273018 Special Functions
Home work #2, due 26.1.2011

(6) (Cf. 10.1.5 on page 595 in my copy of Arfken)

The Maxwells' distribution has the probability density
$$\rho(v) = C v^2 e^{-mv^2/2kT}, \quad 0<v<\infty,$$
where C, m, k, T are certain positive constants. Determine C from the condition $\int_0^\infty \rho(x)\,dx = 1$, and also find the mean and the variance of this distribution.

(7) Show that $\lambda = 0$ is not an eigenvalue of the heat equation treated on pp. 16–22 in the lecture notes. (The only solution for $\lambda = 0$ is identically zero.)

(8) Let $u(x,y) = w(\rho,\varphi)$ where $x = \rho\cos\varphi$ and $y = \rho\sin\varphi$ (that is, the Kartesian coordinates (x,y) are replaced by polar coordinates (ρ,φ)). Suppose that u and w are 2 times continuously differentiable. Show that
$$\frac{\partial w}{\partial\rho} = \cos\varphi\,\frac{\partial u}{\partial x} + \sin\varphi\,\frac{\partial u}{\partial y}$$
$$\frac{\partial w}{\partial\varphi} = -\rho\sin\varphi\,\frac{\partial u}{\partial x} + \rho\cos\varphi\,\frac{\partial u}{\partial y}$$
$$\frac{\partial^2 w}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial w}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 w}{\partial\varphi^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

Hint: The chain rule: $\frac{\partial w}{\partial\rho} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial\rho} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial\rho}$ etc.

(9) Show that it is also possible to solve the Helmholtz equation in cylindrical coordinates by first separating the angle variable φ, then the radial variable ρ, and finally the axial variable z.
(The final solution will be the same as in the lecture notes!?)

SELECTED FORMULAS

CARTESIAN COORDINATES: $u = u(x, y, z)$, $\mathbf{v} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}$

$$\nabla u = \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right)u = \frac{\partial u}{\partial x}\hat{\mathbf{i}} + \frac{\partial u}{\partial y}\hat{\mathbf{j}} + \frac{\partial u}{\partial z}\hat{\mathbf{k}}$$
$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$
$$\nabla\cdot\mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$
$$\nabla\times\mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{i}} - \left(\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z}\right)\hat{\mathbf{j}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{k}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}$$
$$\mathbf{R} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}, \quad d\mathbf{R} = dx\,\hat{\mathbf{i}} + dy\,\hat{\mathbf{j}} + dz\,\hat{\mathbf{k}}$$
$$dA = \begin{cases} dy\,dz \text{ (constant-}x\text{ surface)} \\ dx\,dz \text{ (constant-}y\text{ surface)}, \\ dx\,dy \text{ (constant-}z\text{ surface)} \end{cases} \quad dV = dx\,dy\,dz$$

CYLINDRICAL COORDINATES: $u = u(r, \theta, z)$, $\mathbf{v} = v_r\hat{\mathbf{e}}_r + v_\theta\hat{\mathbf{e}}_\theta + v_z\hat{\mathbf{e}}_z$

$$\nabla u = \left(\hat{\mathbf{e}}_r\frac{\partial}{\partial r} + \hat{\mathbf{e}}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\mathbf{e}}_z\frac{\partial}{\partial z}\right)u = \frac{\partial u}{\partial r}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial u}{\partial\theta}\hat{\mathbf{e}}_\theta + \frac{\partial u}{\partial z}\hat{\mathbf{e}}_z$$
$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\partial^2 u}{\partial z^2}$$
$$\nabla\cdot\mathbf{v} = \frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{1}{r}\frac{\partial}{\partial\theta}v_\theta + \frac{\partial}{\partial z}v_z$$
$$\nabla\times\mathbf{v} = \left(\frac{1}{r}\frac{\partial v_z}{\partial\theta} - \frac{\partial v_\theta}{\partial z}\right)\hat{\mathbf{e}}_r + \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right)\hat{\mathbf{e}}_\theta + \frac{1}{r}\left(\frac{\partial(rv_\theta)}{\partial r} - \frac{\partial v_r}{\partial\theta}\right)\hat{\mathbf{e}}_z$$
$$= \frac{1}{r}\begin{vmatrix} \hat{\mathbf{e}}_r & r\hat{\mathbf{e}}_\theta & \hat{\mathbf{e}}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ v_r & rv_\theta & v_z \end{vmatrix}$$
$$\mathbf{R} = r\hat{\mathbf{e}}_r + z\hat{\mathbf{e}}_z, \quad d\mathbf{R} = dr\,\hat{\mathbf{e}}_r + r\,d\theta\,\hat{\mathbf{e}}_\theta + dz\,\hat{\mathbf{e}}_z$$
$$dA = \begin{cases} r\,d\theta\,dz \text{ (constant-}r\text{ surface)} \\ dr\,dz \text{ (constant-}\theta\text{ surface)}, \\ r\,dr\,d\theta \text{ (constant-}z\text{ surface)} \end{cases} \quad dV = r\,dr\,d\theta\,dz$$