Special Functions, Homework #7, Due 2.3.2011

(27) Find the second order polynomial $P(x)$ which is the best approximation of $\phi(x) = x^3$ in the sense that the integral $\int_0^\infty e^{-x} |\phi(x) - P(x)|^2 dx$ is as small as possible. (This has something to do with Laguerre polynomials.)

12.2.2 By differentiating the generating function, $g(t, x)$, with respect to t, multiplying by $2t$, and then adding $g(t, x)$, show that

(25)

$$\frac{1 - t^2}{(1 - 2tx + t^2)^{3/2}} = \sum_{n=0}^{\infty} (2n + 1) P_n(x) t^n.$$

This result is useful in calculating the charge induced on a grounded metal sphere by a point charge q.

(29) Compute the Fourier-Legendre transform of the function

$$\phi(x) = \begin{cases} 1, & 0 < x < 1, \\ -1, & -1 < x < 0. \end{cases}$$

(Cf. Arfken, Homework 12.3.9)

(30) Let P_n = Legendres polynomial. Show that the functions $u_n(\theta) = P_n(\cos\theta)$ are orthogonal with respect to the inner product

$$\langle \phi, g \rangle = \int_0^\pi \sin(\theta)\, \phi(\theta)\, \overline{g}(\theta)\, d\theta.$$

Also compute the norm of u_n with respect to this inner product (easy).

1

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In[5]:= Plot[-2 Sum[((-1)^n Sin[n x])/n, {n, 1, 40}], {x, -2 π, 2 π}]
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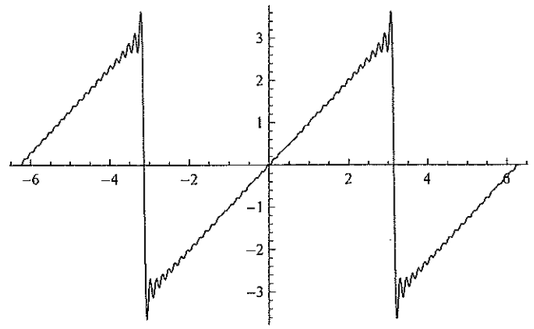
Out[5]=

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In[10]:= Plot[1/π + 1/2 Sin[x] - 2/π Sum[Cos[2 n x]/(4 n^2 - 1), {n, 1, 3}], {x, -2 π, 2 π}]
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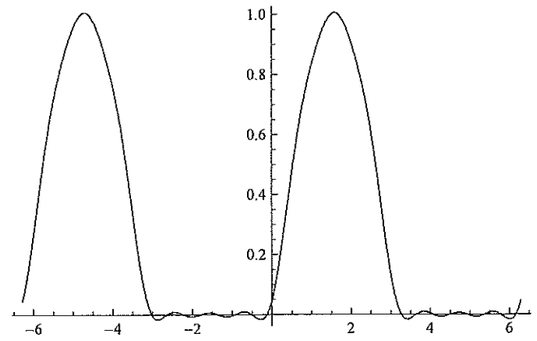
Out[10]=

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In[13]:= Plot[1/π + 1/2 Sin[x] - 2/π Sum[Cos[2 n x]/(4 n^2 - 1), {n, 1, 10}], {x, -2 π, 2 π}]
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Out[13]=