Given two indep. Poisson processes

$\{N_1(t), t \geq 0\}$ param.: λ_1

$\{N_2(t), t \geq 0\}$ param.: λ_2

Call the mth event of first process S_m^1 and the corresp. mth event of the second process S_m^2.

Want to determine

$$\mathbb{P}\{S_m^1 < S_m^2\}$$

1st obs.: $\mathbb{P}(S_1^1 < S_1^2) = \dfrac{\lambda_1}{\lambda_1 + \lambda_2}$

because S_1^i are indep. exponentials with param. λ_i. We solved this before:

$$\iint_{x<y} \lambda_1 e^{-\lambda_1 x} \lambda_2 e^{-\lambda_2 y} \, dx\, dy$$

$$= \int_0^\infty \lambda_1 e^{-\lambda_1 x} \, dx \underbrace{\int_x^\infty \lambda_2 e^{-\lambda_2 y} \, dy}_{e^{-\lambda_2 x}} = \int_0^\infty \lambda_1 e^{-(\lambda_1+\lambda_2)x} \, dx$$

$$= \frac{\lambda_1}{\lambda_1 + \lambda_2}.$$

46 $\mathbb{P}(S_2^1 < S_1^2)$?

Reason as follows: The first event of $N_1 + N_2$ is a point from N_1. Happens with prob. $\frac{\lambda_1}{\lambda_1+\lambda_2}$.

At that point, we may start the processes anew. What happens after the event S_1^1 is probabilistically the same as the original process. This goes for N_1 <u>and</u> also N_2. If the newly started process first has an event from N_1 (which has prob. $\frac{\lambda_1}{\lambda_1+\lambda_2}$) then $S_2^1 < S_1^2$. If the first point of the new process is an event from N_2* then $S_1^1 < S_1^2 < S_2^1$.

*has prob. $\frac{\lambda_2}{\lambda_1+\lambda_2}$

So $S_1^1 < S_2^1 < S_1^2$ has prob. $\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^2$

$S_1^1 < S_1^2 < S_2^1$ $\quad \frac{\lambda_1}{\lambda_1+\lambda_2} \cdot \frac{\lambda_2}{\lambda_1+\lambda_2}$

and

$S_1^2 < S_2^2 < S_1^1$ $\quad \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^2$

$S_1^2 < S_1^1 < S_2^1$ $\quad \frac{\lambda_2}{\lambda_1+\lambda_2} \cdot \frac{\lambda_1}{\lambda_1+\lambda_2}$

Independent coin tosses with probabilities $\frac{\lambda_1}{\lambda_1+\lambda_2}$ and $\frac{\lambda_2}{\lambda_1+\lambda_2}$.

The prob. of the next point being from N_1 or N_2 is independent of what happened before.

$$\mathbb{P}(S_3^1 < S_1^2) = \mathbb{P}(\text{first three events are from } N_1) = \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^3$$

$$\mathbb{P}(S_n^1 < S_m^2) = \mathbb{P}(\text{among first } n+m-1 \text{ points there are at least } n \text{ from } N_1$$
$$(\text{and at most } m-1 \text{ from } N_2))$$

$$= \sum_{k=n}^{n+m-1} \binom{n+m-1}{k} \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n+m-1-k}$$

5.3.5. Conditional Distribution of Arrival Times (or event times)

Suppose we know that there is a point in $(0,t]$, i.e. $N(t)=1$.
Where is it? What is the density of S_1 under the condition that $N(t)=1$?

$N(t)$ PP param.: λ

$$\mathbb{P}\left(S_1 \le s \,\middle|\, N(t) = 1 \right) \qquad 0 < s \le t$$

$$= \frac{\mathbb{P}\left(\text{one point in } (0, s],\ \text{no point in } [s, t] \right)}{\mathbb{P}\left(N(t) = 1 \right)}$$

$$= \frac{e^{-\lambda s} \cdot \frac{(\lambda s)^1}{1!} \cdot e^{-\lambda(t-s)}}{e^{-\lambda t} \cdot \frac{(\lambda t)^1}{1!}} = \frac{s}{t}$$

Uniform distribution in $[0, t]$!

Order statistics

Let $Y_1, Y_2, \cdots, Y_n$ be n (continuous) random variables.

$Y_{(1)}, Y_{(2)}, \cdots, Y_{(n)}$ is the <u>order statistics</u> corresponding to $Y_1, Y_2, \cdots, Y_n$ if

$Y_{(k)}$ is the kth smallest of the $Y_1, \cdots, Y_n$ $(k = 1, 2, \cdots, n)$

Ex. $Y_1 = 5$, $Y_2 = 3{,}1$, $Y_3 = 2$, $Y_4 = 3$

$Y_{(1)} = 2$, $Y_{(2)} = 3$, $Y_{(3)} = 3{,}1$, $Y_{(4)} = 5$

If $Y_1, Y_2, \cdots, Y_n$ are independent, identically distributed with a density $f(y)$, then the joint density of $(Y_{(1)}, Y_{(2)}, \cdots, Y_{(n)})$ is

$$f(y_1, y_2, \cdots, y_n) = n! \prod_{i=1}^{n} f(y_i)$$

$$y_1 < y_2 < \cdots < y_n$$

Note that

$(Y_{(1)}, Y_{(2)}, \cdots, Y_{(n)})$ has the value

$(y_1, y_2, \cdots, y_n)$ if $(Y_1, Y_2, \cdots, Y_n)$ equals any of the $n!$ permutations of $(y_1, y_2, \cdots, y_n)$.

Ex. If $(Y_{(1)}, Y_{(2)}, Y_{(3)}) = (.5, .7, .73)$

then

$$\begin{aligned}(Y_1, Y_2, Y_3) = \; & (.5, .7, .73) \text{ or} \\ & (.7, .5, .73) \text{ or} \\ & (.5, .73, .7) \text{ or} \\ & (.7, .73, .5) \text{ or} \\ & (.73, .5, .7) \text{ or} \\ & (.73, .7, .5).\end{aligned}$$

Ex.

The area of $\{(x,y) \mid 0<x<y<1\}$ is $\frac{1}{2}$

The volume of $\{(x,y,z) \mid 0<x<y<z<1\}$ is $\frac{1}{6}$

The n-dim. generalized volume of

$\{(x_1,x_2,\cdots,x_n) \mid 0<x_1<x_2<\cdots<x_n<1\}$ is $\frac{1}{n!}$

The prob. density of

$$(Y_1, Y_2, \cdots, Y_n) = (y_{i_1}, y_{i_2}, \cdots, y_{i_n})$$

is (by independence)

$$\prod_{j=1}^{n} f(y_{i_j}) = \prod_{j=1}^{n} f(y_j)$$

if $(y_{i_1}, y_{i_2}, \cdots, y_{i_n})$ is a permutation of $(y_1, y_2, \cdots, y_n)$.

If the Y's are <u>uniformly</u> distributed on $[0,t]$, then any individual Y_i has density $\frac{1}{t}$ and so

$$f(y_1,y_2,\cdots,y_n) = \frac{n!}{t^n} \quad (0<y_1<y_2<\cdots<y_n<t)$$

<u>Theorem 5.2</u> Given that $N(t) = n$ then the n points are uniformly distributed on $[0,t]$, i.e. the arrival times $S_1 < S_2 < \dots < S_n$ have the same distribution as the order statistics of n indep. r.v.'s with uniform distribution on $[0,t]$.

Proof. Find the conditional density of $S_1, S_2, \dots, S_n$ given $N(t) = n$:

For $0 < s_1 < s_2 < \dots < s_n < t$ we have

$$S_1 = s_1, \; S_2 = s_2, \; \dots, \; S_n = s_n \iff$$

$$T_1 = s_1, \; T_2 = s_2 - s_1, \; T_3 = s_3 - s_2, \; \dots,$$

$T_n = s_n - s_{n-1}$, $T_{n+1} > t - s_n$ where the T's are the interarrival times.

The T's are independent exponentials so

$$f(s_1, s_2, s_3, \dots, s_n \mid N(t) = n)$$

$$= \lambda e^{-\lambda s_1} \cdot \lambda e^{-\lambda(s_2 - s_1)} \dots \lambda e^{-\lambda(s_n - s_{n-1})} \cdot e^{-\lambda(t - s_n)} \Big/ P(N(t) = n) =$$

$$= \lambda^n e^{-\lambda t} \Big/ e^{-\lambda t} \cdot \frac{(\lambda t)^n}{n!} = \frac{n!}{t^n} .$$

<u>Prop. 5.3</u> Suppose that each event in a PP $N(t)$ (param. λ) is classified as type $1, 2, \ldots, k$ <u>independently of all preceding classifications</u> with probability P_i, $i = 1, 2, \ldots, k$, $\sum P_i = 1$

or, in general, $P_i(y)$, $i = 1, 2, \ldots, k$, $\sum_{i=1}^{k} P_i(y)$, if the event takes place at the time y

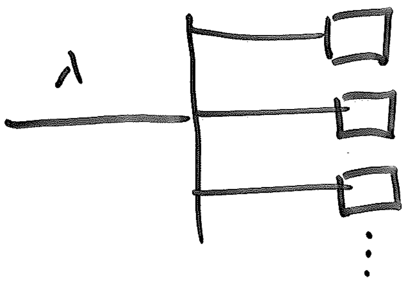
If $N_i(t)$ represents the PP of events classified as type i $(i = 1, 2, \ldots, k)$ then N_i's are independent Poisson Processes with parameter $\lambda \cdot P_i$

in the general case: $N_i(t)$'s are independent Poisson r.v.'s with

$$E\left(N_i(t)\right) = \lambda \int_0^t P_i(s)\, ds$$

Proof will be skipped. In Prop. 5.2. we had binomial dist'n, here multinomial.

Ex 5.18 (An Infinite Server Queue)



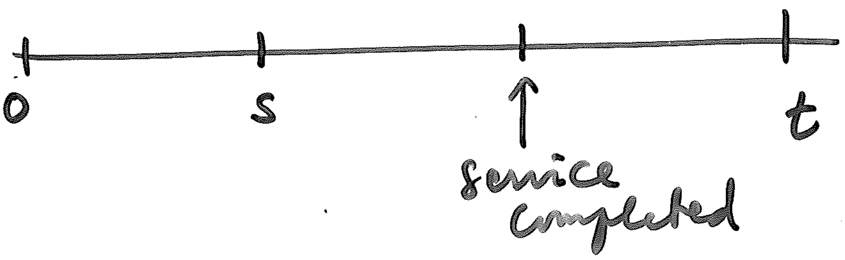
Arrival : PP (λ)
immediately served by a server. Service times indep., distr. G

$X(t)$ no. of customers whose service has been completed by time t

$Y(t)$ no. of customers being served at time t

I-type : customer completely served by time t

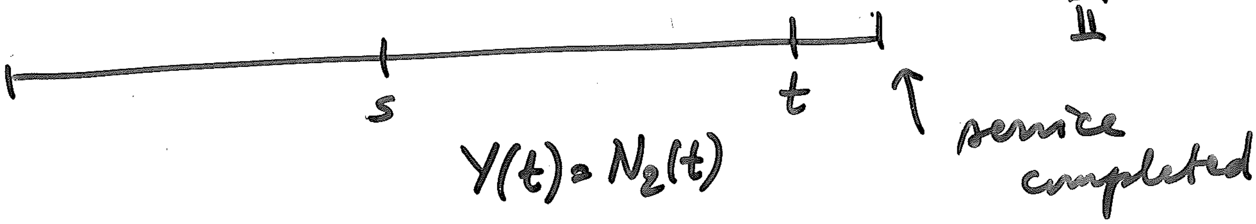
II-type : customer not served by time t



I

$X(t) = N_1(t)$

II



$Y(t) = N_2(t)$

Consider arrival at time $s \le t$

Type I with probability

$$P\{\text{service time} \le t-s\}$$

$$= G(t-s)$$

Type II : $P(\text{service time} > t-s) = \bar{G}(t-s)$

$$= 1 - G(t-s)$$

Prop. 5.3 then says that $N_1(t), N_2(t)$ indep. &

$$E(N_1(t)) = \lambda \int_0^t G(t-s)\,ds$$

$$= \lambda \int_0^t G(y)\,dy$$

$$\mathbb{E}(N_2(t)) = \lambda \int_0^t \bar{G}(y)\,dy$$

Recall: Sum is PP, $\mathbb{E}(N(t)) = \lambda t$.

If $G(y) = 1 - e^{-\mu \cdot y}$ (service time is exponential with mean $\frac{1}{\mu}$), then

$$\int_0^t (1 - e^{-\mu y})\,dy = t - \frac{1}{\mu}(1 - e^{-\mu t})$$

$$\int_0^t e^{-\mu y}\,dy = \frac{1}{\mu} \Big/_0^t - e^{-\mu y} = \frac{1}{\mu}(1 - e^{-\mu t})$$