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5.3.3. Interarrival and Waiting Time Distributions

Let T_1 be the time of the first event. For $n > 1$, let T_n be the time between the $(n-1)^{\text{st}}$ and the n^{th} event.

$T_1, T_2, T_3, \cdots, T_n, \cdots$ <u>interarrival times</u>

<u>Prop 5.1.</u> T_n, $n = 1, 2, \ldots$, are independent exponentially distributed r.v.'s with parameter λ.

<u>Pf.</u> $T_1 > t \iff N(t) = 0$ which has probability $e^{-\lambda t}$. Thus T_1 is exponential with param. λ.

$T_2 > t \iff$ no event between T_1 and $T_1 + t$, <u>no matter what T_1 is</u>

$$\mathbb{P}(T_2 > t \mid T_1 = s) = \mathbb{P}\left(\text{no event in } (s, s+t] \mid T_1 = s\right) = \mathbb{P}\left(\text{no event in } (s, s+t]\right) = e^{-\lambda t}.$$

$\therefore$ T_2 indep. of T_1

T_2 exponential, param. λ

$$S_n \underset{\substack{\uparrow \\ \text{def.}}}{=} \text{the arrival time of the } n^{\text{th}} \text{ event}$$

$$= \underline{\text{the } n^{\text{th}} \text{ waiting time}}$$

$$= T_1 + T_2 + \dots + T_n$$

S_n has gamma-dist'n with param. n and λ (shown before).

We could also calculate this directly:

$$S_n \le t \iff N(t) \ge n$$

The probabily

$$\mathbb{P}(N(t) \ge n) \underset{\substack{\uparrow \\ \text{Poisson}}}{=} e^{-\lambda t} \sum_{k=n}^{\infty} \frac{(\lambda t)^k}{k!}$$

Differentiating this gives us

$$f_{S_n}(t) = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}$$

(elementary calculation)

density of gamma dist'n with parameters n and λ

Ex 5.13 Suppose people immigrate into a territory at a Poisson rate $\lambda = 1$ per day.

(a) What is the expected waiting time until the tenth immigrant arrives.

$$\mathbb{E}(S_n) = \mathbb{E}(T_1 + \dots + T_n)$$
$$= n\, \mathbb{E}(T_1) = \frac{n}{\lambda}$$

$n = 10,\ \lambda = 1$ gives $\mathbb{E}(S_{10}) = 10$ days.

(b) What is the probability that the time between the tenth and eleventh arrival exceeds 2 days.

$$\mathbb{P}(T_{11} > 2) = e^{-\lambda \cdot 2} = e^{-2} \approx 0{,}133.$$

The counting process
$$N(t) = \max\{n \mid S_n \leq t\}$$
is a Poisson process with rate λ (if $S_0 = 0$)

$N(t)$ is exactly counting the no. of points S_n occurring in $(0, t]$.

Remark. We could also find f_{S_n} as follows:

$$\mathbb{P}\left(t < S_n < t+h\right) = \mathbb{P}\Big(N(t) = n-1,$$

one event in $(t, t+h)\Big)$ + $o(h)$

prob. of more than two events in $(t, t+h)$

$$= \mathbb{P}\left(N(t) = n-1\right) \mathbb{P}\left(N(t+h) - N(t) = 1\right) + o(h)$$

$$= e^{-\lambda t} \cdot \frac{(\lambda t)^{n-1}}{(n-1)!} \cdot \left(\lambda h + o(h)\right) + o(h)$$

$$= \lambda e^{-\lambda t} \cdot \frac{(\lambda t)^{n-1}}{(n-1)!} \cdot h + o(h)$$

div. by h, $h \to 0$

Thus $$f_{S_n}(t) = \lambda e^{-\lambda t} \cdot \frac{(\lambda t)^{n-1}}{(n-1)!}$$

5.3.4. *Further Properties of PP ("Thinning")*

Consider a PP with rate λ. Suppose that each event belongs to either type I or II. Type I occurs with probability p (<u>independent</u> of everything else), type II with probability $1-p$.

I and II are sometimes called <u>marks</u>.

increments because the no. of type I events in an interval is calculated by conditioning on the total no. of events in that interval

$$\mathbb{P}\left(N_1(t) - N_1(s) = m \;\middle|\; N(t) - N(s) = m' \right)$$

is distributed as $\sim \mathrm{Bin}(m', p)$ (this is the description of N_1)

so

$\mathbb{P}\left(N_1(t) - N_1(s) = m \right) = \sum$ bin. distr. $\times$ $\mathbb{P}\left(N(t) - N(s) = m' \right)$ which is <u>indep.</u> of what happens in other disjoint intervals. Also, it is dependent on the length of the interval $s-t$ only.

Calc.:

$$\sum_{m' \geq m} \binom{m'}{m} p^m (1-p)^{m'-m} \cdot e^{-\lambda(t-s)} \cdot \frac{(\lambda(t-s))^{m'}}{m'!}$$

3Ya We ___know___ this should be Poisson with param. $p\lambda(t-s)$:

$$e^{-p\lambda(t-s)} \cdot \frac{(\lambda p(t-s))^m}{m!}$$

Verification:

$$\sum_{m' \geq m} p^m (1-p)^{m'-m} \cdot \frac{\cancel{m'!}}{(m'-m)!\ m!} e^{-\lambda(t-s)} \cdot$$

$$\cdot \frac{(\lambda(t-s))^{m'}}{\cancel{m'!}}$$

$$= \sum_{m' \geq m} e^{-\lambda(t-s)} \cdot (\lambda(t-s))^{m'-m} \cdot (\lambda(t-s))^m \cdot (1-p)^{m'-m} \cdot p^m$$

$$\cdot \frac{1}{(m'-m)!} \cdot \frac{1}{m!}$$

$$= e^{-\lambda(t-s)} \frac{(\lambda p(t-s))^m}{m!} \underbrace{\sum_{m' \geq m} \frac{\lambda(t-s)^{m'-m} \cdot (1-p)^{m'-m}}{(m'-m)!}}_{e^{\lambda(1-p)(t-s)}}$$

$$= e^{-\lambda p(t-s)} \frac{(\lambda p(t-s))^m}{m!} .$$

Thus we have shown that $N_1(t)$ is a Poisson process with rate λp.

N_1 and N_2 independent:

The number of I-points in $(s,t]$ is clearly indep. of II-point in $(u,v]$ if the latter is disjoint from $(u,v]$. But what about overlapping intervals? In particular, if it is the same:

$$N_1(t) - N_1(s) \quad \text{and} \quad N_2(t) - N_2(s)$$

$$\mathbb{P}\left(N_1(t) - N_1(s) = m,\ N_2(t) - N_2(s) = n\right)$$

$$= \mathbb{P}\left(N_1(t) - N(s) = m \mid N(t) - N(s) = n+m\right)$$

$$\cdot\ \mathbb{P}\left(N(t) - N(s) = n+m\right)$$

this is Poisson-distr. with param. $\lambda(t-s)$

$$= \mathbb{P}\left(N_1(t) - N(s) = m \mid N(t) - N(s) = n+m\right)$$

$$\cdot\ e^{-\lambda(t-s)} \cdot \frac{(\lambda(t-s))^{n+m}}{(n+m)!} =$$

$$= \binom{n+m}{m} p^m (1-p)^n \cdot e^{-\lambda(t-s)} \cdot \frac{(\lambda(t-s))^n (\lambda(t-s))^m}{(n+m)!}$$

$$= p^m \frac{(\lambda(t-s))^m}{m!} e^{-\lambda p(t-s)} \cdot (1-p)^n \cdot \frac{(\lambda(t-s))^n}{n!} \cdot e^{-\lambda(1-p)(t-s)}$$

$$= \mathbb{P}(N_1(t) - N_1(s) = m) \cdot \mathbb{P}(N_2(t) - N_2(s) = n).$$

Thus $N_1(t) - N_1(s)$ is independent of $N_2(t) - N_2(s)$. Hence $N_1(t)$ and $N_2(t)$ are two independent Poisson processes.

Of course, this can be generalized to three types, four types etc.

Ex 5.14

If immigrants to A arrive as a Poisson process with rate 10 per week and if each immigrant is of English descent (independently of others) with prob. $\frac{1}{12}$, then what is the prob. that no people of English descent immigrate to A during the month of February (= 4 weeks)

$N(t)$ arrivals
$N_1(t)$ — " — of English descent

$$\mathbb{P}(N_1(4) = 0) = e^{-4 \cdot \frac{\lambda}{12}} = e^{-4 \cdot \frac{10}{12}}$$

$$= e^{-10/3} \approx 0{,}0357.$$

Ex. 5.15

You offer to sell an item. Offers to buy arrive as a Poisson process with rate λ. Size of offers are r.v's, independent, with conti density function $f(x)$.
Offer has to be either accepted or rejected at once.
Cost of waiting is c per time unit.

Policy: Accept first offer exceeding the threshold value y. (We call it a y-policy.) Find the optimal y.

Solution:
The expected total return when using y-policy is computed. Then we take the best value of y.

Call $\bar{F}(x) = \mathbb{P}(X > x) = \int_x^\infty f(t)\,dt$

Acceptable offer* arrive as a thinned Poisson process with rate $\lambda \bar{F}(y)$.
* under y-policy

$$\mathbb{P}(T_1 > t) = e^{-\lambda \bar{F}(y)}$$

T_1 is the first arrival of an offer which is acceptable.

Cost: $c\,T_1$

Expected cost: $c\,E(T_1) = \dfrac{c}{\lambda \bar{F}(y)}$

Expected offer $E(X \mid X > y)$

$$= \int_y^\infty x\, f_{X|X>y}(x)\,dx =$$