).$$

Value of $\Gamma(\frac{1}{2})$: By the reflection formula,

$$\Gamma(\tfrac{1}{2})\Gamma(\tfrac{1}{2}) = \Gamma(\tfrac{1}{2})\Gamma(1-\tfrac{1}{2}) = \frac{\pi}{\sin(\pi/2)} = \pi. \quad \square$$

II.4 Stirling's formula

4.1 Thm. As $x \to \infty$ (along the real axis), $\Gamma(x)$ grows like

$$\Gamma(x) \sim \sqrt{2\pi}\, e^{-x} x^{x-1/2},$$

i.e. (= that is / dvs)

$$\lim_{x\to\infty} \frac{\Gamma(x)}{\sqrt{2\pi}\, e^{-x} x^{x-1/2}} = 1$$

Proof: See Arfken (rather long) □

(=ofullständig)

II.5. Incomplete B and Γ functions

These are defined as follows:

$$B_x(z,w) = \int_0^x t^{z-1}(1-t)^{w-1}\,dt, \qquad 0 \le x \le 1$$

$$\gamma(z,x) = \int_0^x e^{-t} t^{z-1}\,dt, \qquad x \ge 0$$

$$\Gamma(z,x) = \int_x^\infty e^{-t} t^{z-1}\,dt, \qquad x \ge 0$$

$$B_1(z,w) = B(z,w)$$

Note: $\gamma(z,x) + \Gamma(z,x) = \Gamma(z), \quad x \ge 0.$

5.1 Example The probability that a $N(0,1)$-distributed random variable takes a value $\le x$ is given by

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-u^2/2}\,du$$

(See MAOL:s tables). Write this as an incomplete Γ-function.

Solution. We know that $\Phi(\infty) = 1$, i.e.

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-u^2/2}\,du = 1, \quad \text{and hence}$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 e^{-u^2/2}\,du = 1/2, \quad \text{so}$$

$$\Phi(x) = \tfrac{1}{2} + \int_0^x e^{-u^2/2}\,du.$$

We get something resembling the incomplete Γ-function by changing the integration variable $u^2/2 = t$, $u = \sqrt{2t}$, $du = \frac{dt}{\sqrt{2t}}$ =

$$\Phi(x) = \tfrac{1}{2} + \frac{1}{\sqrt{2\pi}} \int_0^{x^2/2} e^{-t} t^{-1/2} \frac{1}{\sqrt{2}}\,dt$$

$$= \tfrac{1}{2} + \frac{1}{2\sqrt{\pi}}\, \gamma\left(\tfrac{1}{2}, \tfrac{1}{2}x^2\right).$$

Thus

$$\Phi(x) = \tfrac{1}{2}\left(1 + \tfrac{1}{\sqrt{\pi}}\, \gamma\left(\tfrac{1}{2}, \tfrac{1}{2}x^2\right)\right).$$

(Check: letting $x \to \infty$ we get

$$1 = \Phi(\infty) = \tfrac{1}{2}\left(1 + \tfrac{1}{\sqrt{\pi}}\, \gamma\left(\tfrac{1}{2}, \infty\right)\right)$$

$$= \tfrac{1}{2}\left(1 + \tfrac{1}{\sqrt{\pi}}\, \Gamma\left(\tfrac{1}{2}\right)\right) = 1. \;)$$

Alternative solution: $\gamma\left(\tfrac{1}{2}, \tfrac{1}{2}x^2\right) = \Gamma\left(\tfrac{1}{2}\right) - \Gamma\left(\tfrac{1}{2}, \tfrac{1}{2}x^2\right)$,

$$= \sqrt{\pi} - \Gamma\left(\tfrac{1}{2}, \tfrac{1}{2}x^2\right), \quad \text{so}$$

$$\Phi(x) = \tfrac{1}{2}\left(1 + \tfrac{1}{\sqrt{\pi}}\left(\sqrt{\pi} - \Gamma\left(\tfrac{1}{2}, \tfrac{1}{2}x^2\right)\right)\right)$$

$$= 1 - \frac{1}{2\sqrt{\pi}}\, \Gamma\left(\tfrac{1}{2}, \tfrac{1}{2}x^2\right).$$

Thus: We can use MAOL:s tables for Φ to compute $\gamma\left(\tfrac{1}{2}, y\right)$ and $\Gamma\left(\tfrac{1}{2}, y\right)$.

5.2 Example. The probability density of the Γ-distribution with parameters α and β is given by

$$f(x) = \begin{cases} C x^{\alpha-1} e^{-x/\beta}, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

Compute C (as a function of α and β), and also compute the mean and variance of this distribution.

Solution. a) We know that total probability $=1$.

$$\Rightarrow C\int_0^\infty x^{\alpha-1} e^{-x/\beta}\,dx = 1 \qquad \left(\frac{x}{\beta} = y,\ \frac{dx}{\beta} = dy\right)$$

$$1 = C\int_0^\infty (\beta y)^{\alpha-1} e^{-y} \beta\,dy$$

$$= C\beta^\alpha \int_0^\infty y^{\alpha-1} e^{-y}\,dy = C\beta^\alpha \Gamma(\alpha)$$

$$\Rightarrow \boxed{C = \frac{1}{\beta^\alpha \Gamma(\alpha)}}$$

b) Mean: $\mu = \int_0^\infty x f(x)\,dx$

$$= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty x x^{\alpha-1} e^{-x/\beta}\,dx$$

(same integral as above, but $\left(\frac{x}{\beta} = y\right)$ $\alpha \to \alpha+1$)

$$= \cdots = \frac{\beta^{\alpha+1}\Gamma(\alpha+1)}{\beta^\alpha \Gamma(\alpha)} = \beta\alpha$$

c) Variance: By Steiner's rule:

$$\sigma^2 = \int_0^\infty (x-\mu)^2 f(x)\,dx$$

$$= \text{----} = \int_0^\infty x^2 f(x)\,dx - \mu^2.$$

(Known or not?)

Here $\int_0^\infty x^2 f(x)\,dx = \int_0^\infty x^2 x^{\alpha-1} e^{-x/\beta}\,dx$

same integral again, but $\alpha \to \alpha+2$,

$$= \cdots = \frac{\beta^{\alpha+2}\Gamma(\alpha+2)}{\beta^\alpha \Gamma(\alpha)} = \beta^2\alpha(\alpha+1).$$

Thus $\sigma^2 = \beta^2\alpha(\alpha+1) - \beta^2\alpha^2 = \beta^2\alpha$. $\square$

III. Solutions of PDE-s by separation of variables

III.1 A classical example

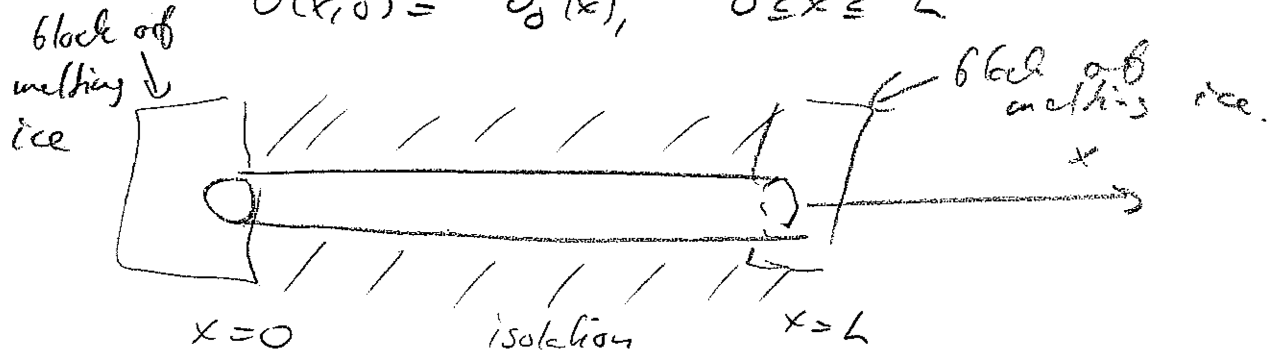
(Kreyszig, section 11.5).

1.1 Problem: A bar (=string) of length L is isolated on its sides. $u(x,t)=$ the temperature at the point x at time t ($0 \leq x \leq L$, $t \geq 0$). We fix the temperature at the end points to be zero (keep it in melting ice):

$$u(0,t) = u(L,t) = 0.$$

We know the initial temperature $u_0(x)$ at each point:

$$u(x,0) = u_0(x), \quad 0 \leq x \leq L$$



According to standard laws of physics: heat flows from higher to lower temperatures. If it accumulates at some point, then the temperature rises:

We get the following set of equation: (over)