# 5.4. Generalizations of Poisson Processes

## 5.4.1. Nonhomogeneous P.P

Def 5.4. The counting process $\{N(t), t \geq 0\}$ is said to be a <u>nonhomogeneous Poisson process</u> with intensity fct $\lambda(t)$, $t \geq 0$, if

(i) $N(0) = 0$

(ii) $\{N(t), t \geq 0\}$ has independent increments

(iii) $\mathbb{P}\big(N(t+h) - N(t) \geq 2\big) = o(h)$
for all $t$

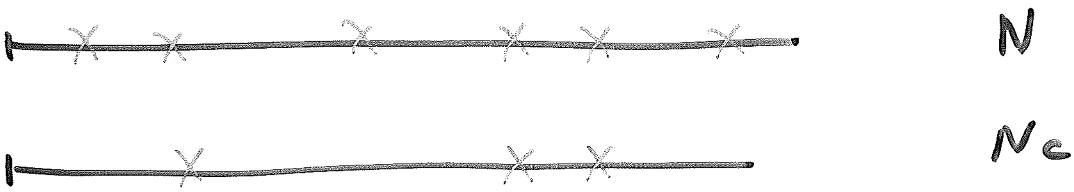
(iv) $\mathbb{P}\big(N(t+h) - N(t) = 1\big) = \lambda(t)h + o(h)$

Of course, $\mathbb{P}\big(N(t+h) - N(t) = 0\big)$
$= 1 - \lambda(t)h + o(h)$

Ex. • Electronic components : failure prob. increase with time
• Life insurance : mortality increases with $t$

Ex. Sampling a PP

Let $N(t)$, $t \geq 0$, be a PP $(\lambda)$. Suppose an event happening at time $t$ is counted with a prob $p(t)$, independent of everything else




Call the resulting process $N_c(t)$, $t \geq 0$.

$N_c$ is a nonhomogeneous PP with intensity $\lambda p(t)$:

1. $N_c(0) = 0$
2. $N_c(s+t) - N_c(s)$ depends only on the events in $(s, s+t]$ and the probabilities $p(u)$, $s < u \leq s+t$. In particular, indep. of $N_c(s)$.
3. $N_c(t+h) - N_c(t) \geq 2$ only if $N(t+h) - N(t) \geq 2$ so it has prob. $o(h)$
4. $\mathbb{P}(N_c(t+h) - N_c(t) = 1) =$
$\mathbb{P}(N_c(t+h) - N_c(t) = 1 \mid \underbrace{N(t+h) - N(t)}_{=1})$.

$$\cdot \, \mathbb{P}(N(t+h) - N(t) = 1) \; +$$

$$\mathbb{P}\Big(N_c(t+h) - N_c(t) = 1 \;\Big|\; N(t+h) - N(t) \geq 2\Big) \cdot$$

$$\cdot \, \mathbb{P}\big(N(t+h) - N(t) \geq 2\big)$$

$$= \mathbb{P}\Big(N_c(t+h) - N(t) = 1 \;\Big|\; N(t+h) - N(t) = 1\Big)$$

$$\cdot \big(\lambda h + o(h)\big) + o(h) = \lambda h \, p(t) + o(h)$$

$p$ cont's assumed in this derivation.

<u>Prop 5.4.</u> Let $\{N(t), t \geq 0\}$ and $\{M(t), t \geq 0\}$ be indep. nonhomogeneous PP with respective intensities $\lambda(t)$ and $\mu(t)$. Let $N^*(t) = N(t) + M(t)$. Then

(a) $\{N^*(t), t \geq 0\}$ is a nonhomogeneous PP with intensity $\lambda(t) + \mu(t)$

(b) Given that an event of the $N^*$-process occurs at time $t$ then, independently of what occurred prior to $t$, the event at $t$ was from the $N(t)$-process with prob. $\dfrac{\lambda(t)}{\lambda(t) + \mu(t)}$.

74 Pf. skipped

Any non-homogeneous PP process with <u>bounded</u> $\lambda(t)$ may be seen as a sampled homogeneous PP :

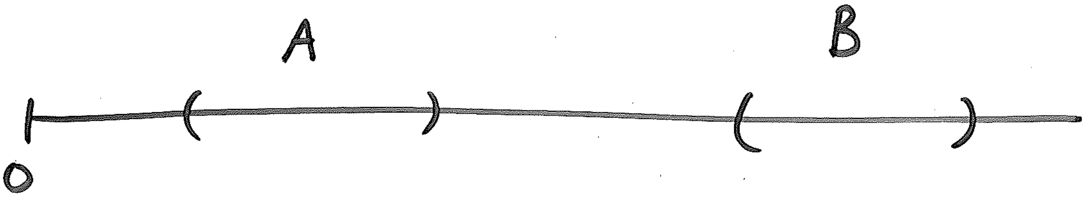
Let $\lambda > \lambda(t)$, all $t$. Let $\mu(t) = \lambda - \lambda(t)$

$$\underset{\substack{\text{intensity:}\\ \lambda(t)}}{N(t)} + \underset{\substack{\text{int.:}\\ \mu(t)\\ (=\lambda-\lambda(t))}}{M(t)} = \underset{\substack{\text{intensity:}\\ \lambda}}{N^*(t)}$$

In this interpretation $N^*(t)$ is the no. of counted events in $(0, t]$. Then mean no. is

$$m(t) = \int_0^t \lambda(y)\, dy$$

$m(t)$ is the <u>mean value function</u> of $N^*(t)$.




# events in A <u>indep.</u> # events in B

No. of events in $(s, t]$ is Poisson with

75 parameter $\int_s^t \lambda(y)\,dy$

Ex 5.24 Siegbert's hot dog stand, opens at 8 a.m. From 8 – 11 customers arrive at random, but at an increasing rate. First 5 cust./h at 8 a.m. and a max. of 20 cust./h at 11 a.m. Between 11 a.m. and 1 p.m. the average rate seems to remain constant at 20 cust./h. From 1 p.m. the ave. arrival rate drops steadily until closing time at 5 p.m. when it is 12 cust./h.

Assuming the no. of customers during disjoint intervals are independent r.v.'s [no regulars coming every day, e.g. !] we may model this arrival process as a non-homogeneous PP.

The rate is 0 when the stand is closed, 20 between 11 and 1, varies linearly between 8 and 11 a.m. and 1 and 5 p.m.

76 rate:

$$\lambda(t) = \begin{cases} 0 & 0 \le t \le 8 \\ 5 + 5(t-8) & 8 < t \le 11 \\ 20 & 11 < t \le 13 \\ 20 - 2(t-13) & 13 < t \le 17 \\ 0 & t > 17 \end{cases}$$

No. of arrivals between 8 and 9 ?

Poisson r.v. with parameter

$$\int_8^9 \lambda(t)\,dt = 5 + \frac{5}{2} = 7{,}5$$

Prob. of no arrival between 8 and 9 ?

$$e^{-7.5}$$

No. of arrivals between 12 and 14 ?

Poisson r.v. with parameter

$$\int_{12}^{14} \lambda(t)\,dt = \int_{12}^{13} 20\,dt + \int_{13}^{14} \big(20 - 2(t-13)\big)\,dt$$

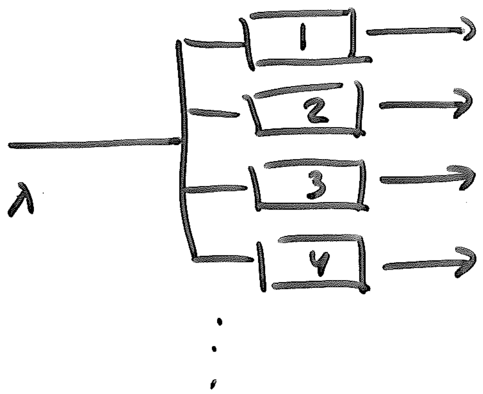
$$= 20 + 20 - 1 = 39$$

Ex. 2.5. The output of a $M/G/\infty$ queue

no. of servers ($\infty$)

Poisson process arrivals ($M$)

general service time dist'n ($G$)



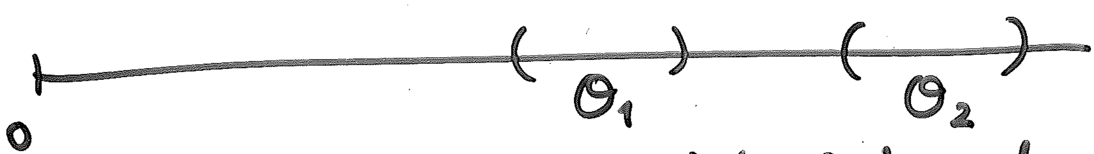

Input: arrivals as a PP $(\lambda)$

Service: general service time, <u>independent</u> of other servers and previous service times, dist'n $G$ (assumed cont's and differentiable)

Claim:

**The output process is a non-homogeneous PP with intensity $\lambda G(t)$**

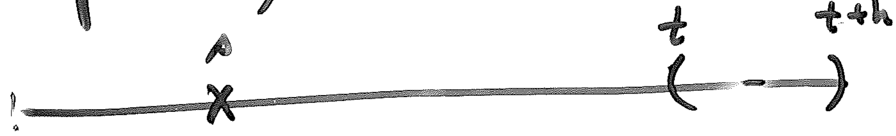
1° The departure process has independent increments




disjoint intervals

Call an arrival (i.e., an event in the PP ($\lambda$) arrival process) a type I event if it departs in $\Theta_1$, type II if it departs in $\Theta_2$. Give rise to two indep. Poisson processes (thinned versions of the original) (Prop. 5.3). Similarly, the departure events falling into $k$ disjoint intervals form $k$ independent PP's.

2° Take an arrival process $N$. Count an arrival event if it departs (= service is completed) in the interval $(t, t+h)$




Consider an arrival at $s$. It is counted if the service time is $\in (t-s, t+h-s]$, which has prob. $G(t+h-s) - G(t-s)$.

Hence the expected no. of departures in $(t, t+h)$ is

$$\lambda \int_0^{t+h} G(t+h-s) - G(t-s) \, ds$$

$$= \lambda \int_0^{t+h} G'(t+h-s) \cdot h \, ds + o(h) =$$

subst. $t+h-s = y$

$$= \lambda h \int_0^{t+h} G'(y)\,dy + o(h) =$$

$$= \lambda h\, G(t) + o(h)$$

$$\mathbb{P}(1 \text{ dep. in } (t, t+h)) = \lambda G(t) h\, e^{-\lambda G(t) h} + o(h)$$

$$\mathbb{P}(0 \quad - \text{"} - \quad) = e^{-\lambda G(t) h} + o(h)$$

$$\mathbb{P}(\geq 2 \quad - \text{"} - \quad) = o(h)$$

$\square$

Similarly, for $S_n$ the $n^{\text{th}}$ event of PP $\lambda(t)$

$$\mathbb{P}(t < S_n < t+h) = \mathbb{P}(N(t) = n-1, \text{ one event in } (t, t+h)) + o(h) = \cdots$$

$$\cdots = \lambda(t) e^{-m(t)} \frac{(mt)^{n-1}}{(n-1)!} h + o(h)$$

[where

$$m(t) = \int_0^t \lambda(s)\,ds \;]$$

and hence its density is

$$f_{S_n}(t) = \lambda(t) e^{-m(t)} \frac{(m(t))^{n-1}}{(n-1)!}$$

79a.

Let $\mu(t) = \int_0^t \lambda(s)\,ds$ where $\lambda$ is positive and continuous. $\mu(t)$ is a strictly increasing function of $t$.

It has an inverse, $g$, say.

$$\mu(g(t)) = g(\mu(t)) = t.$$

Let $Z(t) = N(g(t))$ where $N$ is a PP $(\lambda(t))$. Then $Z$ is a <u>homogeneous</u> PP $(1)$.

$Z(t)$ = no. of events in $(0, g(t))$ in the original process $N$.

This no. is a Poisson r.v. with parameter:

$$\int_0^{g(t)} \lambda(s)\,ds \underset{\substack{\uparrow \\ \text{def.}}}{=} \mu(g(t)) = t$$

Also, $Z$ has indep. increments. Easy!

$Z$ is a <u>time-changed version</u> of $N$.

7.9.b

Ex. $\lambda(t) = 1 + t$

$$\mu(t) = \int_0^t (1+s)\,ds = t + \frac{t^2}{2}$$

$$g(t) = \sqrt{1+2t} - 1$$

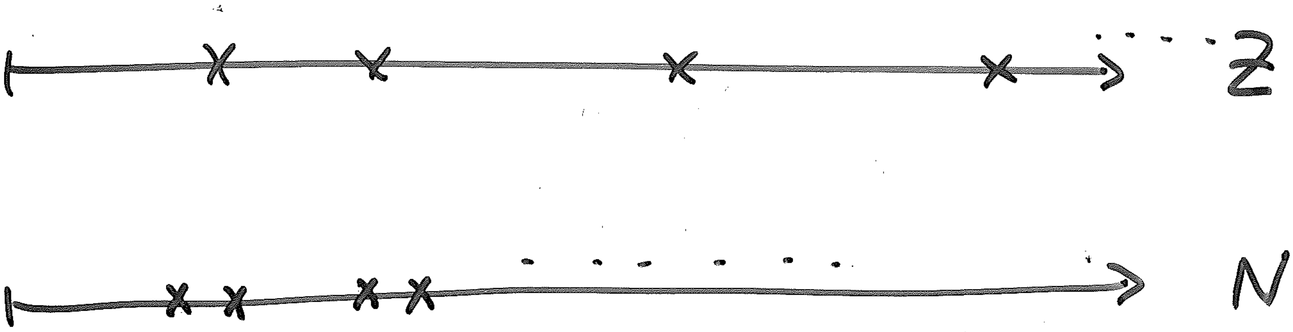
$$\mathbb{E}(Z(1)) = 1$$

$$\mathbb{E}(N(g(1))) = \int_0^{\sqrt{3}-1} (1+s)\,ds = \sqrt{3}-1 + \frac{(\sqrt{3}-1)^2}{2}$$

$$= \sqrt{3}-1 + \frac{3+1-2\sqrt{3}}{2}$$

$$= 1.$$

This can be used for simulation purposes. To simulate N (non-homogeneous), simulate Z instead and perform a time-change



## 5.4.2. Compound Poisson Process

**Def.** $X(t)$, $t \geq 0$ is a <u>compound Poisson process</u> if it can be represented as

$$X(t) = \sum_{i=1}^{N(t)} Y_i , \quad t \geq 0$$

$$= 0 \qquad \text{if } N(t) = 0$$

where $\{N(t), t \geq 0\}$ is a PP and $\{Y_i, i \geq 1\}$ is a family of indep. and identically dist'd r.v.'s.

Ex. · If $Y_i \equiv 1$ all $i$ then it is the ordinary PP

· $\underbrace{\text{Claims}}_{Y_i}$ are indep. with a dist'n $G$.

They occur as a PP $(\lambda)$.

$N(t)$ is the no. of accidents before time $t$, $X(t)$ is the total claim amount accumulated by time $t$

- Assume customers leave a shop as a PP $(\lambda)$. Suppose each customer bought independently, the amount being $Y_i$ i.i.d. Then $X(t)$ is the total amount spent by time $t$

We have

$$(5.24) \quad \mathbb{E}\,X(t) = \lambda t\, \mathbb{E}(Y_1)$$

$$(5.25) \quad \mathrm{Var}\,X(t) = \lambda t\, \mathbb{E}(Y_1^2)$$

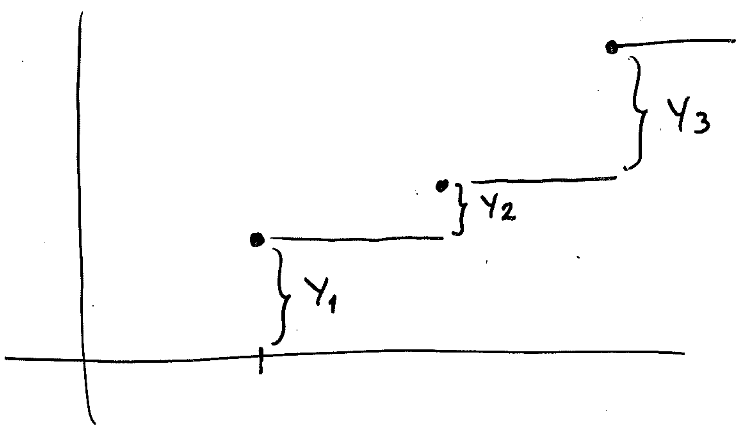
Recall general formula for $S = \sum_{i=1}^{N} Y_i$

$E(S) = E(N)\,E(Y_1)$

$\mathrm{Var}(S) = \sigma^2 E(N) + \mu^2 \mathrm{Var}(N)$

(where $\sigma^2 = \mathrm{Var}(Y_1)$, $\mu = E(Y_1)$)

Here $N(t)$ is Poisson $(\lambda t)$ so $E(N(t)) = \lambda t$ and $\mathrm{Var}\,N(t) = \lambda t$.

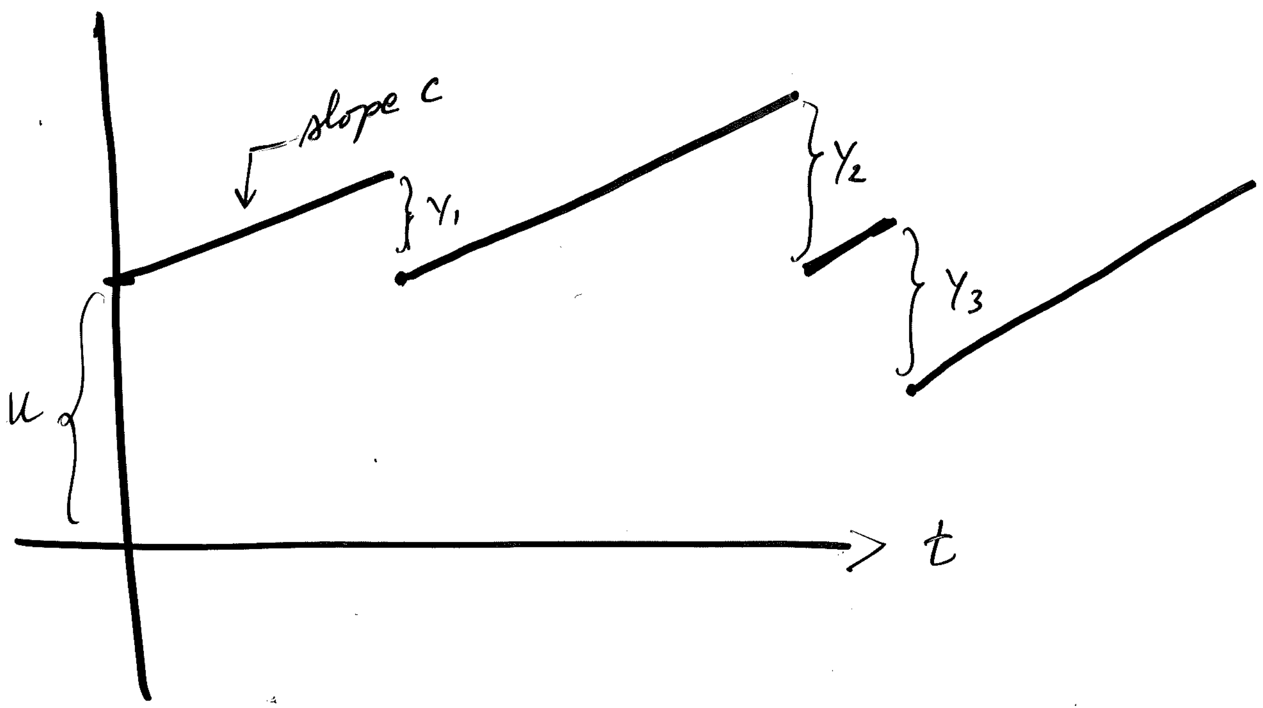


Ex. Assume premiums are accumulated at rate $c$, i.e., the premium income is $ct$ by time $t$.

Initial capital is $u \geq 0$.

Net worth of the insurance company (with this sole insurance product) is

$$-X(t) + u + ct = C(t)$$




Ruin : $C(t) < 0$

Ruin probability :

$$\mathbb{P}(C(t) < 0 \text{ for some } t > 0)$$

If $N, Y, c$ are unchanged we can consider the ruin probability to be a function of the initial capital $u$:

$$R(u) = \mathbb{P}\left\{ u + ct < X(t) \text{ for some } t > 0 \right\}$$

Observations: If $u < 0$ then $R(u) = 1$.

If $\mathbb{E}\, C(t) < 0$ then $R(u) = 1$. Why?

$$\mathbb{E}(C(t)) = u + ct - \lambda t\, \mathbb{E}(Y_1)$$
$$= u + t(c - \lambda \mathbb{E}(Y_1))$$

This is negative, for some $t$, for positive $u$, if and only if

$$c - \lambda \mathbb{E}(Y_1) < 0$$

Then, by the law of large numbers, $C(t) \to -\infty$ and $\mathbb{P}(C(t) < 0$ for some $t) = 1$.

Practically explained: The premium rate $c$ is too low.

84 If $c = \lambda \mathbb{E}(Y_1)$ and $u$ is large, then one might think that the company is safe since $\mathbb{E}(C(t)) = u$. But one can show that $R(u) = 1$ for all $u > 0$.

If $c > \lambda \mathbb{E}(Y_1)$ we collect a large enough premium to offset the claims, on average.

One can show (Lundberg, Cramér,* ...) that $R(u) \approx e^{-\rho u}$ for large $u$.

$\rho > 0$ is the Lundberg exponent

Lundberg, Filip (1876–1965)

Usually one takes

$$c = (1+\theta) \lambda \mathbb{E}(Y_1)$$

where $\theta$ is a safety loading.

The Lundberg exponent $\rho$ depends on the whole distribution of $Y_1$. One can expect larger variance to lead to smaller $\rho$.

*"collective risk theory"