(5.3.2) Def. of the Poisson Process

Def 5.1. The counting process $\{N(t), t \geq 0\}$ is said to be a <u>Poisson process</u> <u>with</u> <u>rate</u> <u>λ</u> ($\lambda > 0$) if

(i) $N(0) = 0$

(ii) the process has independent increments

(iii) the no. of events in any interval of length t is Poisson-distributed with parameter λt.

In other words, for all $t \geq 0$, $s \geq 0$

$$\mathbb{P}\{\underbrace{N(t+s) - N(s) = n}_{\text{No. of events in } (s,\, s+t] \text{ is } = n}\} = e^{-\lambda t} \frac{(\lambda t)^n}{n!} \qquad n = 0, 1, 2, \ldots$$

We will soon derive an alternative description.

Def 5.2. The function f is said to be $o(h)$ "little o of h" if

$$\lim_{h \to 0} \frac{f(h)}{h} = 0$$

"f goes faster to 0 than h"

Ex. x^2 is $o(h)$

$x\sqrt{x}$ is $o(h)$

$5x$ is not $o(h)$

$7x\sqrt{x}$ is $o(h)$

If f is $o(h)$ and g is $o(h)$ then so is $f+g$:

$$\frac{(f+g)(h)}{h} = \frac{f(h)}{h} + \frac{g(h)}{h}$$

Also $c \cdot f$ is $o(h)$:

$$\frac{(cf)(h)}{h} = c \cdot \frac{f(h)}{h}$$

Def 5.3 The counting process $\{N(t), t \geq 0\}$ is said to be a Poisson process with rate λ, $\lambda > 0$, if

(i) $N(0) = 0$

(ii) The process has stationary and independent increments

(iii) $\mathbb{P}(N(h) = 1) = \lambda h + o(h)$

(iv) $\mathbb{P}(N(h) \geq 2) = o(h)$

Theorem 5.1 Definitions 5.1. and 5.3. are equivalent.

Pf. 5.1. ⇒ 5.3. easy

5.3. ⇒ 5.1.

Since (i) and (ii) of Def 5.1. follow automatically we need to work on (iii).

We know from Prob Theory [not proved here] that for a non-negative random variable the dist'n is uniquely determined by its Laplace transform

$$\mathbb{E}(e^{-uX})$$

23 Let
$$g(t) = \mathbb{E}\left(\exp(-u N(t))\right).$$
Then
$$g(t+h) = \mathbb{E}\left(\exp(-u N(t+h))\right)$$
$$= \mathbb{E}\left(\exp(-u N(t)) \cdot \exp\left(-u\left(N(t+h) - N(t)\right)\right)\right)$$
$$\underset{\text{indep.}}{=} \mathbb{E}\left(\exp(-u N(t))\right) \cdot \mathbb{E}\left(\exp\left(-u\left(N(t+h) - N(t)\right)\right)\right)$$
$$\underset{\text{stationary incr.}}{=} g(t) \cdot \mathbb{E}\left(\exp(-u N(h))\right)$$

$$\mathbb{E}\left(\exp(-u N(h))\right) = 1 \cdot (1 - \lambda h + o(h)) + e^{-u} \cdot (\lambda h + o(h)) + o(h)$$

Thus
$$g(t+h) = g(t)\left(1 - \lambda h + e^{-u} \lambda h + o(h)\right)$$
$$\frac{g(t+h) - g(t)}{h} \longrightarrow g(t)(e^{-u} - 1)\lambda + 0$$
$\therefore$ $g'(t)$ exists and $= g(t)(e^{-u}-1)\lambda$

or
$$\frac{g'(t)}{g(t)} = (e^{-u} - 1)\lambda$$

$$\log g(t) = (e^{-u}-1)\lambda t + C$$

$$g(t) = e^{C}\cdot \exp\left(\lambda t\,(e^{-u}-1)\right)$$

Since $g(0)=1$ we get $C=0$

$$\therefore \quad g(t) = \exp\left(\lambda t\,(e^{-u}-1)\right)$$

g is the Laplace transform of $N(t)$ eval. at u

g is the Laplace transform of a Poisson r.v. with parameter λt, eval. at u

Why?

Let X be a Poisson random variable with parameter λt. Then

$$\mathbb{P}(X=n) = e^{-\lambda t}\cdot \frac{(\lambda t)^n}{n!}, \quad n=0,1,2,\dots$$

The Laplace transform $h(u)$ of X is

$$u \longrightarrow \mathbb{E}\left(e^{-uX}\right) =$$

$$= \sum_{n=0}^{\infty} e^{-un}\cdot e^{-\lambda t}\cdot \frac{(\lambda t)^n}{n!}$$

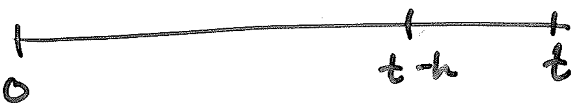
$$= e^{-\lambda t}\sum_{n=0}^{\infty} \frac{(\lambda t)^n (e^{-u})^n}{n!} = e^{-\lambda t}\, e^{\lambda t e^{-u}}$$

$$= \exp\left(\lambda t\,(e^{-u}-1)\right).$$

24a. Note: In Calculus we prove that what we get by taking $h \to 0$, $h > 0$ in

$$\lim \frac{f(x+h) - f(x)}{h} \quad \text{is} \quad f'_+(x)$$

(right derivative). To be exact, we need to consider $h < 0$, $h \to 0$, "too", to obtain an expression for $f'(x)$.



1st observation: $g(t)$ is a decreasing function of t because $N(t)$ is an increasing function of t.

2nd observation: $N(t) =$ no. of events in $[0, t]$,

$\lim_{\substack{h>0 \\ h \to 0}} N(t-h) =$ no. of events in $(0, t)$.

$N(t) - N(t-h) =$ no. of events in $(t-h, t]$

($h > 0$, h small)

$$g(t) = g(t-h) \cdot \underbrace{g(h)}_{1 - \lambda h + e^{-h} \cdot \lambda h + o(h)}$$

As $h \to 0$ $g(t-h) \to g(t)$.

(On p. 25, 26 $p_0(t)$ is also continuous both from the left and from the right. Same argument.)

Then, for $h > 0$, h small, we get, just as for $g(t+h)$ on p. 23:

$$g(t) = g(t-h)\left(1 - \lambda h + e^{-u}\lambda h + o(h)\right)$$

$h \to 0$

$$\frac{g(t) - g(t-h)}{h} \to (-\lambda + \lambda e^{-u}) \cdot g(t)$$

$$\shortparallel$$

$$\frac{g(t-h) - g(t)}{-h}$$

So the left derivative exists and is the same as the right derivative:

$$g_-'(t) = g(t)\left(-\lambda + \lambda e^{-u}\right)$$

(see p. 23)

We will usually do the argument only for $h > 0$, $h \to 0$, but in most cases similar reasoning as above will yield the corresponding expressions for the limits as $h < 0$, $h \to 0$.

Alternative method

Let $p_n(t) = \mathbb{P}\{N(t) = n\}$, $n = 0, 1, 2, \dots$

We want to show that Def 5.3. implies that $p_n(t) = e^{-\lambda t} \cdot \frac{(\lambda t)^n}{n!}$.

Let $n = 0$.

$$\underbrace{\mathbb{P}(N(t+h) = 0)}_{p_0(t+h)} = \mathbb{P}(N(t+h) - N(t) = 0, N(t) = 0)$$

$$\underset{\text{indep.}}{=} \mathbb{P}(N(t+h) - N(t) = 0) \cdot \underbrace{\mathbb{P}(N(t) = 0)}_{= p_0(t)}$$

$$\underset{\text{stat. + (iii) of Def 5.3}}{=} (1 - \lambda h + o(h))\, p_0(t)$$

$\therefore$ $p_0(t+h) - p_0(t) = p_0(t)(-\lambda h + o(h))$

$\therefore$ $\frac{p_0(t+h) - p_0(t)}{h} = p_0(t)\left(-\lambda + \frac{o(h)}{h}\right)$

As $h \to 0$ we get

$$p_0'(t) = -\lambda\, p_0(t) + 0$$

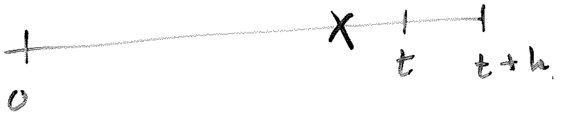
$$p_0(t) = e^c \cdot e^{-\lambda t}$$ but $C = 0$ since $p_0(0) = 1$.

$\underline{n=1}$

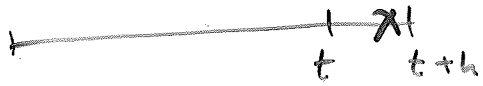
$$p_1(t+h) = \mathbb{P}\big(N(t+h) = 1\big) =$$

$$\mathbb{P}\big(N(t+h) - N(t) = 0,\ N(t) = 1\big) \;+$$

$$\mathbb{P}\big(N(t+h) - N(t) = 1,\ N(t) = 0\big)$$



or



$$= \underbrace{\mathbb{P}\big(N(h) = 0\big)}_{1-\lambda h + o(h)}\ \underset{\cdot\ p_1(t)}{\mathbb{P}\big(N(t) = 1\big)} + \underset{\lambda h + o(h)}{\mathbb{P}\big(N(h)=1\big)}\ \underset{\cdot\ p_0(t)}{\mathbb{P}\big(N(t)=0\big)}$$

$\uparrow$
Def 5.3.
(ii, iii)

Thus

$$p_1(t+h) = \big(1 - \lambda h + o(h)\big)\,p_1(t) + \big(\lambda h + o(h)\big)\,p_0(t)$$

yielding

$$\frac{p_1(t+h) - p_1(t)}{h} = \left(-\lambda + \frac{o(h)}{h}\right)\cdot p_1(t) + \left(\lambda + \frac{o(h)}{h}\right)\cdot p_0(t)$$

When $h \to 0+$ we get

$$p_1'(t) = -\lambda\,p_1(t) + \lambda\,p_0(t)$$

Before solving the eq. for $p_1(t)$ let us take an $n > 1$. Then the same kind of argument as above give a recursion formula for $p_n(t)$:

$$\mathbb{P}(N(t+h)=n) = \mathbb{P}(N(h)=0)\,\mathbb{P}(N(t)=n) + \mathbb{P}(N(h)=1)\,\mathbb{P}(N(t)=n-1) + \mathbb{P}(N(h)=2)\cdot \mathbb{P}(N(t)=n-2) + \cdots$$

$$= (1-\lambda h + o(h))\cdot p_n(t) + (\lambda h + o(h))\cdot p_{n-1}(t)$$

$$+ o(h) \leftarrow \text{because } \mathbb{P}(N(h)\geq 2) = o(h)$$

Hence

$$p_n(t+h) - p_n(t) = -\lambda h\cdot p_n(t) + \lambda h\, p_{n-1}(t) + o(h)$$

Then

$$(*)\quad p_n'(t) = -\lambda p_n(t) + \lambda p_{n-1}(t) \qquad n \geq 1$$

Solution of $(*)$:

Call $Q_n(t) = e^{+\lambda t} p_n(t)$ auxiliary fct

$$Q_n'(t) = +\lambda e^{+\lambda t} p_n(t) + e^{+\lambda t} p_n'(t)$$

$$= +\lambda e^{+\lambda t} p_n(t) + e^{+\lambda t}(-\lambda p_n(t) + \lambda p_{n-1}(t))$$

$$= \lambda Q_{n-1}(t)$$

$$Q_0(t) = 1$$
$$Q_1'(t) = \lambda \cdot 1 \implies Q_1(t) = \lambda t + C$$
$$Q_1(0) = e^{\lambda \cdot 0} \underbrace{p_1(0)}_{=0} = 0$$

$$\therefore \quad Q_1(t) = \lambda t$$
$$Q_2'(t) = \lambda \cdot Q_1(t) = \lambda^2 t$$
$$Q_2(t) = \frac{\lambda^2 t^2}{2} + C \quad ; \quad C = 0 \text{ because } p_2(0) = 0$$

By induction
$$Q_n(t) = \frac{\lambda^n t^n}{n!} \qquad \text{(easy!)}$$

Thus
$$\underbrace{\mathbb{P}(N(t) = n)}_{= p_n(t)} = e^{-\lambda t} \cdot Q_n(t) = e^{-\lambda t} \cdot \frac{(\lambda t)^n}{n!}$$
$$n = 0, 1, 2, \ldots$$

$\therefore$ $N(t)$ is Poisson-distributed with parameter λt. $\square$

Remark. The assumptions in Def 5.3. can be replaced by

(i) $N(0) = 0$
(ii) The process has independent increments
(iii) $\mathbb{P}(N(t+h) - N(t) = 1) = \lambda h + o(h)$
(iv) $\mathbb{P}(N(t+h) - N(t) \geq 2) = o(h)$