273018 Special Functions
Home work #1, due 19.1.2011

10.1.3 Show that

$$\frac{(s-n)!}{(2s-2n)!} = \frac{(-1)^{n-s}(2n-2s)!}{(n-s)!}.$$

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Here s and n are integers with $s < n$. This result can be used to avoid negative factorials such as in the series representations of the spherical Neumann functions and the Legendre functions of the second kind.

10.1.6 By transforming the integral into a gamma function, show that

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$$-\int_0^1 x^k \ln x \, dx = \frac{1}{(k+1)^2}, \qquad k > -1.$$

3 Show that the integral

$$\int_0^1 t^{z-1}(1-t)^{w-1}\,dt$$

converges absolutely if and only if $\operatorname{Re} z > 0$ and $\operatorname{Re} w > 0$. (This integral is $B(z,w)$.)

10.4.2 Verify the following beta function identities:

(a) $B(a,b) = B(a+1,b) + B(a,b+1)$,

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(b) $B(a,b) = \dfrac{a+b}{b} B(a,b+1)$,

(c) $B(a,b) = \dfrac{b-1}{a} B(a+1,b-1)$,

(d) $B(a,b)B(a+b,c) = B(b,c)B(a,b+c)$.

10.4.6 Show, by means of the beta function, that

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$$\int_t^z \frac{dx}{(z-x)^{1-\alpha}(x-t)^\alpha} = \frac{\pi}{\sin \pi\alpha}, \qquad 0 < \alpha < 1.$$

This result is used in Section 16.2 to solve Abel's generalized integral equation.

1. GAMMA- JA BEETTAFUNKTIOT:

1.1. $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt$, $\operatorname{Re}\{z\}>0$

1.2. $\Gamma(z) = \dfrac{\Gamma(z+n)}{z(z+1)\ldots(z+n-1)}$, $n=1,2,\ldots$, $\operatorname{Re}\{z\}>-n$

1.3. $\Gamma(z+1) = z\Gamma(z)$

1.4. $\Gamma(n+1) = n!$, $\Gamma(1) = 1$, $\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$

1.5. $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin \pi z}$

1.6. $\Gamma(2z) = \pi^{-\frac{1}{2}}\cdot 2^{2z-1}\Gamma(z)\Gamma(z+\tfrac{1}{2})$

1.7. $\dfrac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{n=1}^{\infty}\left[(1+\tfrac{z}{n})e^{-\frac{z}{n}}\right]$, missä

$$\gamma = \lim_{m\to\infty}\left\{1+\tfrac{1}{2}+\ldots+\tfrac{1}{m} - \log m\right\} \approx 0.5772157\ldots$$

1.8. $\Gamma(x) = e^{-x}x^{x-\frac{1}{2}}\cdot\sqrt{2\pi}\left[1+\dfrac{1}{12x}+\dfrac{1}{288x^2}-\dfrac{139}{51840x^3}+O(\dfrac{1}{x^5})\right]$

(ns. Stirlingin kaava)

1.9. $B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\,dt$, $\operatorname{Re}\{p\}, \operatorname{Re}\{q\}>0$

1.10. $B(p,q) = \dfrac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$

1.11. $B(p,q) = 2\int_0^{\frac{\pi}{2}} \cos^{2p-1}\varphi\cdot\sin^{2q-1}\varphi\,d\varphi$, $\operatorname{Re}\{p\}, \operatorname{Re}\{q\}>0$

1.12. $B(x,y) = \int_0^\infty t^{x-1}(1+t)^{-(x+y)}\,dt$, $\operatorname{Re}\{x\}, \operatorname{Re}\{y\}>0$

Lähde: Whittaker-Watson: A Course of Modern Analysis, 4th ed., Cambridge, Univ.Press, 1963

Huom: Monikäsitteisissä funktioissa t^z ($z\in\mathbb{C}$, $t\in\mathbb{R}$) valitaan aina päähaara $t^z = e^{z\log t}$, jossa $\log t$ valitaan reaaliseksi