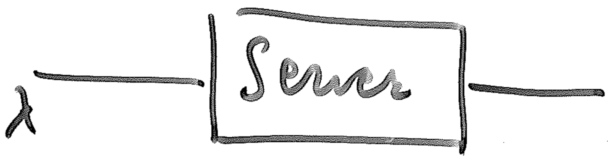
85 Ex. 5.27 Busy Periods in Single-Server Poisson Arrival Queue



1 server

M / G / 1

general

"Markov", i.e., Poisson

Idle period : no customer in system

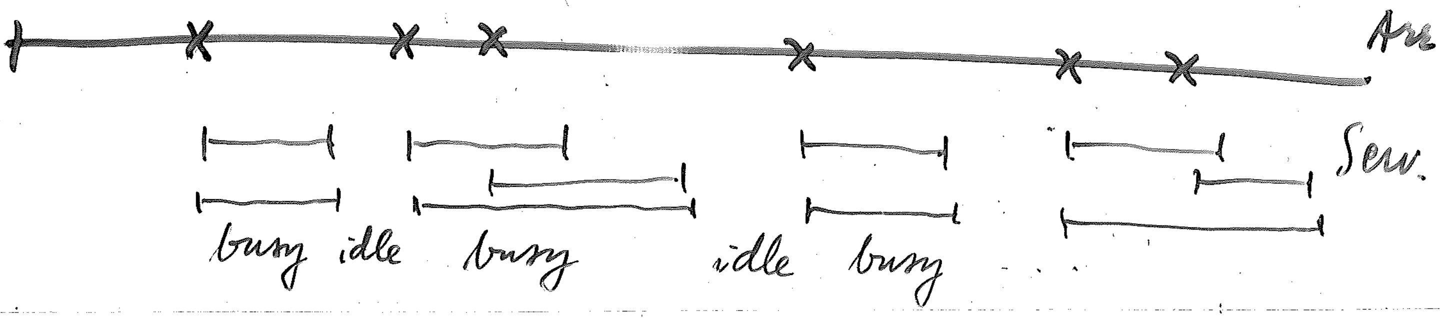
Busy period : customers in system

Arriving customer, PP (λ) :

if no one waiting and no one in service : service can start immediately

if somebody being served : wait in line

Of course, we assume that the service times are indep. with the same dist'n [but not necessarily exponential, in that case M/M/1 queue]



 The busy period will <u>start</u> when a customer enters and there is no line, and no one is being served. Each busy period B has the same distribution. (This is because of the memoryless prop. of the exp. dist'n.) We also say that the start of the busy period is a <u>regeneration</u> point.

Let the service time of this customer be S, and let $N(S)$ be equal to the no. of arrivals during that time.

If $N(S) = 0$ then $B = S$ [and the system becomes idle at the time S]

If $N(S) = 1$, then at time S the service of this one customer begins. The busy period starting at S is a r.v. B_1 <u>with the same dist'n as B.</u>

$$B = S + B_1$$

where, of course, B_1 is indep. of S.

If $N(S) = n > 1$, then at time S there will be n customers, $C_1, C_2, \dots, C_n$ standing in line.

87 To determine the busy period dist'n it does not matter in what order these customers are served.

The order is called the <u>queue</u> <u>discipline</u>.

Standard ones:

FIFO first in, first out

LIFO last in, first out

etc.

We will introduce, for computational purposes only, a peculiar queue discipline.

C_1 is served first

but C_2 is served only <u>after</u> all arrivals after S are served. In other words, C_2 is served when only $C_2, C_3, \dots, C_n$ are left in the queue. C_3 when only $C_3, C_4, \dots, C_n$ remain etc.

C_1: service begins

$N(S_1)$ customers arrive

they are served "out of the way"

⋮

C_2: service begins

$N(S_2)$ customers arrive

they are served ..

C_3: service begins

Time has same dist'n as B

$B_2 \sim B$ indep. of B_1

$$B = S + \sum_{i=1}^{N(S)} B_i$$

where all the B_i's are indep. and identically dist'd. B_i's has the <u>same</u> dist'n as B.

$$E(B) = E\left(E\left(B \mid S\right)\right)$$

$$E\left(B \mid S\right) = S + E\left(\sum_{i=1}^{N(S)} B_i \,\Big|\, S\right)$$

$$= S + \lambda S\, E(B)$$

$$Var(B \mid S) = \lambda S\, E(B^2)$$

$$E(B) = E\left(S + \lambda S\, E(B)\right)$$

$$= E(S) + \lambda E(S)\, E(B)$$

$$E(B) = \frac{E(S)}{1 - \lambda E(S)}$$

if $\lambda E(S) < 1$.

89 Furthermore, one can show that

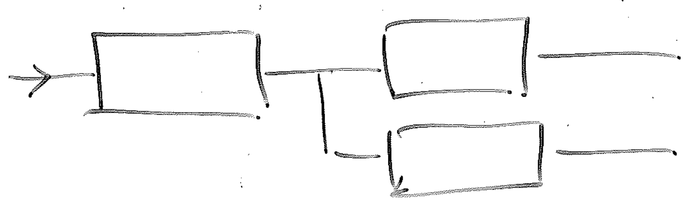
$$\mathrm{Var}(B) = \frac{\mathrm{Var}(S) + \lambda E^3(S)}{(1 - \lambda E(S))^3}$$

Note. The ave. length of the idle period is $\frac{1}{\lambda}$.

In Queueing Theory one studies
M/G/1, M/G/2, M/G/k, M/G/∞
M/M/1, ..., G/G/1, ... queues
with different queue disciplines.

Issues: Waiting time of customer
Queue length
Idle period
Busy period
⋮

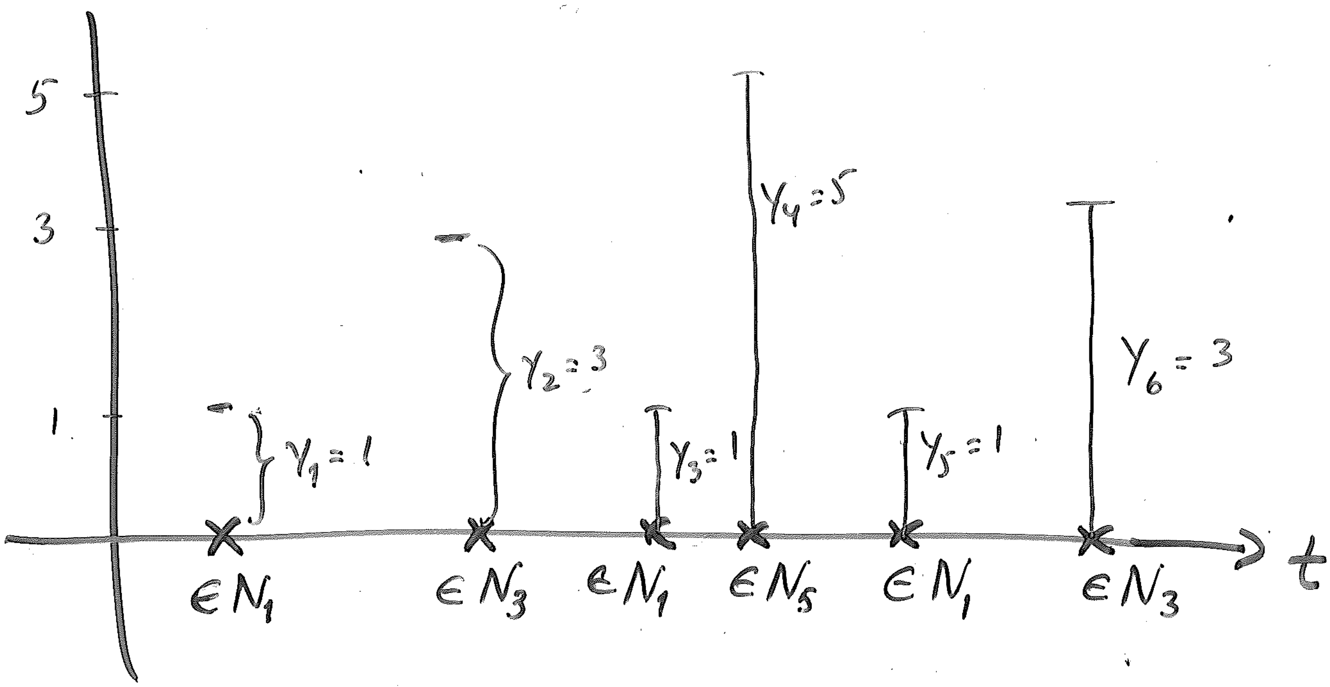
More complicated queue systems, e.g.,



90 Compound PP if Y takes only finitely many values:
(or denumerably)

$$X(t) = \sum_{i=1}^{N(t)} Y_i$$

Let $\mathbb{P}(Y_1 = \alpha_j) = p_j$. Then the PP ($\lambda$) may be written as a sum of PP's N_j where a point of the original process N is classified as belonging to N_j if $Y_i = \alpha_j$:



91 Then

$$X(t) = \sum_j \alpha_j N_j(t) \qquad (5.26)$$

Compute

$EX(t)$ and $Var(X(t))$:

$$E(X(t)) = \sum_j \alpha_j E(N_j(t))$$

$$= \sum_j \alpha_j \lambda p_j t$$

$$= \lambda t \mathbb{E}(Y_1), \quad \text{as it should}$$

$$Var(X(t)) = \sum_j \alpha_j^2 Var(N_j(t)) \quad \text{indep. } N_j\text{'s}$$

$$= \sum_j \alpha_j^2 \lambda t p_j = \lambda t \mathbb{E}(Y_1^2).$$

<u>Cor</u>. As $t \to \infty$, $X(t)$ is approx. normal

<u>Pf</u>. $N_j(t)$ is a Poisson var. When $t \to \infty$ $N_j(t)$ approaches a normal dist'n :

$$N_j(t) = \underbrace{\sum_{k=1}^{t} N_j(k) - N_j(k-1)}_{\text{sum of i.i.d. variables.}} \quad (t \text{ integer})$$

X is a sum of approx. normals, so itself approx. normal.

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Cor. If $X^{(1)}$ and $X^{(2)}$ are compound PP (with λ_1 and λ_2, respectively), and F_1, F_2 are the dist'n of the corresp. Y-r.v.'s

$$X^{(1)}(t) = \sum_{i=1}^{N^{(1)}(t)} Y_i^{(1)}$$

$$X^{(2)}(t) = \sum_{i=1}^{N^{(2)}(t)} Y_i^{(2)}$$

then $X^{(1)} + X^{(2)}$ is also compound Poisson with param. $\lambda_1 + \lambda_2$ and a point of the sum process belongs to $X^{(1)}$ with prob. $\frac{\lambda_1}{\lambda_1+\lambda_2}$ and to $X^{(2)}$ with prob. $\frac{\lambda_2}{\lambda_1+\lambda_2}$

Then sum process may be written

$$\sum_{i=1}^{N^{(1)}(t)+N^{(2)}(t)} Y_i$$

where the dist'n of Y_1 is $\frac{\lambda_1}{\lambda_1+\lambda_2} F_1 + \frac{\lambda_2}{\lambda_1+\lambda_2} F_2$.