MATEMAATTISIA ERIKOISFUNKTIOITA

KAAVAKOKOELMA

Sisällys:

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1. GAMMA- JA BEETTAFUNKTIOT:

1.1. $\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}dt$, $\mathrm{Re}\{z\}>0$

1.2. $\Gamma(z)=\dfrac{\Gamma(z+n)}{z(z+1)\dots(z+n-1)}$, $n=1,2,\dots$, $\mathrm{Re}\{z\}>-n$

1.3. $\Gamma(z+1)=z\Gamma(z)$

1.4. $\Gamma(n+1)=n!$, $\Gamma(1)=1$, $\Gamma(\frac{1}{2})=\sqrt{\pi}$

1.5. $\Gamma(z)\Gamma(1-z)=\dfrac{\pi}{\sin\pi z}$

1.6. $\Gamma(2z)=\pi^{-\frac{1}{2}}\cdot 2^{2z-1}\Gamma(z)\Gamma(z+\frac{1}{2})$

1.7. $\dfrac{1}{\Gamma(z)}=ze^{\gamma z}\prod_{n=1}^{\infty}\left[(1+\frac{z}{n})e^{-\frac{z}{n}}\right]$, missä

$$\gamma=\lim_{m\to\infty}\left\{1+\frac{1}{2}+\dots+\frac{1}{m}-\log m\right\}\approx 0.5772157\dots$$

1.8. $\Gamma(x)=e^{-x}x^{x-\frac{1}{2}}\cdot\sqrt{2\pi}\left[1+\frac{1}{12x}+\frac{1}{288x^2}-\frac{139}{51840x^3}+O(\frac{1}{x^5})\right]$

(ns. Stirlingin kaava)

1.9. $B(p,q)=\int_0^1 t^{p-1}(1-t)^{q-1}dt$, $\mathrm{Re}\{p\}$, $\mathrm{Re}\{q\}>0$

1.10. $B(p,q)=\dfrac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$

1.11. $B(p,q)=2\int_0^{\frac{\pi}{2}}\cos^{2p-1}\varphi\cdot\sin^{2q-1}\varphi\, d\varphi$, $\mathrm{Re}\{p\}$, $\mathrm{Re}\{q\}>0$

1.12. $B(x,y)=\int_0^\infty t^{x-1}(1+t)^{-(x+y)}dt$, $\mathrm{Re}\{x\}$, $\mathrm{Re}\{y\}>0$

Lähde: Whittaker-Watson: A Course of Modern Analysis,
4th ed., Cambridge, Univ.Press, 1963

Huom: Monikäsitteisissä funktioissa t^z ($z\in\mathbb{C}$, $t\in\mathbb{R}$) valitaan aina päähaara $t^z=e^{z\log t}$, jossa $\log t$ valitaan reaaliseksi

2. HERMITEN POLYNOMIT:

2.1. $w''(z)-2zw'(z)+2nw(z)=0$

2.2 $H_n(z)=(-1)^n\cdot e^{z^2}\cdot\dfrac{d^n}{dz^n}(e^{-z^2})=\sum_{k=0}^{[\frac{n}{2}]}\dfrac{(-1)^k n!(2z)^{n-2k}}{k!(n-2k)!}$

2.3. $H_0(z)=1$

$H_1(z)=2z$

$H_2(z)=-2+4z^2$

$H_3(z)=-12z+8z^3$

$H_4(z)=12-48z^2+16z^4$

$H_5(z)=120z-160z^3+32z^5$

2.4. $H_n'(z)=2nH_{n-1}(z)$, $n\in\mathbb{N}_0$

2.5. $zH_n'(z)=nH_{n-1}'(z)+nH_n(z)$, $n\in\mathbb{N}_0$

2.6. $zH_n(z)=\frac{1}{2}H_{n+1}(z)+nH_{n-1}(z)$, $n\in\mathbb{N}_0$

2.7. $e^{z^2-(z-\xi)^2}=\sum_{n=0}^{\infty}\dfrac{H_n(z)}{n!}\xi^n$

2.8. $\int_{-\infty}^{\infty}e^{-x^2}\cdot H_m(x)H_n(x)dx=2^n n!\sqrt{\pi}\,\delta_{mn}$

2.9. $\int_{-\infty}^{\infty}e^{-x^2}\cdot x^m H_n(x)dx=\begin{cases}0 & ; m<n\\ n!\sqrt{\pi} & ; m=n\end{cases}$

Lähde: Sneddon, I.N.: Special Functions of Mathematical Physics
and Chemistry

Huom: Edellä $[\frac{n}{2}]=\begin{cases}\frac{n}{2} & \text{, kun n on parillinen ja}\\ \frac{n-1}{2} & \text{, kun n on pariton}\end{cases}$

Sopimus: $H = 0$

3. LAGUERREN POLYNOMIT:

3.1. $zw''(z) + (1-z)w'(z) + nw(z) = 0$

3.2. $L_n(z) = \frac{e^z}{n!}\frac{d^n}{dz^n}(e^{-z}z^n) = \sum_{k=0}^{n} \frac{(-1)^k n! z^k}{(k!)^2(n-k)!}$

3.3. $(-1)^0 0! L_0(x) = 1 \qquad (= L_0(x))$

$(-1)^1 1! L_1(x) = x - 1 \qquad (= -L_1(x))$

$(-1)^2 2! L_2(x) = x^2 - 4x + 2$

$(-1)^3 3! L_3(x) = x^3 - 9x^2 + 18x - 6$

$(-1)^4 4! L_4(x) = x^4 - 16x^3 + 72x^2 - 96x + 24$

$(-1)^5 5! L_5(x) = x^5 - 25x^4 + 200x^3 - 600x^2 + 600x - 120$

3.4. $L_n'(z) - L_{n-1}'(z) + L_{n-1}(z) = 0 \quad , n \in \mathbb{N}_0$

3.5. $zL_n'(z) = n L_n(z) - L_{n-1}(z) \quad , n \in \mathbb{N}_0$

3.6. $(n+1)L_{n+1}(z) + (z-2n-1)L_n(z) + nL_{n-1}(z) = 0 \quad , n \in \mathbb{N}_0$

3.7. $L_n'(z) = -\sum_{k=0}^{n-1} L_k(z)$

3.8. $\frac{1}{1-\xi}\exp\left(\frac{-z\xi}{1-\xi}\right) = \sum_{k=0}^{\infty} L_k(z)\xi^k$

3.9. $\int_0^\infty e^{-x}L_m(x)L_n(x)dx = \delta_{mn}$

3.10. $\int_0^\infty e^{-x}x^m L_n(x)dx = \begin{cases} 0 & , m<n \\ (-1)^n n! & , m=n \end{cases}$

Lähde: Rainville E.D.: Special Functions, MacMillan, N.Y., 1960

Huom: Sovimme $L_{-1} \equiv 0$

4. LAGUERREN LIITTOPOLYNOMIT:

4.1. $zw''(z) + (1+p-z)w'(z) + nw(z) = 0$

4.2. $L_n^{(p)}(z) = \frac{1}{n!} z^{-p}e^z \frac{d^n}{dz^n}(e^{-z}z^{n+p}) \quad , p \in \mathbb{C}, n \in \mathbb{N}_0$

$L_n^0(z) = L_n(z)$

4.3. $L_n^{(p)}(z) = \sum_{k=0}^{n} \frac{(-1)^k \Gamma(p+n+1) z^k}{k!(n-k)!\Gamma(p+k+1)} \quad , p \neq -1,-2, \ldots$

4.4. $L_{n-m}^{(m)}(z) = (-1)^m \frac{d^m}{dz^m}L_n(z) \quad , m = 0,1,2, \ldots , n$

4.5. $L_0^{(p)}(z) = 1$

$L_1^{(p)}(z) = 1+p-z$

$L_2^{(p)}(z) = \frac{1}{2}(1+p)(2+p) - (2+p)z + \frac{1}{2}z^2$

$L_3^{(p)}(z) = \frac{1}{6}(1+p)(2+p)(3+p) - \frac{1}{2}(2+p)(3+p)z + \frac{1}{2}(3+p)z^2 - \frac{1}{6}z^3$

4.6. $(n+1)L_{n+1}^{(p)}(z) - (2n+p+1-z)L_n^{(p)}(z) + (n+p)L_{n-1}^{(p)}(z) = 0 \quad , n \in \mathbb{N}_0$

4.7. $\frac{d}{dz}\left(L_n^{(p)}(z) - L_{n-1}^{(p)}(z)\right) = -L_{n-1}^{(p)}(z) \quad , n \in \mathbb{N}_0$

4.8. $z\frac{d}{dz}L_n^{(p)}(z) = nL_n^{(p)}(z) - (n+p)L_{n-1}^{(p)}(z) \quad , n \in \mathbb{N}_0$

4.9. $\frac{d}{dz}L_n^{(p)}(z) = -L_{n-1}^{(p+1)}(z) \quad , n \in \mathbb{N}_0$

4.10. $\frac{d}{dz}L_n^{(p)}(z) = -\sum_{k=0}^{n-1} L_k^{(p)}(z) \quad , n \in \mathbb{N}$

$L_n^{(p+1)}(z) = \sum_{k=0}^{n} L_k^{(p)}(z) \quad , n \in \mathbb{N}_0$

4.11. $z^n = \sum_{k=0}^{n} \frac{(-1)^k n!}{(n-k)!} \frac{(n+p+1)}{(n+p+1)} L_k^{(p)}(z) \quad , \mathrm{Re}\{p\} > -1$

4.12. $\frac{1}{(1-\xi)^{p+1}} \exp\left(\frac{-z\xi}{1-\xi}\right) = \sum_{k=0}^{\infty} L_k^{(p)}(z)\xi^k$

4.13. $\int_0^\infty e^{-x}x^p L_m^{(p)}(x)L_n^{(p)}(x)dx = \frac{1}{n!}(n+p+1)\delta_{mn} \quad , \mathrm{Re}\{p\} > -1$

Lähde: Kuten edellisessä kohdassa

Huom: 1. $L_{-1}^{(p)} = 0$ (sopimus) 2. kun $p=0 \Rightarrow$ pykälän 3. kaavat

5. <u>BESSELIN FUNKTIOT</u>:

5.1. $w''(z) + \frac{1}{z}w'(z) + (1 - \frac{p^2}{z^2})w(z) = 0$

5.2. $J_p(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(p+k+1)}\left(\frac{z}{2}\right)^{2k+p}$, $p \neq -1,-2, \ldots$

$= (-1)^p J_{-p}(z)$, $p = -1,-2, \ldots$

5.3 $\frac{d}{dz}(z^{-p}J_p(z)) = -z^{-p}J_{p+1}(z)$

5.4. $\frac{d}{dz}(z^p J_p(z)) = z^p J_{p-1}(z)$

5.5. $\frac{d}{dz}J_p(z) = \frac{p}{z}J_p(z) - J_{p+1}(z)$

5.6. $\frac{d}{dz}J_p(z) = -\frac{p}{z}J_p(z) + J_{p-1}(z)$

5.7. $pJ_p(z) = \frac{z}{2}(J_{p-1}(z) + J_{p+1}(z))$

5.8. $\frac{d}{dz}J_p(z) = \frac{1}{2}(J_{p-1}(z) - J_{p+1}(z))$

5.9. $J_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\sin z$, $J_{\frac{-1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\cos z$

5.10. $J_n(z) = \frac{1}{\pi}\int_0^{\pi}\cos(n\varphi - z\sin\varphi)d\varphi$, $n \in \mathbb{Z}$

(ns. <u>Besselin integraali</u>)

5.11. $\exp \frac{1}{2}z(\zeta - \frac{1}{\zeta}) = \sum_{k=-\infty}^{\infty} J_k(z)\zeta^k$

5.12. $\int_0^1 xJ_n(\alpha_k x)J_n(\alpha_l x)dx = \frac{1}{2}J_{n+1}^2(\alpha_k)\cdot\delta_{kl}$

$J_n(\alpha_k) = J_n(\alpha_l) = 0$

6. <u>MUIDEN LAJIEN BESSELIN FUNKTIOITA</u>:

6.1. $Y_p(z) = \frac{\cos p\pi \cdot J_p(z) - J_{-p}(z)}{\sin p\pi}$, $p \neq 0, \pm 1, \ldots$

$Y_n(z) = \frac{2}{\pi}\left(\log(\frac{z}{2}) + \gamma\right)J_n(z)$

$+ \frac{1}{\pi}\sum_{k=0}^{\infty}\left\{\frac{(-1)^{k+1}}{k!(n+k)!}\left[\sum_{l=1}^{n+k}\frac{1}{l} + \sum_{l=1}^{k}\frac{1}{l}\right]\right\}\cdot(\frac{z}{2})^{n+2k}$

$- \frac{1}{\pi}\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}(\frac{z}{2})^{2k-n}$, $n=1,2, \ldots$

$Y_0(z) = \frac{2}{\pi}\left(\log(\frac{z}{2}) + \gamma\right)J_0(z)$

$+ \frac{2}{\pi}\sum_{k=0}^{\infty}\left\{\frac{(-1)^{k+1}}{(k!)^2}\sum_{l=1}^{k}\frac{1}{l}\right\}(\frac{z}{2})^{2k}$

6.2. $H_p^{(1)}(z) = J_p(z) + iY_p(z)$

$H_p^{(2)}(z) = J_p(z) - iY_p(z)$

6.3. $I_p(z) = i^{-p}I_p(iz) = \sum_{k=0}^{\infty}\frac{1}{k!\Gamma(p+k+1)}(\frac{z}{2})^{p+2k}$, $p \neq -1,-2, \ldots$

6.4. $K_p(z) = \frac{\pi}{2}\frac{I_{-p}(z) - I_p(z)}{\sin p\pi}$, $p \neq 0, \pm 1, \pm 2, \ldots$

$K_n(z) = \left(\log\frac{z}{2} + \gamma\right)I_n(z)$

$- \frac{1}{2}\sum_{k=0}^{\infty}\left\{\frac{1}{k!(n+k)!}\left[\sum_{l=1}^{n+k}\frac{1}{l} + \sum_{l=1}^{k}\frac{1}{l}\right]\right\}(\frac{z}{2})^{n+2k}$

$- \frac{1}{2}\sum_{k=0}^{n-1}\frac{(-1)^{n-k}(n-k-1)!}{k!}(\frac{z}{2})^{2k-n}$

6.5. $\mathrm{ber}_p x + i\,\mathrm{bei}_p x = I_p(i^{\frac{3}{2}}x)$, $p \in \mathbb{R}$

6.6. $\mathrm{ker}_p x + i\,\mathrm{kei}_p x = i^{-p}K_p(i^{\frac{1}{2}}x)$, $p \in \mathbb{R}$

6.7. $\mathrm{her}_p x + i\,\mathrm{hei}_p x = H_p^{(1)}(i^{\frac{3}{2}}x)$, $p \in \mathbb{R}$

6.8. $\mathrm{ner}_p x + i\,\mathrm{nei}_p x = Y_p(i^{\frac{3}{2}}x)$, $p \in \mathbb{R}$

6.9. $x^2y''(x) + (2\alpha+1)xy'(x) + (\beta^2\gamma^2x^{2\beta} - (\beta^2p^2 - \alpha^2))y(x) = 0$

$y(x) = C_1x^{-\alpha}J_p(\gamma x^{\beta}) + C_2x^{-\alpha}Y_p(\gamma x^{\beta})$

Lähde pykäliin 5 ja 6: Kuten pykälässä 1 ja lisäksi <u>Laasonen</u>: Matemaattisia erikoisfunktioita

Huom: Kaavat 5.2. ja 6.3. pätevät myös indekseillä $p=-1,-2,\ldots$, jos asianomaisessa sarjakehitelmässä merkitään $\left[\Gamma(p+k+1)\right]^{-1} = 0$, kun p+k+1 on negatiivinen kokonaisluku.

Kaavoissa 6.1. ja 6.4. esiintyvä vakio on sama kuin kaavassa 1.7. eli ns. <u>Eulerin</u> (tai <u>Mascheronin</u>) <u>vakio</u>.

Kaavasta 6.4. saadaan $K_0(z)$ merkitsemällä n=0 ja sopimalla, että $\sum_{k=0}^{-1} = 0$ (tämä huomautus sopii myös kohtaan 6.1.).

Voidaan osoittaa

$$Y_n(z) = \lim_{\varepsilon\to 0} Y_{n+\varepsilon}(z) \quad , \quad K_n(z) = \lim_{\varepsilon\to 0} K_{n+\varepsilon}(z)$$

7. LEGENDREN POLYNOMIT:

7.1. $$(1-z^2)w''(z) - 2zw'(z) + n(n+1)w(z) = 0$$

7.2. $$P_n(z) = \frac{1}{2^n n!}\frac{d^n}{dz^n}(z^2-1)^n \qquad \text{(Rodrigues)}$$

$$= \frac{1}{2^n}\sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k(2n-2k)!}{k!(n-k)!(n-2k)!} z^{n-2k}$$

$$= \frac{(2n)!}{2^n(n!)^2}\left\{z^n - \frac{n(n-1)}{2(2n-1)} z^{n-2} + \ldots\right\}$$

$$\left[\frac{n}{2}\right] = \begin{cases} \frac{n}{2} & \text{, kun n on parillinen} \\ \frac{n-1}{2} & \text{, kun n on pariton} \end{cases}$$

7.3. $$P_0(z) = 1$$

$$P_1(z) = z$$

$$P_2(z) = \frac{1}{2}(3z^2 - 1)$$

$$P_3(z) = \frac{1}{2}(5z^3 - 3z)$$

$$P_4(z) = \frac{1}{8}(35z^4 - 30z^2 + 3)$$

$$P_5(z) = \frac{1}{8}(63z^5 - 70z^3 + 15z)$$

7.4. $$P_n(1) = 1 \ , \ P_n(-1) = (-1)^n \ , \ P_{2n+1}(0) = 0$$

7.5. $$Q_n(z) = \frac{1}{2}P_n(z)\log\frac{z+1}{z-1} - \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \frac{2n-4k-1}{(2k+1)(n-k)} P_{n-2k-1}(z) \qquad , n \geq 1$$

7.6. $$Q_0(z) = \frac{1}{2}\log\frac{z+1}{z-1}$$

$$Q_1(z) = \frac{z}{2}\log\frac{z+1}{z-1} - 1$$

$$Q_2(z) = \frac{1}{4}(3z^2 - 1)\log\frac{z+1}{z-1} - \frac{3}{2}z$$

7.7. $$(n+1)P_{n+1}(z) - z(2n+1)P_n(z) + nP_{n-1}(z) = 0 \qquad , n \in \mathbb{N}_0$$

7.8. $$P'_{n+1}(z) + P'_{n-1}(z) = P_n(z) + 2zP'_n(z) \qquad , n \in \mathbb{N}_0$$

7.9. $$\begin{cases} P'_{n-1}(z) = zP'_n(z) - nP_n(z) \\ P'_{n+1}(z) = zP'_n(z) + (n+1)P_n(z) \end{cases} \qquad , n \in \mathbb{N}_0$$

7.10. $$P'_{n+1}(z) - P'_{n-1}(z) = (2n+1)P_n(z) \qquad , n \in \mathbb{N}_0$$

7.11. $$(z^2-1)P'_n(z) = nzP_n(z) - nP_{n-1}(z) \qquad , n \in \mathbb{N}_0$$

7.12. $$P_n(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(z + \sqrt{z^2-1}\cos\varphi)^n d\varphi$$

7.13. $$z^n = \frac{n!}{2^n}\sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(2n-4k+1)\Gamma(\frac{3}{2})}{k!\Gamma(n+\frac{3}{2}-k)} P_{n-2k}(z)$$

7.14. $$\int_{-1}^{1} x^m P_n(x)\,dx = \begin{cases} 0 & \text{, kun } m=0,1,\ldots,n-1 \\ \frac{(n!)^2 2^{n+1}}{(2n+1)!} & \text{, } m=n \end{cases}$$

7.15. $$\int_{-1}^{1} P_m(x)P_n(x)\,dx = \frac{1}{n+\frac{1}{2}}\delta_{mn}$$

7.16. $$(1 - 2z\zeta + \zeta^2)^{-\frac{1}{2}} = \sum_{k=0}^{\infty} P_k(z)\zeta^k \qquad , |2z\zeta| + |\zeta^2| < 1$$

Lähteet: Kuten aikaisemmin

8. LEGENDREN LIITTOFUNKTIOT:

8.1. $$(1-z^2)w''(z) - 2zw'(z) + \left[n(n+1) - \frac{m^2}{1-z^2}\right]w(z) = 0$$

$$w(z) = (1-z^2)^{\frac{m}{2}}\left(C_1\frac{d^m}{dz^m}P_n(z) + C_2\frac{d^m}{dz^m}Q_n(z)\right), \text{ kun } m=0,1,\ldots,n$$

8.2. $$P_n^m(z) = (1-z^2)^{\frac{m}{2}}\frac{d^m}{dz^m}P_n(z) \quad , P_n^0(z) = P_n(z)$$

$$Q_n^m(z) = (1-z^2)^{\frac{m}{2}}\frac{d^m}{dz^m}Q_n(z)$$

8.3. $$P_0^0(z) = 1$$

$$P_1^0(z) = P_1(z) \quad , P_1^1(z) = \sqrt{1-z^2}$$

$$P_2^0(z) = P_2(z) \quad , P_2^1(z) = 3z\sqrt{1-z^2}\, , P_2^2(z) = 3(1-z^2)$$

$$P_3^0(z) = P_3(z) \quad , P_3^1(z) = \tfrac{3}{2}(5z^2-1)\sqrt{z^2-1}$$

$$, P_3^2(z) = 15z(1-z^2),$$

$$, P_3^3(z) = 15(1-z^2)^{\frac{3}{2}}, \text{ jne.}$$

8.4. $$(2n+1)\sqrt{1-z^2}\,P_n^{m-1}(z) = P_{n+1}^m(z) - P_{n-1}^m(z) \quad , m,n\in\mathbb{N}_0 \, , m\le n$$

8.5. $$(2n+1)zP_n^m(z) = (n-m+1)P_{n+1}^m(z) + (n+m)P_{n-1}^m \quad , m,n\in\mathbb{N}_0, m\le n$$

8.6. $$(2n+1)\sqrt{1-z^2}\,P_n^{m+1}(z) = \quad , m,n\in\mathbb{N}_0, m\le n$$

$$= (n+m)(n+m+1)P_{n-1}^m(z)-(n-m)(n-m+1)P_{n+1}^m(z)$$

8.7. $$(2n+1)(1-z^2)\frac{d}{dz}P_n^m(z) = \quad , m,n\in\mathbb{N}_0, m\le n$$

$$= (n+m)(n+1)P_{n-1}^m(z) - n(n-m+1)P_{n+1}^m(z)$$

8.8. $$P_n^m(z) = \frac{1}{\pi}\frac{(n+m)!}{n!}\int_0^{\pi}(z+\sqrt{z^2-1}\cos\varphi)^n\cos m\varphi\, d\varphi$$

8.9. $$\int_{-1}^{1} P_n^m(x)P_k^m(x)dx = \frac{1}{n+\frac{1}{2}}\frac{(n+m)!}{(n-m)!}\,\delta_{nk}$$

Lähteet: Kuten aikaisemmin

Lisäksi Kurki-Suonio: Matemaattiset apuneuvot II osa 1

9. HARMONISET PALLOFUNKTIOT:

9.1. $$\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial u}{\partial\vartheta}\right) + \frac{1}{\sin^2\vartheta}\frac{\partial^2 u}{\partial\varphi^2} + n(n+1)u = 0$$

9.2. $$u_{nm}(\vartheta,\varphi) = P_n^m(\cos\vartheta)\cos m\varphi \quad , m=0,1,2,\ldots,n$$

$$v_{nm}(\vartheta,\varphi) = P_n^m(\cos\vartheta)\sin m\varphi \quad , m=1,2,3,\ldots,n$$

9.3. $$(u_{nm}, v_{kl}) = 0$$

$$(u_{nm}, u_{kl}) = 0 \quad , \text{ kun } n\ne k \text{ tai } m\ne l$$

$$(v_{nm}, v_{kl}) = 0 \quad , \text{ kun } n\ne k \text{ tai } m\ne l$$

9.4. $$(u_{n0}, u_{n0}) = \frac{2\pi}{n+\frac{1}{2}}$$

$$(u_{nm}, u_{nm}) = (v_{nm}, v_{nm}) = \frac{\pi}{n+\frac{1}{2}}\frac{(m+n)!}{(n-m)!} \quad , m\ge 1$$

$$(f,g) = \int_{(4\pi)} f(\vartheta,\varphi)g(\vartheta,\varphi)d\Omega =$$

$$= \int_0^{2\pi}\partial\varphi\int_0^{\pi} f(\vartheta,\varphi)g(\vartheta,\varphi)\sin\vartheta\,\partial\vartheta$$

10. HYPERGEOMETRISET JA KONFLUENTTIHYPERGEOMETRISET FUNKTIOT:

10.1. $$z(z-1)w''(z) + \left[(a+b+1)z-c\right]w'(z) + abw(z) = 0$$

$$w(z) = C_1F(a,b,c;z) + C_2z^{1-c}F(a-c+1,b-c+1,2-c;z) \quad , \text{ jos } c\notin\mathbb{Z}, \; |z|<1$$

10.2. $$F(a,b,c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad , \mathrm{Re}\{c-a-b\}>0,\; c\ne 0,-1,-2,\ldots$$

10.3. $$F(a,b,c;z) = 1 + \sum_{n=1}^{\infty}\frac{a(a+1)\ldots(a+n-1)b(b+1)\ldots(b+n-1)}{c(c+1)\ldots(c+n-1)}\frac{z^n}{n!},$$

$$c\ne 0,-1,\ldots \quad , |z|<1$$

10.4. $$\frac{d}{dz}F(a,b,c;z) = \frac{ab}{c}F(a+1,b+1,c+1;z)$$

10.5. $$(1+z)^n = F(-n,\beta,\beta;-z)$$

$$\log(1+z) = zF(1,1,2;-z)$$

$$P_n(z) = F(-n,n+1,1;\tfrac{1}{2}(1-z))$$

$$T_n(z) = F(n,-n,\tfrac{1}{2};\tfrac{1}{2}(1-z))$$

$$Q_n(z) = \frac{\sqrt{\pi}}{2^{n+1}} \frac{\Gamma(n+1)}{\Gamma(n+\ )} \frac{1}{z^{n+1}} F(\tfrac{n+1}{2},\tfrac{n+2}{2},\tfrac{n+3}{2};\tfrac{1}{z^2})$$

10.6. $zw''(z) + (c-z)w'(z) - aw(z) = 0$

$$w(z) = C_1F(a,c;z) + C_2z^{1-c}F(a-c+1,2-c;z) \qquad , c \notin \mathbb{Z}$$

10.7. $F(a,c;z) = 1 + \sum_{n=1}^{\infty} \frac{a(a+1)\ldots(a+n-1)}{c(c+1)\ldots(c+n-1)} \frac{z^n}{n!} \quad , c \neq 0,-1,\ldots \quad , |z|<\infty$

10.8. $H_{2m}(z) = (-1)^m \cdot \frac{(2m)!}{m!} F(-m,\tfrac{1}{2};z^2)$

$$H_{2m+1}(z) = (-1)^m \cdot \frac{(2m+1)!}{m!} zF(-m,\tfrac{3}{2};z^2)$$

$$I_n(z) = \frac{e^{-iz}}{\Gamma(n+1)} (\tfrac{z}{2})^n F(n+\tfrac{1}{2},2n+1;2iz)$$

$$L_n^m(z) = \frac{\Gamma(n+m+1)}{n!\Gamma(m+1)} F(-n,m+1;z)$$

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11. <u>TŠEBYSEVIN POLYNOMIT</u>:

11.1. $(1-x^2)y''(x) - xy'(x) + n^2y(x) = 0$

11.2. $T_n(x) = \cos(n\arccos x)$

11.3. $U_n(x) = (1-x^2)^{\frac{-1}{2}}\sin((n+1)\arccos x) = (1-x^2)^{\frac{-1}{2}}V_{n+1}(x)$

11.4. $2T_n(x)\,T_m(x) = T_{n+m}(x) + T_{n-m}(x) \qquad , n \geq m$

11.5. $$\int_{-1}^{1} T_m(x)T_n(x)\frac{dx}{\sqrt{1-x^2}} = \begin{cases} 0 & , m\neq n \\ \frac{\pi}{2} & , m=n\neq 0 \\ \pi & , m=n=0 \end{cases}$$

11.6. $T_0(x) = 1$

$$T_1(x) = x \qquad\qquad T_n(x) = \sum_{k=0}^{[\frac{n}{2}]} \frac{n!x^{n-2k}(x^2-1)^k}{(2k)!(n-2k)!}$$

$$T_2(x) = 2x^2 - 1$$

$$T_3(x) = 4x^3 - 3x$$

$$T_4(x) = 8x^4 - 8x^2 + 1$$

$$T_5(x) = 16x^5 - 20x^3 + 5x$$

11.7. $T_{n+1}(x) - 2xT_n(x) + T_{n-1}(x) = 0$

11.8. $$x^{2n} = \frac{1}{2^{2n-1}} \cdot \sum_{k=0}^{n-1}\binom{2n}{k}T_{2n-2k}(x) + \frac{1}{2^{2n-2}}\binom{2n}{n}T_0(x)$$

$$x^{2n+1} = \frac{1}{2^{2n}} \sum_{k=0}^{n}\binom{2n+1}{k}T_{2n+1-2k}(x)$$

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