273018 Special Functions
Home work #3, due 2.2.2011

(10)
8.3.5 An atomic (quantum mechanical) particle is confined inside a rectangular box of sides a, b, and c. The particle is described by a wave function ψ which satisfies the Schrödinger wave equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi = E\psi.$$

The wave function is required to vanish at each surface of the box (but not to be identically zero). This condition imposes constraints on the separation constants and therefore on the energy E. What is the smallest value of E for which such a solution can be obtained?

$$\text{ANS.}\quad E = \frac{\pi^2\hbar^2}{2m}\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right).$$

(11)
8.5.2 A series solution of Eq. (8.80) is attempted, expanding about the point $x = x_0$. If x_0 is an ordinary point show that the indicial equation has roots $k = 0, 1$.

(12)
8.5.16 If the parameter a^2 in Eq. (8.110d) is equal to 2, Eq. (8.110d) becomes

$$y'' + \frac{1}{x^2}y' - \frac{2}{x^2}y = 0.$$

From the indicial equation and the recurrence relation *derive* a solution $y = 1 + 2x + 2x^2$. Verify that this is indeed a solution by substituting back into the differential equation.

(13)
8.6.11 Transform our linear, second-order, differential equation

$$y'' + P(x)y' + Q(x)y = 0$$

by the substitution

$$y = z\exp\left[-\frac{1}{2}\int^x P(t)\,dt\right]$$

and show that the resulting differential equation for z is

$$z'' + q(x)z = 0,$$

where

$$q(x) = Q(x) - \tfrac{1}{2}P'(x) - \tfrac{1}{4}P^2(x).$$

Note. This substitution can be *derived* by the technique of Exercise 8.6.24.

(14)
8.6.15 Given that one solution of

$$R'' + \frac{1}{r}R' - \frac{m^2}{r^2}R = 0$$

is $R = r^m$, show that Eq. (8.127) predicts a second solution, $R = r^{-m}$.

$$(8.127)\quad y_2(x) = y_1(x)\int^x \frac{\exp\left[-\int^{x_2} P(x_1)\,dx_1\right]}{[y_1(x_2)]^2}\,dx_2$$

(samma som formeln (4e) på sida 45 i föreläsningsanteckningarna.)

Table 8.4

Equation	Regular singularity $x =$	Irregular singularity $x =$
1. Hypergeometric $x(x-1)y'' + [(1+a+b)x - c]y' + aby = 0.$	$0, 1, \infty$	—
2. Legendre* $(1-x^2)y'' - 2xy' + l(l+1)y = 0.$	$-1, 1, \infty$	—
3. Chebyshev $(1-x^2)y'' - xy' + n^2y = 0.$	$-1, 1, \infty$	—
4. Confluent hypergeometric $xy'' + (c-x)y' - ay = 0.$	0	∞
5. Bessel $x^2y'' + xy' + (x^2 - n^2)y = 0.$	0	∞
6. Laguerre* $xy'' + (1-x)y' + ay = 0.$	0	∞
7. Simple harmonic oscillator $y'' + \omega^2y = 0.$	—	∞
8. Hermite $y'' - 2xy' + 2\alpha y = 0.$	—	∞

* The associated equations have the same singular points.

which shows that point $x = 0$ is a regular singularity. By inspection we see that there are no other singular points in the finite range. As $x \to \infty$ ($z \to 0$), from Eq. (8.78) we have the coefficients

$$\frac{2z - z}{z^2}\quad\text{and}\quad\frac{1 - n^2z^2}{z^4}.$$

Since the latter expression diverges as z^4, point $x = \infty$ is an irregular or essential singularity.

The ordinary differential equations of Section 8.3, plus two others, the hypergeometric and the confluent hypergeometric, have singular points, as shown in Table 8.4.

It will be seen that the first three equations in the preceding tabulation, hypergeometric, Legendre, and Chebyshev, all have three regular singular points. The hypergeometric equation with regular singularities at 0, 1, and ∞ is taken as the standard, the canonical form. The solutions of the other two may then be expressed in terms of its solutions, the hypergeometric functions. This is done in Chapter 13.

In a similar manner, the confluent hypergeometric equation is taken as the canonical form of a linear second-order differential equation with one regular and one irregular singular point.