

(i)  $H$  contains an idempotent if and only if  $Y$  meets each equivalence class of  $X \text{ mod } \pi$  in exactly one element (i.e.,  $Y$  is a "cross-section" of  $\pi$ ).

(ii) If  $H$  contains an idempotent, then  $H$  induces and is isomorphic with the symmetric group  $\mathcal{G}_Y$  on  $Y$ .

**PROOF.** (i) Let  $\epsilon$  be an idempotent element of  $H$ . Thus  $Y = X\epsilon$ ,  $\pi = \pi_\epsilon$ , and  $\epsilon^2 = \epsilon$ . The mapping  $\epsilon$  leaves every element of  $Y$  fixed, and moves every element of  $X \setminus Y$ . Let  $x \in X$ . Since  $x\epsilon = (x\epsilon)\epsilon$  it follows from  $\pi = \pi_\epsilon$  that  $x\pi x\epsilon$ . On the other hand, if  $y$  and  $y'$  are elements of  $Y$  such that  $y\pi y'$ , then  $y = y\epsilon = y'\epsilon = y'$ . Hence each equivalence class of  $X \text{ mod } \pi$  contains exactly one element of  $Y$ , and  $\epsilon$  maps every element of  $y\pi^\perp$  ( $y$  in  $Y$ ) upon  $y$ .

Conversely, assume that  $Y$  is a cross-section of  $\pi$ . Then the element of  $\mathcal{T}_X$  which maps each element  $x$  of  $X$  upon the element  $y$  of  $Y$  such that  $x\pi y$  is clearly an idempotent element of  $H$ .

(ii) Assume that  $H$  contains an idempotent  $\epsilon$ . Let  $\alpha \in H$ . For each  $x$  in  $X$ ,  $x\alpha \in X\alpha = Y = X\epsilon$ , and so  $x\alpha\epsilon = x\alpha$ ; this implies that  $\alpha\epsilon = \alpha$ . For each  $x$  in  $X$ ,  $x\pi x\epsilon$  (shown above), and since  $\pi_\alpha = \pi = \pi_\epsilon$ , we have  $x\alpha = (x\epsilon)\alpha$ ; this implies that  $\epsilon\alpha = \alpha$ .

We now show that  $\alpha$  induces a permutation of  $Y$ . If  $y\alpha = y'\alpha$  ( $y, y'$  in  $Y$ ) then  $y\pi y'$ , and so  $y = y'$ . Given  $y$  in  $Y = X\alpha$ , there exists  $x$  in  $X$  such that  $x\alpha = y$ . Then  $(x\epsilon) \in Y$  and  $(x\epsilon)\alpha = x\alpha = y$ . Hence,  $(\alpha|Y) \in \mathcal{G}_Y$ .

Every element  $\phi$  of  $\mathcal{G}_Y$  is induced by some element  $\alpha$  of  $H$ , namely that defined by  $x\alpha = (x\epsilon)\phi$ . Moreover,  $\alpha$  is uniquely determined by  $\phi$ . For if  $y\alpha = y\beta$  for all  $y$  in  $Y$ , with  $\alpha$  and  $\beta$  in  $H$ , then  $x\epsilon\alpha = x\epsilon\beta$  for all  $x$  in  $X$ , so that  $\alpha = \epsilon\alpha = \epsilon\beta = \beta$ . Hence the mapping  $\alpha \rightarrow \phi = \alpha|Y$  is a one-to-one mapping of  $H$  upon  $\mathcal{G}_Y$ , evidently an isomorphism. Hence  $H$  is a subgroup of  $\mathcal{T}_X$  isomorphic with  $\mathcal{G}_Y$ .

As an example, we write out all the  $\mathcal{D}$ -classes of  $\mathcal{T}_4$  ( $\mathcal{T}_X$  with  $|X| = 4$ ). Let  $X = \{1, 2, 3, 4\}$ . We shall write  $(i\ j\ k\ l)$  for the mapping  $1 \rightarrow i, 2 \rightarrow j, 3 \rightarrow k, 4 \rightarrow l$ . There are four  $\mathcal{D}$ -classes  $D_r$  ( $r = 1, 2, 3, 4$ ), where  $D_r$  is the set of all elements of rank  $r$ . The headings for the rows are partitions of  $\{1, 2, 3, 4\}$ ; those for the columns are subsets of  $\{1, 2, 3, 4\}$ . We omit  $D_4$ , which consists of a single  $\mathcal{H}$ -class; it is just the symmetric group of degree 4 (order 24) on  $\{1, 2, 3, 4\}$ . Starred elements are idempotent; these show which cells are groups. Table 4 gives the whole  $\mathcal{D}$ -picture of  $\mathcal{T}_4$ , the numbers 1, 2, 6, 24 giving merely the number of elements in each  $\mathcal{H}$ -class.

TABLE 1

$D_1$	{1}	{2}	{3}	{4}
{1234}	(1111)*	(2222)*	(3333)*	(4444)*

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Theory of Geodesy, Vol. I  
Classification : The Affine

$D_3$	{123}	{124}	{134}	{234}
{1}	{(2)} {34}	{(234)} * {(2133)} {(1233)} * {(2133)}	{(2344)} {(3244)} {(2344)} *	{(3242)} {(4322)} {(3242)}
{1}	{(3)} {24}	{(1232)} * {(2131)} {(1232)} * {(2131)}	{(1234)} * {(2141)} {(1234)} * {(2141)}	{(1222)} {(3221)} {(1222)} {(3221)}
{1}	{(4)} {23}	{(1223)} {(2113)} {(1223)} {(2113)}	{(1224)} * {(2114)} {(1224)} * {(2114)}	{(1224)} * {(2112)} {(1224)} * {(2112)}
{2}	{(3)} {14}	{(1231)} * {(2132)} {(1231)} * {(2132)}	{(1232)} {(2132)} {(1232)} {(2132)}	{(1233)} {(2132)} {(1233)} {(2132)}
{2}	{(4)} {13}	{(1213)} {(2123)} {(1213)} {(2123)}	{(1214)} * {(2124)} {(1214)} * {(2124)}	{(1214)} * {(2121)} {(1214)} * {(2121)}
{3}	{(4)} {12}	{(1123)} {(2213)} {(1123)} {(2213)}	{(1124)} * {(2214)} {(1124)} * {(2214)}	{(1124)} * {(2211)} {(1124)} * {(2211)}

TABLE 3

$D_2$	{12}	{13}	{14}	{23}	{24}	{34}
{1}	{(234)}	{(1222)} *	{(2111)}	{(1333)} *	{(1444)} *	{(2444)}
{2}	{(134)}	{(1211)} *	{(1311)}	{(1411)}	{(2422)}	{(3222)}
{3}	{(124)}	{(1121)}	{(2212)}	{(3313)}	{(4414)}	{(3343)}
{4}	{(123)}	{(1112)}	{(2221)}	{(3331)}	{(4441)}	{(3334)}
{12}	{(34)}	{(1122)}	{(2222)}	{(3332)}	{(4442)}	{(3334)}
{13}	{(24)}	{(1212)} *	{(2233)} *	{(3322)}	{(4444)}	{(3343)}
{14}	{(23)}	{(1221)} *	{(2244)} *	{(3321)}	{(4441)}	{(3341)}

TABLE 2

2.2  $\mathcal{G}$ -STRUCTURE OF FULL SEMIGROUP  $\mathcal{G}^x$  ON A SET X 55

TABLE 4

“EGG-BOX PICTURE” OF  $\mathcal{T}_4$ 

NUMBER OF ELEMENTS

$D_4$	$24$	$1 \cdot 1 \cdot 24 = 24$																																										
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$D_3$	<table border="1"> <tbody> <tr><td>6</td><td>6</td><td>6</td><td>6</td></tr> <tr><td>6</td><td>6</td><td>6</td><td>6</td></tr> <tr><td>6</td><td>6</td><td>6</td><td>6</td></tr> </tbody> </table>	6	6	6	6	6	6	6	6	6	6	6	6	$7 \cdot 6 \cdot 2 = 84$																														
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$D_1$	<table border="1"> <tbody> <tr><td>1</td><td>1</td><td>1</td><td>1</td></tr> </tbody> </table>	1	1	1	1	Total $4^4 = \underline{\underline{256}}$																																						
1	1	1	1																																									

## EXERCISES FOR §2.2

1. (a) Let  $Y$  be a non-empty subset of a set  $X$ . There exists in  $\mathcal{T}_X$  at least one projection of  $X$  upon  $Y$ , that is, a mapping  $\epsilon$  of  $X$  upon  $Y$  leaving each element of  $Y$  fixed. The idempotent elements in the  $\mathcal{L}$ -class of  $\mathcal{T}_X$  corresponding to  $Y$  by Theorem 2.9 (iv) are just the projections of  $X$  upon  $Y$ .

(b) Let  $\pi$  be any partition of  $X$ . There exists (by the Axiom of Choice) at least one cross-section  $Y$  of  $\pi$ . For each  $x$  in  $X$ , let  $x\epsilon$  be the element  $y$  of  $Y$  such that  $x\pi y$ . We call  $\epsilon$  a representative mapping of  $\pi$ . The idempotents in the  $\mathcal{R}$ -class of  $\mathcal{T}_X$  corresponding to  $\pi$  by Theorem 2.9 (v) are just the representative mappings of  $\pi$ .

(c) By Lemma 1.13,  $\mathcal{T}_X$  is regular (cf. Exercise 1 of §1.9).

2. (a) Each  $\mathcal{L}$ -class of rank  $r$  in  $\mathcal{T}_n$  ( $= \mathcal{T}_X$  with  $|X| = n$ ) contains  $r^{n-r}$  idempotents.