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$$f_{X|X>y}(x) = \frac{f(x)}{\bar{F}(y)}$$

Expected gain:

$$\frac{\int_y^\infty x \cdot f(x)\,dx}{\bar{F}(y)} - \frac{c}{\lambda \bar{F}(y)} \quad (*)$$

For $y=0$ we get $E(X) - \frac{c}{\lambda}$ ($y=0$ corresponds to accepting the first offer, without any threshold condition)

Differentiate the expected return $(*)$

$$\left\{ \left(- y \cdot f(y) \right) \cdot \bar{F}(y) - \left(\int_y^\infty x f(x)\,dx - \frac{c}{\lambda}\right) \cdot -f(y) \right\} \cdot \frac{1}{(\bar{F}(y))^2}$$

Derivative is 0 if

$$y\,\bar{F}(y) = \int_y^\infty x f(x)\,dx - \frac{c}{\lambda}$$

or

$$y\int_y^\infty f(x)\,dx = \int_y^\infty x\,f(x)\,dx - \frac{c}{\lambda}$$

or

$$\int_y^\infty (x-y)f(x)\,dx = \frac{c}{\lambda}$$

or

$$\mathbb{E}(X-y)^+ = \frac{c}{\lambda} \qquad (5.14)$$

LHS ~~nonincreasing~~ decreasing if $f>0$ function of y
largest value: EX (at $y=0$)

$EX < \frac{c}{\lambda}$: Take any offer. Return negative anyway.
(i.e., expected return < 0)

$EX \geq \frac{c}{\lambda}$: Take y to be unique solution to (5.14).

Ex. X is exponential with mean $\frac{1}{\mu} > \frac{c}{\lambda}$.

$$\mathbb{E}(X-y)^+ = e^{-\mu\cdot y}\cdot\frac{1}{\mu}$$

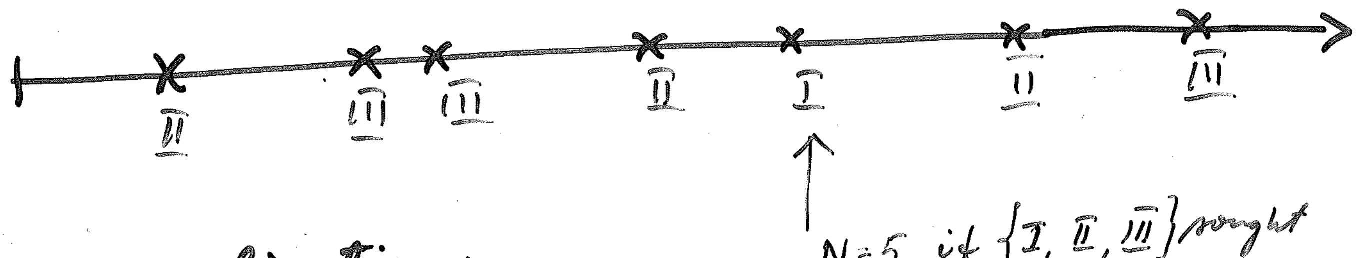
Ex. 5.17 (The Coupon Collecting Problem)

There are m different types of coupons. Each time a person gets a coupon it is, independently of all other events, a type j coupon with prob. p_j, $\sum_{j=1}^{m} p_j = 1$. N is the no. of coupons needed to collect at least one of each.

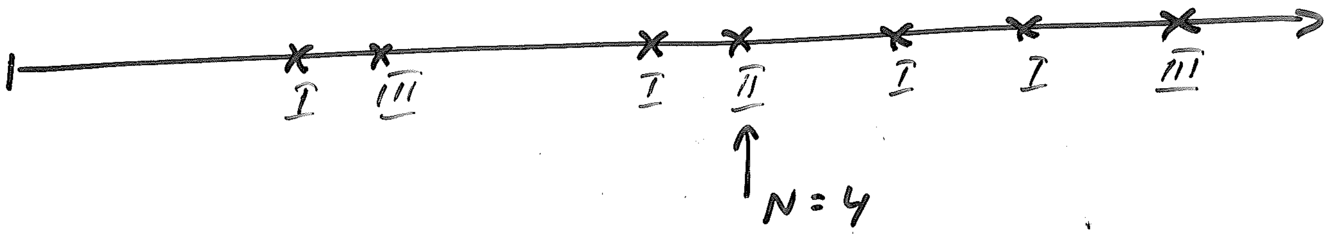
Find $E(N)$.

Usually the m coupons are equally likely, $p_j = \frac{1}{m}$. Ex.: $p_1 = p_2 = \frac{1}{2}$, $E(N) = 3$.

Solution using Poisson processes:



Another realization:



Our Poisson process is assumed to have rate 1. E(no. of events in $(0,t]$) $= t$.

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$$N_j(t) \quad , \; j = 1, \cdots, m$$

of events with coupon j. [The total process is the sum of the N_j - processes.]

Let X_j be the <u>first</u> event in the N_j - process. Then

$$X \equiv \max_{1 \le j \le m} X_j$$

is the time when <u>all</u> coupons have been collected.

$$X = \underset{\substack{\uparrow \\ \text{random}}}{W_N} = T_1 + T_2 + \cdots + \underset{\substack{\uparrow \\ \text{random}}}{T_N}$$

where $T_1, T_2, \cdots$ are interarrival times in the total process with intensity 1.

$$E(X) = E(N) \cdot \underbrace{E(T_1)}_{= 1} \quad \left(= \frac{1}{\lambda}\right)$$

Thus $E(N)$ is $= E(X)$ which may be calculated :

$$\mathbb{P}(X \le t) = \mathbb{P}\left(\max_{1\le j\le m} X_j \le t\right)$$

$$\underset{\text{indep.}}{\overset{\uparrow}{=}} \prod_{j=1}^{m} \mathbb{P}(X_j \le t)$$

$$= \prod_{j=1}^{m} \left(1 - e^{-p_j \cdot t}\right)$$

$$\mathbb{P}(X > t) = 1 - \prod_{j=1}^{m}\left(1 - e^{-p_j t}\right)$$

Thus

$$E(N) = E(X) = \int_0^\infty \left(1 - \prod_{j=1}^{m}\left(1-e^{-p_j t}\right)\right) dt$$

Check $p_1 = p_2 = \frac{1}{2}$

$$\int_0^\infty \left(1 - \left(1-e^{-\frac{1}{2}t}\right)\left(1-e^{-\frac{1}{2}t}\right)\right) dt =$$

$$\int_0^\infty \left(2 \cdot e^{-\frac{1}{2}t} - e^{-t}\right) dt = 2 \cdot 2 - 1 = 3.$$

Ex. $p_1 = \frac{5}{6}$, $p_2 = \frac{1}{6}$. Calculation gives

$$\int_0^\infty \left(e^{-\frac{5}{6}t} + e^{-\frac{1}{6}t} - e^{-t}\right) dt = \frac{6}{5} + 6 - 1 = \underline{\underline{6\tfrac{1}{5}}}$$

Classical case : $p_j = \frac{1}{m}$, $j = 1, \dots, m$

We can also reason as follows

step 1 : first coupon
wait for next coupon (prob : $\frac{m-1}{m}$ at each step) for $\frac{m}{m-1}$ steps on average
wait for third coupon $\frac{m}{m-2}$ steps on ave.

$\vdots$

Total no. of steps to wait (on ave.)

$$1 + \frac{m}{m-1} + \frac{m}{m-2} + \cdots + \frac{m}{2} + m$$

$$= m\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m-1} + \frac{1}{m}\right)$$

$\approx m \log m$ for large values of m