of $\mathscr{T}_X$ such that γ has rank $r+1$, δ has rank $n-1$, and $\beta = \gamma\delta$. (We can choose δ to be idempotent, and γ so as to differ from β at only one point of X.) By induction, every element of $\mathscr{T}_X$ of defect k $(1 \leqq k \leqq n-1)$ is expressible as the product of an element of $\mathscr{G}_X$ and k (idempotent) elements of defect 1.

(b) If α is an element of $\mathscr{T}_X$ of defect 1, then every other element of $\mathscr{T}_X$ of defect 1 can be expressed in the form $\lambda\alpha\mu$ with λ and μ in $\mathscr{G}_X$.

(c) If α is an element of $\mathscr{T}_X$ of defect 1, then $\langle \mathscr{G}_X, \alpha \rangle = \mathscr{T}_X$. (Vorobev [1953b].)

1.2 Light's associativity test

To test a finite groupoid $S(\cdot)$ for associativity, when the operation $(\cdot)$ is given by a Cayley multiplication table, is usually quite a tedious business. The following procedure was suggested to one of the authors by Dr. F. W. Light in 1949.

This procedure is to be carried out for each element a of the groupoid S. However, we shall show below that it suffices to carry it out for each element a of a set of generators of S.

Consider the two binary operations $(*)$ and $(\circ)$ defined in S as follows:

$$x * y = x \cdot (a \cdot y), \quad x \circ y = (x \cdot a) \cdot y.$$

Associativity holds in $S(\cdot)$ if and only if, for each fixed element a of S, these two binary operations coincide. The idea is essentially to construct the Cayley tables for $(*)$ and $(\circ)$ and then see if they are the same.

The $(*)$-table is obtained from the original $(\cdot)$-table by replacing, for each y in S, the y column by the $a \cdot y$ column. Similarly, to make up the $(\circ)$-table we simply copy down in the x row the $x \cdot a$ row of the $(\cdot)$-table. However, we do not need to write out the $(\circ)$-table, since we can check directly whether the x row of the $(*)$-table coincides with the $x \cdot a$ row of the $(\cdot)$-table.

For convenience in performing the test, we replace the top index line of the $(*)$-table by the a row of the $(\cdot)$-table, and the left-hand index column by the a column of the $(\cdot)$-table. For each entry $a \cdot y$ in the a row of the $(\cdot)$-table tells us what column of the $(\cdot)$-table to copy down as the y column of the $(*)$-table, and each entry $x \cdot a$ in the a column of the $(\cdot)$-table tells us which row of the $(\cdot)$-table should be compared with x row of the $(*)$-table. For example, let $S(\cdot)$ be defined by the table:

$\cdot$	a	b	c	d	e
a	a	a	a	d	d
b	a	b	c	d	d
c	a	c	b	d	d
d	d	d	d	a	a
e	d	e	e	a	a

The set $\{c, e\}$ generates S, since $a = e \cdot e$, $b = c \cdot c$, and $d = c \cdot e$. The $(*)$-tables (with index rows and columns modified as described above) for the elements c and e are as follows:

c	a	c	b	d	d
a	a	a	a	d	d
c	a	c	b	d	d
b	a	b	c	d	d
d	d	d	d	a	a
e	d	e	e	a	a

e	d	e	e	a	a
d	d	d	d	a	a
d	d	d	d	a	a
d	d	d	d	a	a
a	a	a	a	d	d
a	a	a	a	d	d

Thus, to form the c-table, copy the c row $(a\ c\ b\ d\ d)$ from the $(\cdot)$-table into the upper index line, and similarly the c column into the left-hand index column. Now copy the columns of the $(\cdot)$-table in the order specified by the upper index line, i.e., the a column, the c column, etc. We now verify that the rows of the c-table thus formed are just those of the $(\cdot)$-table labelled by the left-hand index column. One may prefer to copy the rows of the $(\cdot)$-table as specified by the left-hand index column, and then check that the columns are correctly labelled. Since this is found to check for both the c-table and the e-table, we conclude that $S(\cdot)$ is associative.

That it suffices to carry out Light's procedure only for a set of generators of S is an immediate consequence of the fact that *the set of all elements a of a groupoid S that associate with all elements of S, in the sense that $x(ay) = (xa)y$ for all x, y in S, is a subsemigroup of S.* For if we assume that a and b are elements of S such that $x(ay) = (xa)y$ and $x(by) = (xb)y$ for all x, y in S, then

$$\begin{aligned} x((ab)y) &= x(a(by)) = (xa)(by) \\ &= ((xa)b)y = (x(ab))y. \end{aligned}$$

Thus if a and b associate with all elements of S, so does ab.

EXERCISES FOR §1.2

1. Check for associativity:

	e	f	g	a
e	e	e	e	e
f	f	f	f	f
g	g	g	g	g
a	e	e	f	e

2. Check for associativity:

	e	f	g	a	0
e	e	a	e	a	0
f	0	f	g	0	0
g	g	f	g	f	0
a	0	a	e	0	0
0	0	0	0	0	0

Clifford - Preston The Algebraic Theory of Semigroups, Vol. I