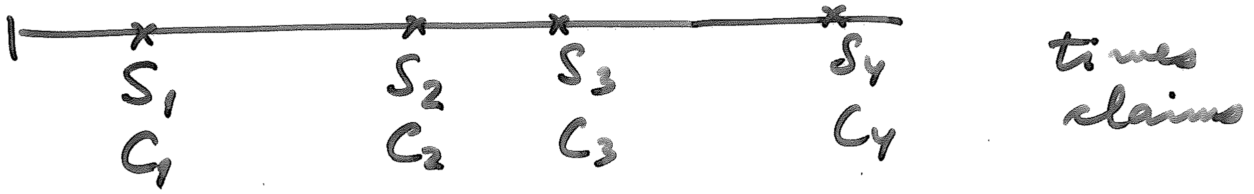
Ex. 5.21

Insurance claims are made as a PP (λ).
The claim amounts are indep. r.v.,
dist'n G with mean μ



$D(t)$ = total discounted cost by time t

$$\sum_{i=1}^{N(t)} e^{-\alpha S_i} C_i$$

If $\alpha = 0$ this is a <u>compound Poisson process</u> to be studied later.

α discount rate

$$\mathbb{E}\, D(t) = \sum \mathbb{E}\big(D(t) \,\big|\, N(t)=n\big) \cdot \mathbb{P}(N(t)=n)$$

$$e^{-\lambda t} \frac{(\lambda t)^n}{n!}$$

$\mathbb{E}\big(D(t) \,\big|\, N(t)=n\big)$ [is $n\mu$ if $\alpha=0$]

$$= \mathbb{E}\Big(\sum_{i=1}^{n} e^{-\alpha S_i} C_i \,\Big|\, N(t)=n\Big)$$

$$= \mathbb{E}\left[\sum_{i=1}^{n} C_i e^{-\alpha U_{(i)}}\right]$$

$U_{(1)}, \dots, U_{(n)}$ uniform r.v's on $[0, t]$, ordered

indep. $\to$ "

$$\sum_{i=1}^{n} \mathbb{E}(C_i)\, \mathbb{E}\left(e^{-\alpha U_{(i)}}\right)$$

$$\parallel$$

$$\mu \sum_{i=1}^{n} \mathbb{E}\left(e^{-\alpha U_i}\right) = \circledast$$

$U_1, U_2, \dots, U_n$ uniform r.v.'s on $[0, t]$

N.B. $U_{(1)}, U_{(2)}, \dots, U_{(n)}$ are the same as $U_1, U_2, \dots, U_n$, only in different order

$$\circledast = \mu \cdot n \cdot \mathbb{E}\left(e^{-\alpha U_1}\right)$$

$$= \mu\, n\, \frac{1}{t} \int_0^t e^{-\alpha s}\, ds$$

$$= \frac{\mu n}{t} \frac{1}{\alpha}\left(1 - e^{-\alpha t}\right)$$

$$= n \frac{\mu}{\alpha t}\left(1 - e^{-\alpha t}\right)$$

lim when $\alpha \to 0$ is $n\mu$ as it should.

$$\mathbb{E}\big(D(t)\,\big|\,N(t)\big) = N(t)\cdot\frac{\mu}{\alpha t}\left(1-e^{-\alpha t}\right)$$

exp.: λt

$$\mathbb{E}\big(D(t)\big) = \frac{\lambda\mu}{\alpha}\left(1-e^{-\alpha t}\right)$$

Ex. 5.22

Arrivals PP (λ)
At T they are all sent away (dispatching time). What is a good intermediate time of dispatch U, $0<U<T$, when the objective is to minimize the waiting time for the items.

item arrives before U: waits $U-s$
at s

— " — after U: $T-s$

Total waiting time:
Take any arrival before U. Its arrival time is <u>uniformly</u> distributed on $[0,U]$, thus its mean is $\frac{U}{2}$.
Take an arrival in $(U,T]$. Again it

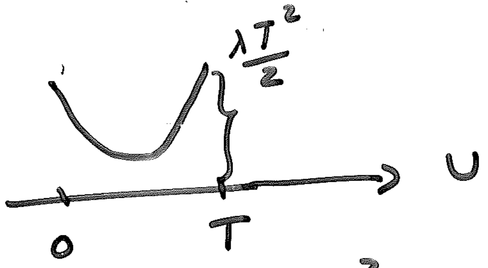
64 has uniform distribution in $(U, T]$, so the mean is $\frac{T+U}{2}$, and waiting time $T - \frac{T+U}{2}$

No. of arrivals before U has expectation λU. No of arrivals before T (but after U) has expected value $\lambda(T-U)$.
Total waiting time has expectation

$$\frac{U}{2}\cdot\lambda U + \lambda(T-U)\cdot\frac{T-U}{2}$$

$$= \frac{\lambda U^2}{2} + \frac{\lambda(T-U)^2}{2}$$

Quadratic poly. in U
Minimum at $\frac{T}{2}$



Minimal value, for $U = \frac{T}{2}$, is $\frac{\lambda T^2}{4}$.

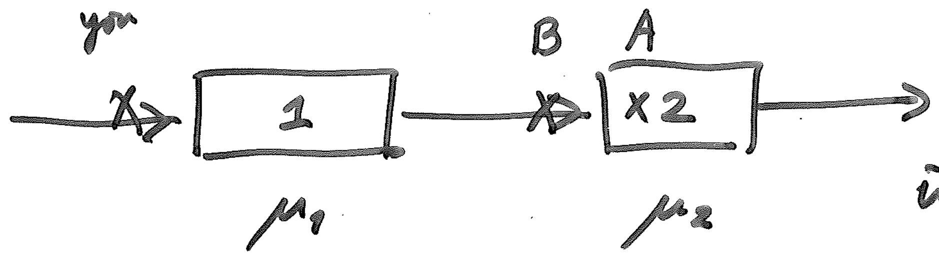
Prop. 5.4. Given that $S_n = t$, the set $S_1, \dots, S_{n-1}$ has the dist'n of a set of (ordered) indep. uniformly distributed r.v.'s on $(0, t)$.

"Pf." $S_1, \dots, S_{n-1} \mid N(s) = n-1 \sim U_{(1)}, \dots, U_{(n-1)}$ on $(0, s)$

has the same distribution

Let $s \to t^-$.

65 Exercise 20, p. 348



indep. exp. service

1 is free, but 2 is busy + one customer waiting

(a) P_A : A is still in service when you are ready from server 1

$$\mathbb{P}\left(A\text{:s service} > \text{your service} \,\middle|\, \begin{array}{l}A\text{:s service}\\ \text{started first}\end{array}\right)$$

$$= \mathbb{P}\left(A\text{:s service time} > \text{your service time}\right)$$

$$= \frac{\mu_1}{\mu_1+\mu_2}$$

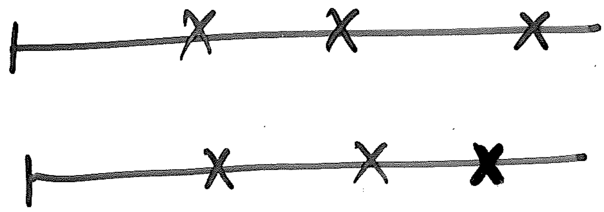
$\uparrow$ no memory

(b) P_B : B is still in the system when you are ready with your service at 1

Call S_1^2 the time when 2 is ready with A, S_2^2 the time when 2 is ready with B. S_1^2, S_2^2 are the first two events of a PP (μ_2). Call S_1^1 the time when 1 is ready with you.

$$P_B = \mathbb{P}\left(S_1^1 < S_2^2\right) =$$

$$= \frac{\mu_1}{\mu_1+\mu_2} + \frac{\mu_2}{\mu_1+\mu_2} \cdot \frac{\mu_1}{\mu_1+\mu_2}$$



(c) Find $E(T)$, T is the total time you spend in the system.

Hint: $T = S_1 + S_2 + W_A + W_B$

where S_i your service time at i

W_A amount of time waiting for A to be ready

W_B —— " —— for B to be ready

$S_1 = S_1'$ has mean $\frac{1}{\mu_1}$

S_2 has mean $\frac{1}{\mu_2}$

$W_A = 0$ with prob. $\frac{\mu_2}{\mu_1+\mu_2}$

is exponential (μ_2) with prob. $\frac{\mu_1}{\mu_1+\mu_2}$

$W_B = 0$ with prob. $1 - P_B$

is exponential (μ_2) with prob. P_B

$$E(T) = \frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{1}{\mu_2} \cdot \frac{\mu_1}{\mu_1 + \mu_2} + \frac{1}{\mu_2}\left(\frac{\mu_1}{\mu_1+\mu_2} + \frac{\mu_2 \mu_1}{(\mu_1+\mu_2)^2}\right)$$

$$= \frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{2\mu_1}{\mu_2(\mu_1+\mu_2)} + \frac{\mu_1}{(\mu_1+\mu_2)^2}$$

<u>Exercise 2.3, p. 349</u>

A flashlight needs two batteries to be operational. There are n batteries available. First we use batteries 1 and 2. Then they are replaced on failure by the lowest numbered battery not in use. Thus, e.g.,

first used 1, 2
if 2 fails it is replaced by 3, then
if 3 fails it is replaced by 4, then
if 1 fails it is replaced by 5, etc.

The lifetimes of the batteries are indep. exponentials with param. μ. At a random time T a battery will fail and there are no more reserves. Exactly one battery is still functional.

68 (After T the flashlight is no longer operational). The lone operational battery is called X.

(a) $\mathbb{P}(X = n)$?

(b) $\mathbb{P}(X = 1\}$?

(c) $\mathbb{P}(X = i)$?

(d) $E(T)$

(e) Distribution of T.

T occurs when the $(n-1)$ st. battery fails.

Thus $E(T) = (n-1)/2\mu$

And T being the sum of $n-1$ indep. failure times:

Dist'n of T is gamma with parameters $n-1$ and 2μ.

Why 2μ? Replacement occurs when one (= the first) of two batteries fail, so the dist'n is the same as for

$$X_{(1)} = X_1 \wedge X_2 \quad (\min(X_1, X_2))$$

where X_1 and X_2 are $\mathrm{Exp}(\mu)$:

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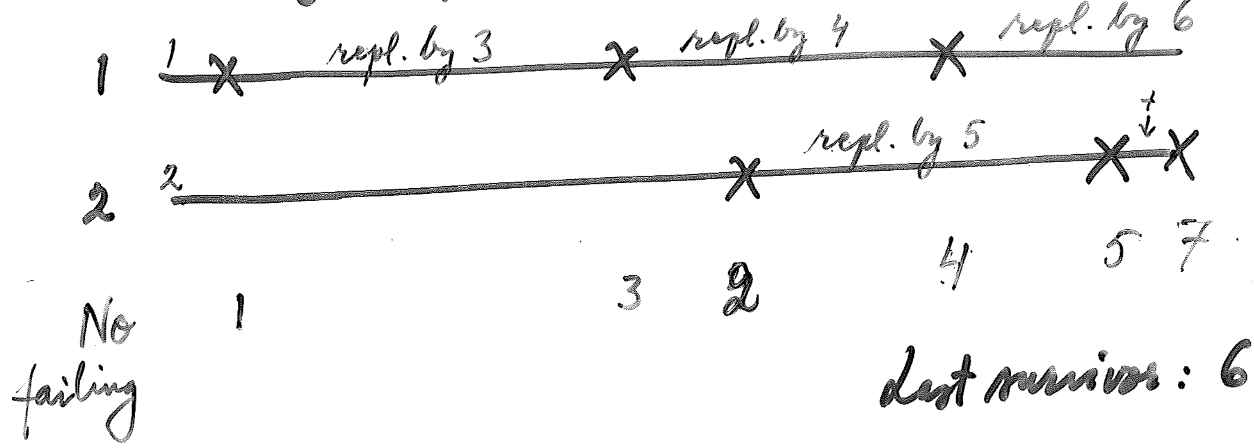
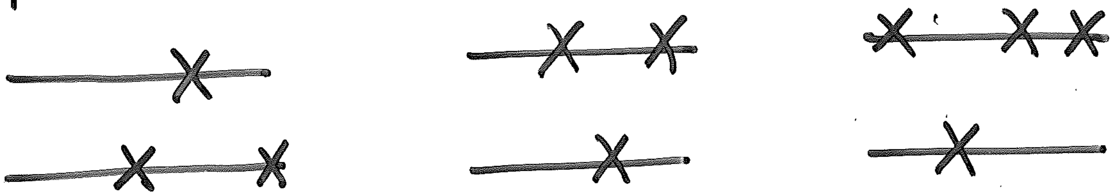
Battery in place   x failure $n = 7$



Last survivor : 6

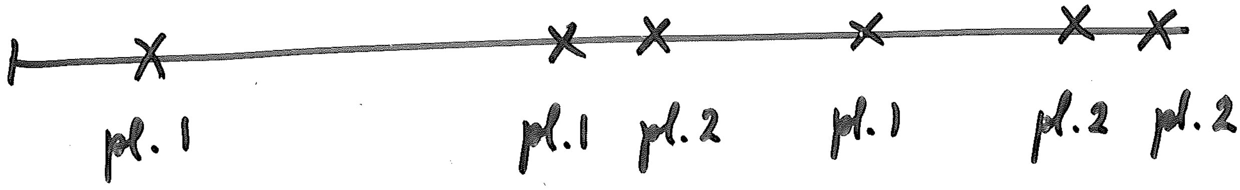
Pattern



Last survivor:

n n $n-1$

The points (events) have prob. $\frac{1}{2}$ of being a place 1 - failure.



$\mathbb{P}(X = n) = \mathbb{P}($last two failures are in different places$) = \mathbb{P}($last two failures are 1 in pl. 1 and 1 in pl. 2$) = \frac{1}{2}$.

$= 2 \cdot \left(\frac{1}{2}\right)^2$

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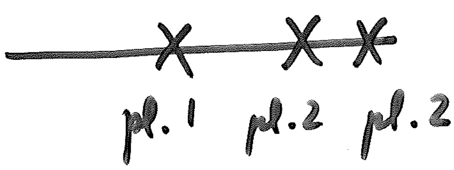
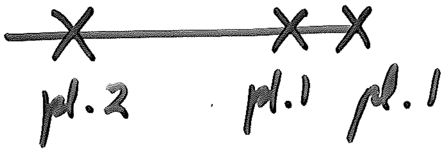
$$\mathbb{P}(X=1) = \mathbb{P}(\text{all failures are in pl. 2})$$
$$= \left(\frac{1}{2}\right)^{n-1}$$

$$\mathbb{P}(X=2) = \left(\frac{1}{2}\right)^{n-1}$$

$$\mathbb{P}(X=n-1) = 2\cdot\left(\frac{1}{2}\right)^{3} = \left(\frac{1}{2}\right)^{2}$$





$$\mathbb{P}(X=i) = \left(\frac{1}{2}\right)^{n-i+1} \qquad i = 2, \dots, n$$