93 Conditional or Mixed PP

Draw a λ at random from a dist'n.
Then form the PP (λ).
In other words, there is a positive r.v. L such that, conditional on $L = \lambda$,

$N(t)$ is a PP with parameter λ.

This process is called conditional or mixed Poisson process. (L is the mixing dist'n.)

$$\mathbb{P}\big(N(t+s) - N(s) = n\big) =$$

$$\int \mathbb{P}\big(N(t+s) - N(s) = n \,\big|\, L = \lambda\big)\, g(\lambda)\, d\lambda = (*)$$

where g is the (continuous) density of L.

If L takes two values: λ_1 and λ_2 with prob. p and $q = 1-p$ we get

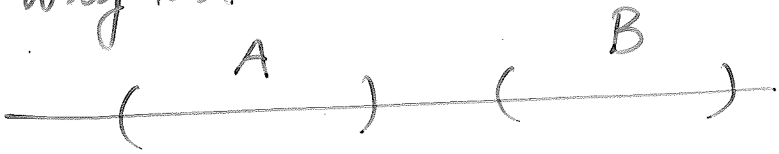
$$\mathbb{P}\big(N(t+s) - N(s) = n\big) =$$

$$p \cdot e^{-\lambda_1 t} \frac{(\lambda_1 t)^n}{n!} + (1-p)\, e^{-\lambda_2 t} \frac{(\lambda_2 t)^n}{n!}$$

$$\circledast = \int e^{-\lambda t}\,\frac{(\lambda t)^n}{n!}\cdot g(\lambda)\,d\lambda \qquad (5.27)$$

A mixed PP is <u>not</u> a PP in general.*
It has stationary increments, though, as shown by (5.27).

* Why not?



The no. of points in A gives us indications as to what the value of λ is, affecting the probability distn of the no. of points in B.

Ex. Assume we have two kinds of customers in an insurance company. Type I are accident-prone $\lambda_1 = 10$, type II are prudent, $\lambda_2 = 1$. Probability of a random customer being of type I or II is equal, $= \frac{1}{2}$.
Assume $N(1) = 10$, calculate $E\big(N(2) - N(1)\big)$.

$N(1) = 10$ given.

$$\mathbb{P}\left(\lambda = \lambda_1 \mid N(1) = 10\right) = \frac{\mathbb{P}\left(\lambda = \lambda_1 \text{ and } N(1) = 10\right)}{\mathbb{P}\left(N(1) = 10\right)} =$$

$$= \frac{e^{-10} \dfrac{10^{10}}{10!}}{\mathbb{P}(N(1)=10, \lambda=\lambda_1) + \mathbb{P}(N(1)=10, \lambda=\lambda_2)}$$

$$= \frac{e^{-10} \cdot \dfrac{10^{10}}{10!}}{e^{-10} \cdot \dfrac{10^{10}}{10!} + e^{-1} \cdot \dfrac{1}{10!}} \equiv p_1$$

$p_2 \equiv 1 - p_1$

$\mathbb{E}\left(N(2) - N(1) \mid \lambda = \lambda_1\right) = 10$ and

$\mathbb{E}\left(N(2) - N(1) \mid \lambda = \lambda_2\right) = 1$

$\mathbb{E}\left(N(2) - N(1) \mid N(1) = 10\right) = p_1 \cdot 10 + p_2 \cdot 1$

Example 5.29

$$g(\lambda) = \theta e^{-\theta\lambda} \frac{(\theta\lambda)^{m-1}}{(m-1)!} \qquad \lambda > 0$$

gamma (m, θ)

In this case $N(t)$ has a dist'n which can be explicitly calculated :

$$\int_0^\infty e^{-\lambda t}\frac{(\lambda t)^n}{n!}\,\theta\, e^{-\theta\lambda}\frac{(\theta\lambda)^{m-1}}{(m-1)!}\,d\lambda$$

$$= \frac{1}{n!\,(m-1)!}\,t^n\theta^m\int_0^\infty e^{-(t+\theta)\lambda}\,\lambda^{n+m-1}\,d\lambda$$

$$= \frac{1}{n!\,(m-1)!}\,\frac{t^n\theta^m\,(n+m-1)!}{(t+\theta)^{n+m}}\cdot$$

$$\int_0^\infty \frac{(t+\theta)\,e^{-(t+\theta)\lambda}\,\lambda^{n+m-1}\,(t+\theta)^{n+m-1}}{(n+m-1)!}\,d\lambda$$

$\uparrow$ density of Gamma$(n+m,\ t+\theta)$

$$= \binom{n+m-1}{n}\left(\frac{\theta}{t+\theta}\right)^m\left(\frac{t}{t+\theta}\right)^n\cdot 1$$

negative binomial distr.

with param. $\frac{\theta}{t+\theta}$

= no. of failures before m successes

in a binomial trial with

$p = \frac{\theta}{t+\theta}$ prob. of success.

97 Influence of observations: If we observe $N(t)$ for some time we get better and better estimate of the λ chosen.

$f_{L|N(t)}$ the cond. density of L given $N(t)$ sought

$$\mathbb{E}(N(t)\,|\,L) = Lt \quad \text{gives } EN(t) = E(L)\cdot t$$

$$Var(N(t)\,|\,L) = Lt$$

$$Var(N(t)) = E(L)t + t^2 \, Var(L)$$

Why?

$$Var(N(t)) = E(N(t)^2) - \underbrace{(E(N(t)))^2}_{t^2(E(L))^2}$$

$$E(N(t)^2) = E\left(E(N(t)^2\,|\,L)\right)$$

$$= E\left(Lt + L^2t^2\right) = t\,E(L) + t^2E(L^2)$$

$$= t\,E(L) + t^2\left(Var(L) + (E(L))^2\right)$$

so

$$Var(N(t)) = t\,E(L) + t^2\,Var(L)$$

$f_{L|N(t)}$?

$$\mathbb{P}\left(L \le x \mid N(t) = m\right) = \frac{\mathbb{P}\left(L \le x,\ N(t) = m\right)}{\mathbb{P}\left(N(t) = m\right)}$$

$$= \frac{\int_0^\infty \mathbb{P}\left(L \le x,\ N(t) = m \mid L = \lambda\right) g(\lambda)\, d\lambda}{\mathbb{P}\left(N(t) = m\right)}$$

$$= \frac{\int_0^x \mathbb{P}\left(N(t) = m \mid L = \lambda\right) g(\lambda)\, d\lambda}{\mathbb{P}\left(N(t) = m\right)}$$

$$= \frac{\int_0^x e^{-\lambda t} \frac{(\lambda t)^m}{\cancel{m!}}\, g(\lambda)\, d\lambda}{\int_0^\infty e^{-\lambda t} \frac{(\lambda t)^m}{\cancel{m!}}\, g(\lambda)\, d\lambda}$$

density! differentiate w.r.t. x

$$\frac{e^{-xt} \cdot (x\cancel{t})^m\, g(x)}{\int_0^\infty e^{-\lambda t} \cdot (\lambda \cancel{t})^m\, g(\lambda)\, d\lambda}$$

 Evaluate at $x = \lambda$

$$f_{L \mid N(t)}(\lambda \mid m) =$$

$$(5.28) \qquad \frac{e^{-\lambda t}\, \lambda^{m}\, g(\lambda)}{\int_0^\infty e^{-\lambda t}\, \lambda^{m}\, g(\lambda)\, d\lambda} \qquad \lambda \geq 0$$

Discrete case $\mathbb{P}(L = \lambda_j) = g_j \quad j = 1, 2, \dots$

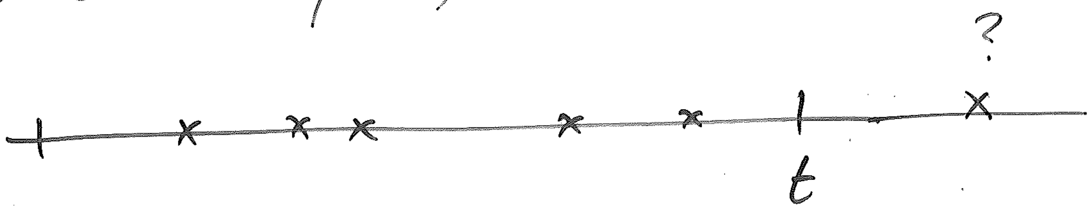
$$\frac{e^{-\lambda_j t}\, \lambda_j^{m}\, g_j}{\sum_{j=1}^{\infty} e^{-\lambda_j t}\, \lambda_j^{m}\, g_j}$$

Case $\lambda_1 = \lambda_2 = \frac{1}{2}$

$$\frac{e^{-\lambda_j t}\, \lambda_j^{m}}{e^{-\lambda_1 t}\, \lambda_1^{m} + e^{-\lambda_2 t}\, \lambda_2^{m}}$$

Ex. 5.30 Insurance company. Customer rating "λ" means that claims will be initiated at the rate λ (per year). The prob. dist'n of λ is uniform over $(0,1)$. Given that a customer has made n claims during his/her first t years, what is the cond'l dist'n of the time to the next claim.

Note. If we know the rate is λ then the time is $\mathrm{Exp}(\lambda)$.



$$\mathbb{P}(T > x \mid N(t) = n)$$

T is the time until the next claim

$$= \int_0^1 \mathbb{P}(T > x \mid L = \lambda,\ N(t) = n)\, f_{L\mid N(t)}(\lambda \mid n) \cdot d\lambda$$

$g(\lambda) \equiv 1$
$0 < \lambda < 1$

$$= \frac{\int_0^1 e^{-\lambda x} \cdot e^{-\lambda t} \cdot \lambda^n \, d\lambda}{\int_0^1 e^{-\lambda t} \lambda^n \, d\lambda}$$