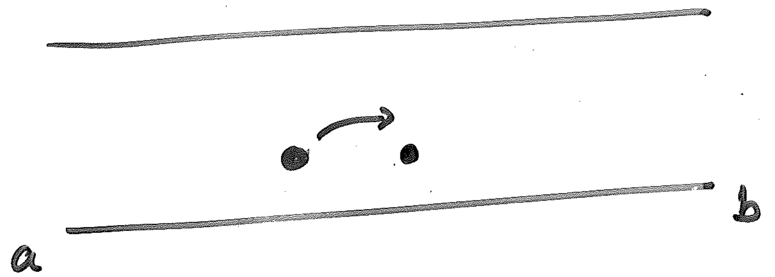
55 Ex. 5.19. Minimizing The No. of Encounters



* uniform on the highway, once chosen

Cars enter a one-way highway at a and depart from it at b. Cars enter as a PP (λ). Their speed is random*, independent of each other, dist'n G. If all cars have the same speed: no need to pass. But if the speed varies then an encounter may take place: either you pass the preceding car(s) or you are passed by succeeding car(s).

Let $d = b - a$. If your speed is x, then the time spent on the highway is

$$\frac{d}{x}.$$

Suppose you enter at time s. Then you exit at $s + t_0$ where $t_0 = \frac{d}{x}$.

Other cars enter at random exp. times. If the speed is X, the travel time is

$\frac{d}{X}$, a r.v. The dist'n of the travel time T:

$$F(t) = \mathbb{P}(T \le t) = \mathbb{P}\left(\frac{d}{X} \le t\right)$$

$$= \mathbb{P}\left(\frac{d}{t} \le X\right) = 1 - G\left(\frac{d}{t}\right)$$

(cont's dist'n assumed)

Suppose a car enters at time t. Classify the event as type I if there is an encounter with your car.

$t < s$ and $t + T > s + t_0$
i.e., passes your car

$s + t_0 > t > s$ and $t + T < s + t_0$
i.e., is passed by your car

Type II event: No encounter with your car.

$\mathbb{P}(\text{Type I}) = p(t)$

$$= \begin{cases} \bar{F}(s + t_0 - t) & \text{if } t < s \\ F(s + t_0 - t) & \text{if } s < t < s + t_0 \\ 0 & \text{if } t > s + t_0 \end{cases}$$

57 Total no. of encounters $N_1(\infty)$ is Poisson with mean

$$\lambda \int_0^\infty p(t)\,dt = \lambda \int_0^s \bar{F}(s+t_0-t)\,dt +$$

$$\lambda \int_s^{s+t_0} F(s+t_0-t)\,dt \;\overset{\substack{y = s-t+t_0 \\ dy = -dt}}{=}$$

$$= \lambda\left(\int_{t_0}^{s+t_0} \bar{F}(y)\,dy + \int_0^{t_0} F(y)\,dy\right)$$

Try to choose t_0 to minimize this expression. ∂t_0 derivative

$$\lambda\left(\bar{F}(s+t_0) - \bar{F}(t_0) + F(t_0)\right)$$

When s is large (the process has been going on for a long time when you enter) $\bar{F}(s+t_0) \approx 0$. Then the derivative is

0 if $\quad \bar{F}(t_0) = F(t_0)$

$$1 - F(t_0) = F(t_0)$$

$$F(t_0) = \frac{1}{2}$$

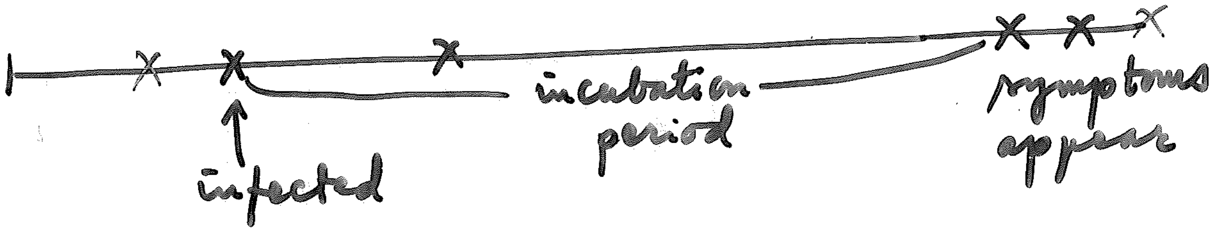
t_0 should be the <u>median</u> travel time.

$$P(T \le t_0) = \frac{1}{2} \iff P\left(\frac{d}{X} \le t_0\right) = \frac{1}{2}$$

$$\iff P\left(X \ge \frac{d}{t_0}\right) = \frac{1}{2} = \bar{G}(d/t_0)$$

So the speed X chosen should be the <u>median</u> of the speed distribution G, the sol'n of $G(x) = \frac{1}{2}$.

Ex. 5.20 (Tracking the no. of HIV infections)



Ass.: infections as a PP (λ)
incubation period (long) with dist'n G
incubation periods of diff. individuals are independent

$N_1(t)$ = individuals who have shown symptoms by time t

$N_2(t)$ = infected but have not shown symptoms yet

N_1 PP with param. λp
N_2 PP $\lambda(1-p)$

59 If point at s then classified as I with prob $\mathbb{P}(\text{incub. time} \le t-s) = G(t-s)$ and II with prob
$$\mathbb{P}(\text{incub. time} > t-s) = \bar{G}(t-s)$$

Hence (Prop. 5.3)
$$\mathbb{E}(N_1(t)) = \lambda \int_0^t G(t-s)\, ds$$

$$\mathbb{E}(N_2(t)) = \lambda \int_0^t \bar{G}(t-s)\, ds$$

Total, as it should, is λt.

λ is usually unknown. How to estimate it from data. We assume that N_1 is known. And G (medical fact).
$$\mathbb{E}(N_1(t)) \approx N_1(t)$$

Thus
$$\hat{\lambda} = N_1 \Big/ \int_0^t G(y)\, dy$$

Estimate of $N_2(t)$ is then
$$\frac{N_1(t)}{\int_0^t G(y)\,dy} \cdot \int_0^t \bar{G}(y)\, dy$$

EX.

If G is <u>exponential</u> with param $\frac{1}{\mu}$ we get

$$\int_0^t \bar{G}(y)\,dy = \mu\left(1 - e^{-t/\mu}\right)$$

$$\int_0^t G(y)\,dy = t - \mu\left(1 - e^{-t/\mu}\right)$$

For $t = 16$, $\mu = 10$ yrs, $N_1(16) = 220\,000$ we get the estimate

$$\frac{2200\left(1 - e^{-1.6}\right) \times 10^3}{16 - 10\left(1 - e^{-1.6}\right)} = 218\,960$$

So the total no. of infected is 439 000 of which 220 000 show symptoms and 219 000 are symptomless.